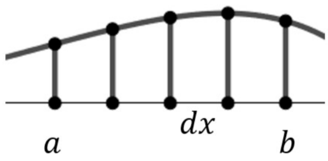
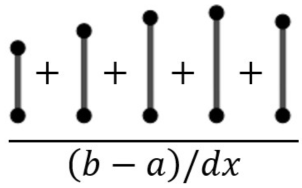
Average Value

Average Value of a Function

The average height of a continuous function over a interval.

The average of a finite list of heights is found by adding up all the of heights, then dividing by how many heights are in the list. The average for an infinite list would require adding up infinitely many heights and dividing by this infinite number of heights. To do this we take an alternate route:



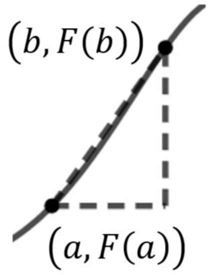


For the interval $x \in [a, b]$, we divide it up into chunks of width $\mathrm{d}x$.

The number of heights in the list would be $\dfrac{b-a}{\mathrm{d}x}$.

The average would then be

$$\lim_{dx\to 0}\frac{\sum_{x=a}^{b} f(x)}{\left(\dfrac{b-a}{\mathrm{d}x}\right)} = \lim_{dx\to 0}\frac{\sum_{x=a}^{b} f(x)\mathrm{d}x}{b-a} = \frac{\int_a^b f(x)\mathrm{d}x}{b-a} = \frac{F(b)-F(a)}{b-a}$$



So the average value of a function is equal to the average rate of change of the primitive function over the same interval.

Think: $f(x)$ is the gradient of $F(x)$, so the average of all the gradients on the interval is the gradient between the two endpoints, which is much easier to calculate.

Warning

Be careful when doing average questions that you check whether it is average value or average rate.

$$\text{average value} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{F(b)-F(a)}{b-a} \qquad \text{average rate of change} = \frac{f(b)-f(a)}{b-a}$$

Example VCAA 2006 Exam 2 Question 8

The average value of the function $y = \cos(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

$$\frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}}\cos(x)\,\mathrm{d}x = \frac{2}{\pi}[\sin(x)]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) = \frac{2}{\pi}(1-0) = \frac{2}{\pi}$$

Average Value and Area

Alternatively, the average value of a function is the height at which a rectangle between two points, a and b, has the same area as the definite integral between the same two points, a and b.

f_{ave} = average value of f

$$\int_a^b f(x)\mathrm{d}x = (b-a)\times f_{\text{ave}}$$

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$