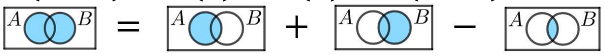
Events from a Sample Space

The Addition Rule

When finding the probability of a union of events, that is the probability that either event or both events occur, we count up all the outcomes.

In terms of probabilities, we add the probability of each event. However, this will double count any intersection there is between the events. To compensate, we remove the intersection once.

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$



	B	B'	Total
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
Total	$\Pr(B)$	$\Pr(B')$	1

$$\Pr(A \cup B) = \Pr(A \cap B) + \Pr(A' \cap B) + \Pr(A \cap B')$$

Using a Karnaugh map, addition statements can be formed involving intersections and unions:

$$\Pr(A \cap B) + \Pr(A' \cap B) + \Pr(A \cap B') + \Pr(A' \cap B') = 1$$

	B	B'	Total
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
Total	$\Pr(B)$	$\Pr(B')$	1

$$\Pr(A \cup B) = 1 - \Pr(A' \cap B')$$

	B	B'	Total
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
Total	$\Pr(B)$	$\Pr(B')$	1

Example Modified VCAA 2007 Exam 1 Question 6

Two events, A and B, from a given event space, are such that $\Pr(A) = \frac{1}{5}$ and $\Pr(B) = \frac{1}{3}$.

If $\Pr(A \cap B) = \frac{1}{8}$ then $\Pr(A \cup B)$ is

	B	B'	Total
A	$\frac{1}{8} = \frac{15}{120}$		$\frac{1}{5} = \frac{24}{120}$
A'			
Total	$\frac{1}{4} = \frac{40}{120}$		$1 = \frac{120}{120}$

$$\frac{24}{120} + \frac{40}{120} - \frac{15}{120} = \frac{49}{120}$$

	B	B'	Total
A	$\frac{15}{120}$	$\frac{9}{120}$	$\frac{24}{120}$
A'	$\frac{25}{120}$	$\frac{71}{120}$	$\frac{96}{120}$
Total	$\frac{40}{120}$	$\frac{80}{120}$	$\frac{120}{120}$

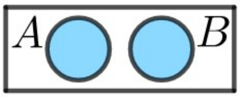
$$\frac{120}{120} - \frac{71}{120} = \frac{49}{120}$$

	B	B'	Total
A	$\frac{15}{120}$	$\frac{9}{120}$	$\frac{24}{120}$
A'	$\frac{25}{120}$	$\frac{71}{120}$	$\frac{96}{120}$
Total	$\frac{40}{120}$	$\frac{80}{120}$	$\frac{120}{120}$

$$\frac{15}{120} + \frac{25}{120} + \frac{9}{120} = \frac{49}{120}$$

Mutually Exclusive

Two events are mutually exclusive when they that cannot occur at the same time. The probability of $\Pr(A \cap B) = 0$. Hence, $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.



Example Modified VCAA 2007 Exam 1 Question 6

Two events, A and B, from a given event space, are such that $\Pr(A) = \frac{1}{5}$ and $\Pr(B) = \frac{1}{3}$.

when $\Pr(A \cap B) = \frac{1}{8}$

	B	B'	Total
A	$\frac{1}{8}$	$\Pr(A \cap B')$	$\frac{1}{5}$
A'	$\Pr(A' \cap B)$		$\frac{4}{5}$
Total	$\frac{1}{3}$	$\frac{2}{3}$	1

$$\Pr(A' \cap B) = \frac{1}{3} - \frac{1}{8} = \frac{40}{120} - \frac{15}{120} = \frac{25}{120} = \frac{5}{24}$$

$$\Pr(A \cap B') = \frac{1}{5} - \frac{1}{8} = \frac{24}{120} - \frac{15}{120} = \frac{9}{120} = \frac{3}{40}$$

