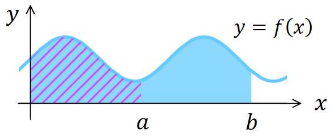
Area Bound by a Curve

Areas Bound by a Curve and the x-Axis

The area between the curve $y = f(x)$ and the x-axis is equal to the definite integral of f over the specified domain. That is, the area from $x = a$ to $x = b$ is $\int_a^b y\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$ units^2

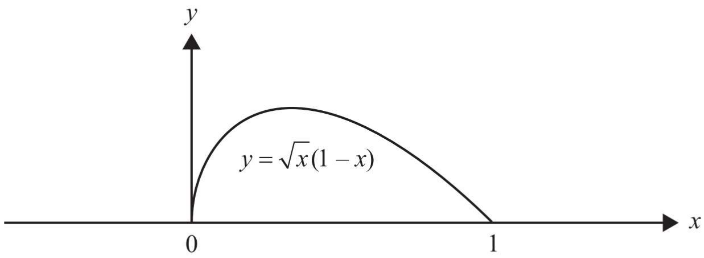


If the question asks for the area specifically, don't forget to write the unit!
If no unit is specified in the question use "units^2" or "square units".

You can use symmetry properties to help calculate areas. If one area of the required region is repeated, you can work on the smaller area and multiply it to fill the required region.

Example VCAA 2017 Exam 1 Question 9a

The graph of $f: [0,1] \to R, f(x) = \sqrt{x}(1-x)$ is shown below.



The area between the graph of f and the x-axis is

$$A = \int_0^1 \sqrt{x}(1-x)\,\mathrm{d}x = \int_0^1 \left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^1 = \left(\frac{2}{3} - \frac{2}{5} - (0)\right) = \frac{4}{15} \text{ square units.}$$

Example VCAA 2008 Exam 1 Question 5

The area of the region bounded by the y-axis, the x-axis, the curve $y = e^{2x}$ and the line $x = C$, where C is a positive real constant, is $\frac{5}{2}$. The value of C is

$$\int_0^C e^{2x}\mathrm{d}x = \frac{5}{2} \Rightarrow \left[\frac{1}{2}e^{2x}\right]_0^C = \frac{5}{2} \Rightarrow e^{2C} - 2^0 = 5 \Rightarrow e^{2C} = 6 \Rightarrow 2C = \log_e(6) \Rightarrow C = \frac{1}{2}\log_e(6)$$

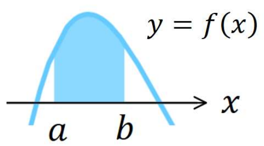
Signed Area of Regions Bounded by a Curve and an Axis

Regions above the x-axis have a positive signed area.
Regions below the x-axis have a negative signed area.

To change from negative to positive signed area either multiply by -1 or swap the bounds around.
If the function is partially below and partially above then split the integral into sections.

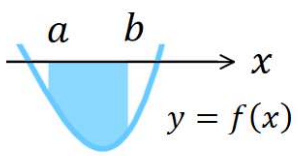
The signed area under the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$

$f(x) \geq 0$

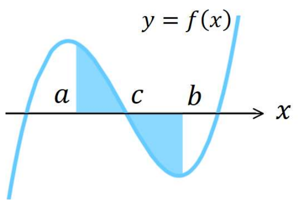


$A = \int_a^b f(x)\mathrm{d}x$ units^2

$f(x) < 0$



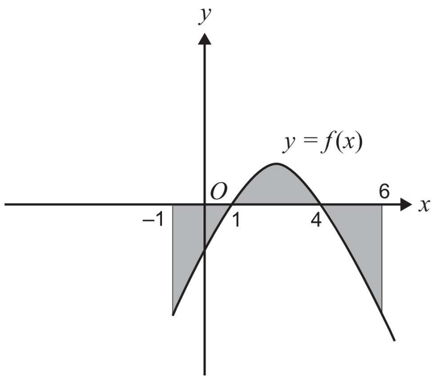
$A = -\int_a^b f(x)\mathrm{d}x = \int_b^a f(x)\mathrm{d}x$ units^2



$A = \int_a^c f(x)\mathrm{d}x - \int_c^b f(x)\mathrm{d}x$ units^2

Example VCAA 2006 Exam 2 Question 15

The total area of the shaded regions in the diagram is given by



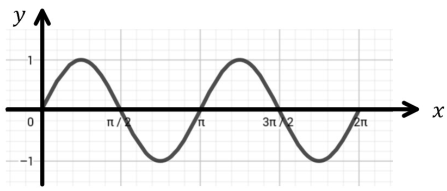
$$-\int_{-1}^{1} f(x)\mathrm{d}x + \int_{1}^{4} f(x)\mathrm{d}x - \int_{4}^{6} f(x)\mathrm{d}x$$

$$\int_{1}^{-1} f(x)\mathrm{d}x + \int_{1}^{4} f(x)\mathrm{d}x + \int_{6}^{4} f(x)\mathrm{d}x$$

$$\int_{1}^{4} f(x)\mathrm{d}x - 2\int_{4}^{6} f(x)\mathrm{d}x,$$ if f is symmetric

