Finding a Function from a Rate of Change

Integration to Find a Function from a Rate of Change

Given a derivative function $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ and a boundary condition such as a coordinate (x_0, y_0), the original function can be determined using integration or anti-differentiation.

Anti-Differentiation

Find the primitive function, then substitute the coordinate to find the constant of integration.

$$F(x) = \int f(x)\mathrm{d}x + c$$

$$y_0 = F(x_0) + c \quad \Rightarrow \quad c = y_0 - F(x_0)$$

Integration

We know that $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$ regardless of the constant of integration.

If we let $F(x)$ be the antiderivative function with the correct constant of integration, then using a dummy variable t: $\int_{x_0}^{x} f(t)\mathrm{d}t = F(x) - F(x_0)$.

However, we do not want $F(x) - F(x_0)$, we want $F(x)$.

If we know that (x_0, y_0) lies on the graph of $F(x)$ then $F(x_0) = y_0$.

If we add y_0 to the integral we shall get $F(x)$. That is, $F(x) = \int_{x_0}^{x} f(t)\, dt + y_0$

Example VCAA 2006 Sample Exam 1 Question 2b

If $f\colon (-\infty, 2) \rightarrow R$ is such that $f'(x) = \frac{1}{x-2}$ and $f(1) = 6$. The rule of f is

Using Anti-Differentiation

$\int \frac{1}{x-2}\mathrm{d}x = \log_e\big(-(x-2)\big) + c$ since $x \in (-\infty, 2)$

$$f(1) = \log_e\big(-(1-2)\big) + c$$

$$6 = \log_e(1) + c$$

$$c = 6$$

$$\therefore f(x) = \log_e\big(-(x-2)\big) + 6$$

Using Integration

$f(x) = \int_1^x \frac{1}{t-2}\mathrm{d}t + 6 = \big[\log_e\big(-(t-2)\big)\big]_1^x + 6,$ since $t \in (-\infty, 2)$

$$f(x) = \log_e\big(-(x-2)\big) - \log_e\big(-(1-2)\big) + 6$$

$$f(x) = \log_e\big(-(x-2)\big) - \log_e(1) + 6$$

$$f(x) = \log_e\big(-(x-2)\big) + 6$$