when A and B are mutually exclusive events

	B	B'	Total
A	0	$\Pr(A \cap B')$	$\frac{1}{5}$
A'	$\Pr(A' \cap B)$		$\frac{4}{5}$
Total	$\frac{1}{3}$	$\frac{2}{3}$	1

$$\Pr(A' \cap B) = \frac{1}{3} - 0 = \frac{1}{3}$$

$$\Pr(A' \cap B) = \frac{1}{5} - 0 = \frac{1}{5}$$

Independent and Dependent Events

Independent Compound Events

Two events are independent if the events have no effect on the probability of the other occurring.
The probability of $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
The distribution follows a binomial distribution. Sampling with replacement is independent.

Dependent Compound Events

Two events are dependent if the events have an effect on the probability of the other occurring
The probability of $\Pr(A \cap B) = \Pr(A) \times \Pr(A|B)$, where $\Pr(A|B)$ is the probability that A occurs given B has occured. Sampling without replacement is dependent

Example VCAA 2008 Exam 2 Question 15

The sample space when a fair die is rolled is $\{1, 2, 3, 4, 5, 6\}$, with each outcome being equally likely.
For which of the following pairs of events are the events independent?

$\{1, 2, 3\}$ and $\{1, 2\}$ $\quad$ $\{1, 2\}$ and $\{3, 4\}$ $\quad$ $\{1, 3, 5\}$ and $\{1, 4, 6\}$ $\quad$ $\{1, 2\}$ and $\{1, 3, 4, 6\}$ $\quad$ $\{1, 2\}$ and $\{2, 4, 6\}$

For independent events $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. $\therefore A \cap B = \{1, 2\}$
$\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{1}{3}, \Pr(A \cap B) = \frac{1}{3} \neq \frac{1}{2} \times \frac{1}{3}$
$\therefore$ Dependent events

Let $A = \{1, 2\}$ and $B = \{3, 4\}$. $\therefore A \cap B = \{\ \}$
$\Pr(A) = \frac{1}{3}, \Pr(B) = \frac{1}{3}, \Pr(A \cap B) = 0 \neq \frac{1}{3} \times \frac{1}{3}$
$\therefore$ Dependent events

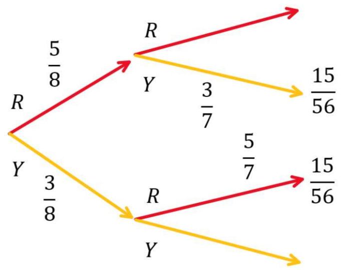
Let $A = \{1, 3, 5\}$ and $B = \{1, 4, 6\}$. $\therefore A \cap B = \{1\}$
$\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{1}{2}, \Pr(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \times \frac{1}{2}$
$\therefore$ Dependent events

Let $A = \{1, 2\}$ and $B = \{1, 3, 4, 6\}$. $\therefore A \cap B = \{1\}$
$\Pr(A) = \frac{1}{3}, \Pr(B) = \frac{2}{3}, \Pr(A \cap B) = \frac{1}{6} \neq \frac{1}{3} \times \frac{2}{3}$
$\therefore$ Dependent events

Let $A = \{1, 2\}$ and $B = \{2, 4, 6\}$. $\therefore A \cap B = \{2\}$
$\Pr(A) = \frac{1}{3}, \Pr(B) = \frac{1}{2}, \Pr(A \cap B) = \frac{1}{6} = \frac{1}{3} \times \frac{1}{2}$
$\therefore$ Independent events

Example VCAA 2017 Exam 2 Question 3

A box contains five red marbles and three yellow marbles. Two marbles are drawn at random from the box without replacement. The probability that the marbles are of different colours is the same as the probability of getting one red marble when selecting two marbles.



$$\Pr(R = 1) = \left(\frac{5}{8} \times \frac{3}{7}\right) + \left(\frac{3}{8} \times \frac{5}{7}\right) = \frac{15}{56} + \frac{15}{56} = \frac{30}{56} = \frac{15}{28}$$