Example VCAA 2003 Exam 1 Question 16

The total area of the regions enclosed by the graph of the function with equation $y = \sin(2x)$, and the x-axis between $x = 0$ and $x = 2\pi$, is equal to

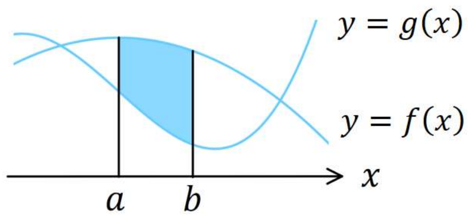


$$A = \int_{0}^{\frac{\pi}{2}} \sin(2x)\,\mathrm{d}x - \int_{\frac{\pi}{2}}^{\pi} \sin(2x)\,\mathrm{d}x + \int_{\pi}^{\frac{3\pi}{2}} \sin(2x)\,\mathrm{d}x - \int_{\frac{3\pi}{2}}^{2\pi} \sin(2x)\,\mathrm{d}x = 4\int_{0}^{\frac{\pi}{2}} \sin(2x)\,\mathrm{d}x$$

$$= 4\left[-\frac{1}{2}\cos(2x)\right]_{0}^{\frac{\pi}{2}} = -2(\cos(\pi) - \cos(0)) = -2(-1-1) = 4 \text{ square units}$$

Area Between Curves

The area between the curves $y = f(x)$ and $y = g(x)$ if $f(x) > g(x)$ for $x \in [a, b]$ will be



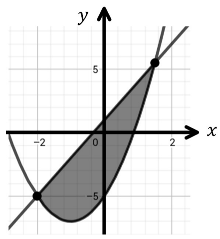
$$\int_{a}^{b} (f(x) - g(x))\mathrm{d}x = \int_{a}^{b} (y_1 - y_2)\mathrm{d}x$$

$$= \int_{a}^{b} f(x)\mathrm{d}x - \int_{a}^{b} g(x)\mathrm{d}x = \int_{a}^{b} (f - g)(x)\mathrm{d}x$$

Example VCAA 2002 Exam 1 Question 8b

The x-coordinates of the points of intersection of the graphs of $y = 3x + 1$ and $y = 2x^2 + 4x - 5$ are $x = -2$ and $\frac{3}{2}$.

The area of the region bounded by the line with equation $y = 3x + 1$ and the parabola with equation $y = 2x^2 + 4x - 5$ is



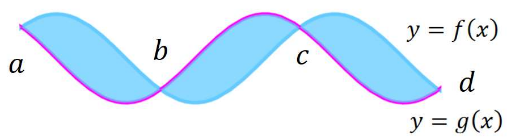
$$A = \int_{-2}^{\frac{3}{2}} [(3x+1) - (2x^2 + 4x - 5)]\mathrm{d}x = \int_{-2}^{\frac{3}{2}} (-2x^2 - x + 6)\mathrm{d}x = \left[-\frac{2}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_{-2}^{\frac{3}{2}}$$

$$= -\frac{2}{3}\left(\frac{3}{2}\right)^3 - \frac{1}{2}\left(\frac{3}{2}\right)^2 + 6\left(\frac{3}{2}\right) - \left(-\frac{2}{3}(-2)^3 - \frac{1}{2}(-2)^2 + 6(-2)\right)$$

$$= -\frac{9}{4} - \frac{9}{8} + 9 - \left(\frac{16}{3} - 2 - 12\right) = \frac{45}{8} + \frac{26}{3} = \frac{343}{24} = 14.2916666667 \approx 14.292 \text{ square units}$$

Signed Area Between Curves

When the curves cross and the curve on top and bottom switch, the area between them becomes negative. Therefore, to determine the area (not the integral), the same precautions can be taken as signed areas (changing the sign or switching the bounds), in addition to switching the top and bottom functions.



Top Function − Bottom Function

$$A=\int_a^b \big(f(x)-g(x)\big)\mathrm{d}x+\int_b^c \big(g(x)-f(x)\big)\mathrm{d}x+\int_c^d \big(f(x)-g(x)\big)\mathrm{d}x$$

Change the sign from b to c

$$A=\int_a^b \big(f(x)-g(x)\big)\mathrm{d}x-\int_b^c \big(f(x)-g(x)\big)\mathrm{d}x+\int_c^d \big(f(x)-g(x)\big)\mathrm{d}x$$

Change the bounds from b to c

$$A=\int_a^b \big(f(x)-g(x)\big)\mathrm{d}x+\int_c^b \big(f(x)-g(x)\big)\mathrm{d}x+\int_c^d \big(f(x)-g(x)\big)\mathrm{d}x$$

