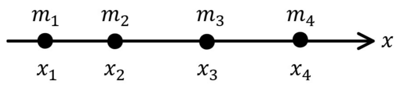
Discrete Random Variables

Discrete Random Variable and the Probability Mass Function (pmf)

A discrete random variable can assume only countable or integer values.

For a discrete random variable, we define a function (the probability mass function or pmf) that describes the mass of the probability, that is the value of the probability at different positions. That is, $\Pr(X = x) = p(x)$.



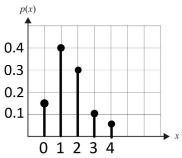
To obtain the total mass we add up the masses.
This is also true of the probability mass function.

$$\text{Total Mass (Probability)} = \sum p(x)$$

The pmf can be given as a function, table, or graph. The values of a pmf must be all be positive and total one. That is, $0 \le \Pr(X = x) \le 1$ and $\sum \Pr(X = x) = 1$

Example VCAA 2005 E1 Q2

A graph of the probability distribution of the random variable X is shown.



Example VCAA 2018 NHT E1 Q6

The discrete random variable X has the probability mass function

$$\Pr(X = x) = \begin{cases} \dfrac{x}{12} & x \in \{1,4,6\} \\ \dfrac{1}{12} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

Example Modified VCAA 2017 NHT Exam 2 Question 19

The first digit of each house number in a large address database is a random variable D.

The possible values of D are $1, 2, \dots, 9$ and $\Pr(D = d) = \log_{10}\left(1 + \dfrac{1}{d}\right)$.

Example VCAA 2004 Exam 2 Question 2a

A LUCKYDIPZ is a toy that is packaged inside an egg-shaped chocolate.
A certain manufacturer produces 4 different types of LUCKYDIPZ toy – car, ship, plane and ring, in the proportions given in the table below.
Find the exact value of k.

car	$4k^2 + 4k$
ship	$5k^2 + 2k$
plane	$k^2 + k$
ring	$2k$

The total probability must be 1. Therefore,
$4k^2 + 4k + 5k^2 + 2k + k^2 + k + 2k = 1 \quad \Rightarrow \quad 10k^2 + 9k - 1 = 0$
The solutions to the quadratic are $k = -1$ and $k = 0.1$.

Since all the probabilities must be between 0 and 1, $k \neq -1$ as the probability of ring would equal -2. When $k = 0.1$ the probabilitity distribution is:

car	0.44
ship	0.25
plane	0.11
ring	0.2

Example VCAA 2003 Exam 1 Question 24

Valid probability distributions of a discrete random variable

w	-2	-1	0	1
$\Pr(W = w)$	0.2	0.3	0.3	0.2

$\sum \Pr(W = w) = 0.2 + 0.3 + 0.3 + 0.2 = 1$
$0 \le \Pr(W = w) \le 1, \qquad w = -2, -1$ is fine

x	10	20	30	40
$\Pr(X = x)$	0.4	0.3	0.2	0.1

$\sum \Pr(X = x) = 0.4 + 0.3 + 0.2 + 0.1 = 1$
$0 \le \Pr(X = x) \le 1$

z	1	2	3	4
$\Pr(Z = z)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$

$\sum \Pr(Z = z) = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} = 1$
$0 \le \Pr(Z = z) \le 1$

Invalid probability distributions of a discrete random variable

v	2	3	4	5
$\Pr(V = v)$	0.1	0.2	0.4	0.5

$\sum \Pr(V = v) = 0.1 + 0.2 + 0.4 + 0.5 = 1.2$
$0 \le \Pr(V = v) \le 1, \ \sum \Pr(V = v) \neq 1$

y	0	1	2	3
$\Pr(Y = y)$	-0.1	0.4	0.5	0.2

$\sum \Pr(Y = y) = -0.1 + 0.4 + 0.5 + 0.2 = 1$
$\Pr(Y = 0) = -0.1 < 0$

Calculating Probabilities using Discrete Random Variables

Individual probabilities can be read off or evaluated from the probability mass function.
Other probabilities can be determined using the addition rule with mutually exclusive x values or using the multiplication rule with independent x values.

Example VCAA 2004 Exam 1 Question 1

Mollie has constructed a spinner that will randomly display an integer between 0 and 4, inclusive, with the following probabilities.

Number	x	0	1	2	3	4
Probability	$\Pr(X = x)$	0.2	0.3	0.15	0.25	0.1

If Mollie spins the spinner once, the probability of obtaining an odd number is
$\Pr(\text{odd}) = \Pr(X = 1) + \Pr(X = 3) = 0.3 + 0.25 = 0.55$

Example Modified VCAA 2016 Sample Exam 1 Question 4 / VCAA 2012 Exam 1 Question 4

Let X be the random variable that represents the number of phone calls that Daniel receives on any given day with the probability distribution given by the table below.

x	0	1	2	3
$\Pr(X = x)$	0.2	0.2	0.5	0.1

The probability that Daniel receives 1 phone call on Monday, 2 on Tuesday, and 3 on Wednesday

$\Pr(X = 1) \times \Pr(X = 2) \times \Pr(X = 3)$
$= 0.2 \times 0.5 \times 0.1 = 0.01$

The probability that Daniel receives only one phone call on each of three consecutive days

$[\Pr(X = 1)]^3 = (0.2)^3 = 0.008$

The probability that Daniel receives the same number of phone calls on each of two consecutive days

$[\Pr(X = 0)]^2 + [\Pr(X = 1)]^2 + [\Pr(X = 2)]^2 + [\Pr(X = 3)]^2$
$= (0.2)^2 + (0.2)^2 + (0.5)^2 + (0.1)^2$
$= 0.04 + 0.04 + 0.25 + 0.01 = 0.34$

Example Modified VCAA 2017 NHT Exam 2 Question 19

The first digit of each house number in a large address database is a random variable D.

The possible values of D are $1, 2, \ldots, 9$ and $\Pr(D = d) = \log_{10}\left(1 + \frac{1}{d}\right)$.

The probability that D is equal to 4 is $\Pr(D = 4) = \log_{10}\left(1 + \frac{1}{4}\right) = \log_{10}\left(\frac{5}{4}\right)$

Inequalities and Probabilities

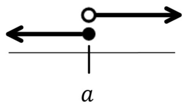
Once a pmf (or the required probabilities of the pmf) have been calculated, they are added together for probabilities that use $>, \geq, <$, and $\leq$. For $>$ and $<$, the probability at the boundary is not added.

Some common phrases involving inequalities include:

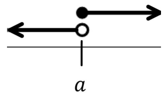
- The probability of no more than a: $\Pr(X \leq a)$
- The probability of fewer than a: $\Pr(X < a)$
- The probability of at least a / of no fewer than a: $\Pr(X \geq a)$
- The probability of more than a: $\Pr(X > a)$

Complementary Probabilities with Inequalities

$1 = \Pr(X \leq a) + \Pr(X > a)$
$\Rightarrow \Pr(X \leq a) = 1 - \Pr(X > a)$
$\Rightarrow \Pr(X > a) = 1 - \Pr(X \leq a)$



$1 = \Pr(X < a) + \Pr(X \geq a)$
$\Rightarrow \Pr(X < a) = 1 - \Pr(X \geq a)$
$\Rightarrow \Pr(X \geq a) = 1 - \Pr(X < a)$



Example

The random variable X has this probability distribution.

X	0	1	2	3	4
$\Pr(X = x)$	0.15	0.3	0.4	0.05	0.1

$\Pr(X \leq 2) = \Pr(X < 3)$
$= 0.15 + 0.3 + 0.4 = 0.85$
$= 1 - \Pr(X > 2) = 1 - \Pr(X \geq 3)$

X	0	1	2	3	4
$\Pr(X = x)$	0.15	0.3	0.4	0.05	0.1

$\Pr(X < 2) = \Pr(X \leq 1)$
$= 0.15 + 0.3 = 0.45$
$= 1 - \Pr(X \geq 2) = 1 - \Pr(X > 1)$

X	0	1	2	3	4
$\Pr(X = x)$	0.15	0.3	0.4	0.05	0.1

$\Pr(X \geq 2) = \Pr(X > 1)$
$= 0.4 + 0.05 + 0.1 = 0.55$
$= 1 - \Pr(X < 2) = 1 - \Pr(X \leq 1)$