Example VCAA 2007 Exam 2 Question 3b

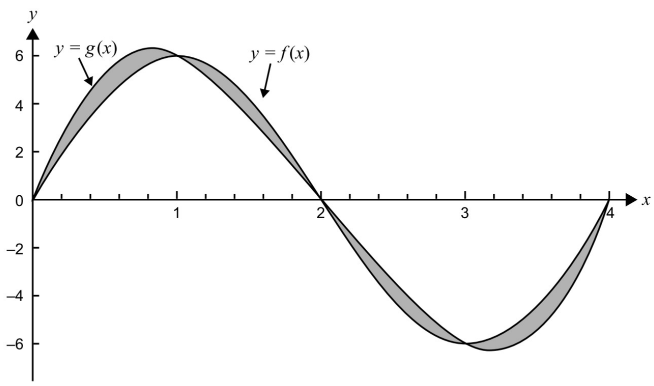
Shown below are the graphs of the functions

$f\colon[0,4]\to R, f(x)=6\sin\left(\frac{\pi x}{2}\right)$ and

$g\colon[0,4]\to R, g(x)=2x(x-2)(x-4)$
$=2(x^3-6x^2+8x)$

The point $(1,6)$ lies on both graphs.

The total area of these shaded regions is



$$A=\int_0^1 \big(g(x)-f(x)\big)\mathrm{d}x-\int_1^2 \big(g(x)-f(x)\big)\mathrm{d}x+\int_2^3 \big(g(x)-f(x)\big)\mathrm{d}x-\int_3^4 \big(g(x)-f(x)\big)\mathrm{d}x$$

$$\int\big(g(x)-f(x)\big)\mathrm{d}x=\int\left(2x^3-12x^2+16x-6\sin\left(\frac{\pi x}{2}\right)\right)\mathrm{d}x=\frac{1}{2}x^4-4x^3+8x^2+\frac{12}{\pi}\cos\left(\frac{\pi x}{2}\right)$$

Let $I(x)=\int\big(g(x)-f(x)\big)\mathrm{d}x=\frac{1}{2}x^4-4x^3+8x^2+\frac{12}{\pi}\cos\left(\frac{\pi x}{2}\right)$

$$I(0)=\frac{12}{\pi},\quad I(1)=\frac{9}{2},\quad I(2)=8-\frac{12}{\pi},\quad I(3)=\frac{9}{2},\quad I(4)=\frac{12}{\pi}$$

$$\therefore A=I(1)-I(0)-\big(I(2)-I(1)\big)+I(3)-I(2)-\big(I(4)-I(3)\big)$$

$$=\frac{9}{2}-\frac{12}{\pi}-\left(8-\frac{12}{\pi}-\frac{9}{2}\right)+\frac{9}{2}-\left(8-\frac{12}{\pi}\right)-\left(\frac{12}{\pi}-\frac{9}{2}\right)=2\text{ square units}$$

Areas Bound by the y-Axis

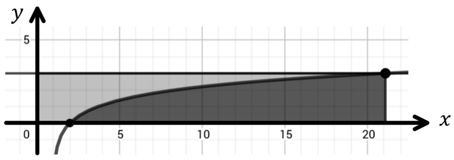
There are multiple ways of finding areas bound by the y-axis between two y values. This includes

- finding the area between the $y = b$ (the top y value) and the curve (which is equivalent to subtracting the area between the curve and the x-axis from the area of the surrounding rectangle)
- finding the area between the curve's inverse and the x-axis (ensure that you use the y values as the bounds of the inverse not the x values, that is the values of the inverse at the bounds)

Choosing an appropriate method is important as some functions are easier to integrate than others, some functions are difficult to find inverses of, and some areas are hard to set up integrals for in the wrong form.

Example VCAA 2009 Exam 2 Question 22

Consider the region bounded by the x-axis, the y-axis, the line with equation $y = 3$ and the curve with equation $y = \log_e(x - 1)$. The exact value of the area of this region is



Finding the area between the curve and the x-axis requires integrating a log function which is not straight forward. So using the inverse is suggested

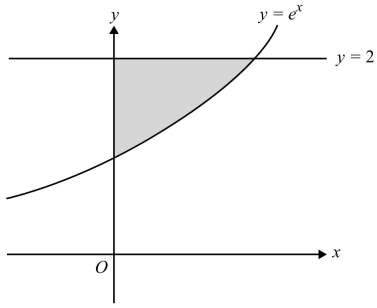
$y^{-1} = e^x + 1, \qquad y^{-1}(3) = e^3 + 1$

Area using the inverse of y: $y^{-1} = e^x + 1$.

$$\int_0^3 (e^x + 1)\mathrm{d}x = e^3 + 1 - (e^0 + 0) = e^3 + 2 \text{ square units}$$

Example VCAA 2004 Exam 1 Question 26

Parts of the graphs with equations $y = e^x$ and $y = 2$ are shown below. The graphs intersect at $(\log_e(2), 2)$. The total area, bounded by the y-axis, the line $y = 2$ and the curve with equation $y = e^x$ is given by



Area of rectangle − area under $y = \log_e(2) \times 2 - \int_0^{\log_e(2)} e^x \mathrm{d}x$

Area between the curves $y = 2$ and $y = e^x = \int_0^{\log_e(2)} (2 - e^x)\mathrm{d}x$

Area using the inverse of y: $y^{-1} = \log_e(x) \Rightarrow A = \int_1^2 \log_e(x)\, \mathrm{d}x$