X	0	1	2	3	4
$\Pr(X = x)$	0.15	0.3	0.4	0.05	0.1

$\Pr(X > 2) = \Pr(X \geq 3)$
$= 0.05 + 0.1 = 0.15$
$= 1 - \Pr(X \leq 2) = 1 - \Pr(X < 3)$

X	0	1	2	3	4
$\Pr(X = x)$	0.15	0.3	0.4	0.05	0.1

$\Pr(1 < X \leq 3) = \Pr(2 \leq X < 4)$
$= 0.4 + 0.05 = 0.45$

X	0	1	2	3	4
$\Pr(X = x)$	0.15	0.3	0.4	0.05	0.1

$\Pr(1 \leq X < 3) = \Pr(0 < X \leq 2)$
$= 0.3 + 0.4 = 0.7$

X	0	1	2	3	4
$\Pr(X = x)$	0.15	0.3	0.4	0.05	0.1

Example VCAA 2012 Exam 2 Question 20

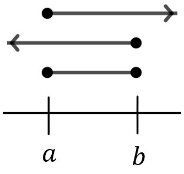
A discrete random variable X has the probability function $\Pr(X = k) = (1-p)^k p$, where $k \in Z^+ \cup \{0\}$. $\Pr(X > 1)$ is equal to

$$\begin{aligned}
\Pr(X > 1) &= 1 - \Pr(X = 0) - \Pr(X = 1) \\
&= 1 - \left((1-p)^0 p\right) - \left((1-p)^1 p\right) \\
&= 1 - p - (p - p^2) \\
&= 1 - 2p + p^2 \\
&= (1-p)^2
\end{aligned}$$

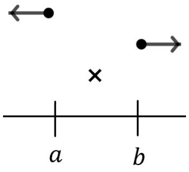
Conditional Probabilities with Inequalities

Until now, the intersection of the events is either given or can be deduced from the information. However, when the events are inequalities the intersections can be determined using number lines.

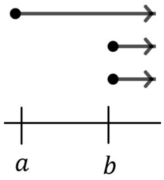
$X \geq a \cap X \leq b$
$= a \leq X \leq b$



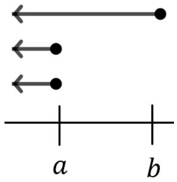
$X \leq a \cap X \geq b$
$= X \in \emptyset$



$X \geq a \cap X \geq b$
$= X \geq b$



$X \leq a \cap X \leq b$
$= X \leq a$



Example VCAA 2009 E1 Question 7a

The random variable X has this probability distribution.

X	0	1	2	3	4
$\Pr(X = x)$	0.1	0.2	0.4	0.2	0.1

$$\Pr(X > 1 | X \leq 3) = \frac{\Pr(X > 1 \cap X \leq 3)}{\Pr(X \leq 3)} = \frac{\Pr(1 < X \leq 3)}{1 - \Pr(X = 4)} = \frac{0.4 + 0.2}{1 - 0.1} = \frac{0.6}{0.9} = \frac{2}{3}$$

Example VCAA 2017 NHT Exam 2 Question 19

The first digit of each house number in a large address database is a random variable D.

The possible values of D are $1, 2, \ldots, 9$ and $\Pr(D = d) = \log_{10}\left(1 + \frac{1}{d}\right)$.

The probability that D is greater than 8, given that D is greater than 7, is

$$\Pr(D > 8 | D > 7) = \frac{\Pr(D > 8 \cap D > 7)}{\Pr(D > 7)} = \frac{\Pr(D > 8)}{\Pr(D > 7)} = \frac{\Pr(D = 9)}{\Pr(D = 8) + \Pr(D = 9)}$$

$$= \frac{\log_{10}\left(\frac{10}{9}\right)}{\log_{10}\left(\frac{9}{8}\right) + \log_{10}\left(\frac{10}{9}\right)} = \frac{\log_{10}\left(\frac{10}{9}\right)}{\log_{10}\left(\frac{10}{8}\right)} = \frac{\log_{10}(10) - \log_{10}(9)}{\log_{10}(10) - \log_{10}(8)} = \frac{1 - \log_{10}(9)}{1 - \log_{10}(8)}$$