Example VCAA 2015 Exam 1 Question 8c

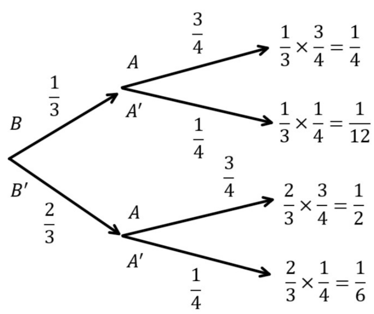
For events A and B from a sample space, $\Pr(A|B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

If events A and B are independent, calculate $\Pr(A \cup B)$.

$\Pr(A) = \Pr(A|B) = \frac{3}{4} \;\Rightarrow\; \Pr(A') = 1 - \frac{3}{4}$

$\Pr(B') = 1 - \frac{1}{3} = \frac{2}{3}, \qquad \Pr(A' \cap B') = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$

$\Rightarrow \Pr(A \cup B) = 1 - \frac{1}{6} = \frac{5}{6}, \qquad \Pr(A \cup B) = \frac{1}{4} + \frac{1}{12} + \frac{1}{2} = \frac{5}{6}$



Conditional Probability

Conditional probability can be considered as the probability with a reduced sample space. That is, you are considering the probability an event occurs given that you can only choose out of a smaller set.

The probability that:

A occurs given that B has already occurred is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{n(A \cap B)}{n(B)}$$

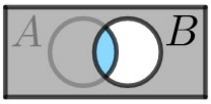
B occurs given that A has already occurred is

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{n(A \cap B)}{n(A)}$$

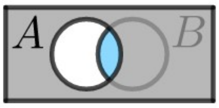
Conditional Probability and Venn Diagrams

To find conditional probabilities using a Venn diagram, restrict to one circle in the diagram.

$\Pr(A|B)$



$\Pr(B|A)$



Words that indicate conditional probability include: If…then…; Given…; Of…; Knowing that…

Conditional Probability and Two Way Tables and Karnaugh Maps

To find conditional probabilities using a two way table, restrict to one column or row of the table.

	B	B'	Total
A	$n(A \cap B)$	$n(A \cap B')$	$n(A)$
A'	$n(A' \cap B)$	$n(A' \cap B')$	$n(A')$
Total	$n(B)$	$n(B')$	$n(\text{Total})$

	B	B'	Total
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
Total	$\Pr(B)$	$\Pr(B')$	1

The Multiplication Rule

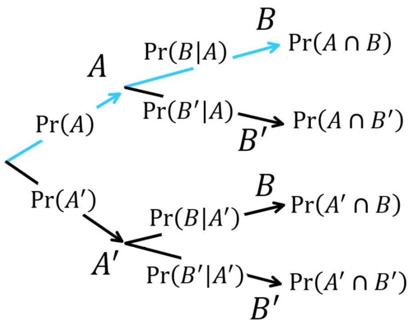
The conditional probability formula can also be rearranged to the multiplication rule.

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$$
$$\Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$$

To clarify this, consider the tree diagram. This is useful as tree diagrams are relatively intuitive to set up and allows the understanding that you multiply down a chain to be continued to be used.

Conditional Probability and Tree Diagrams

To find conditional probabilities using tree diagrams, it is the dependent event in a chain.

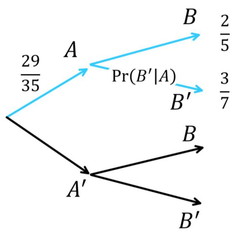


Example VCAA 2012 Exam 2 Question 13

A and B are events of a sample space S. $\Pr(A \cap B) = \frac{2}{5}$ and $\Pr(A \cap B') = \frac{3}{7}$. $\Pr(B'|A)$ is equal to

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B') = \frac{2}{5} + \frac{3}{7} = \frac{29}{35}, \quad \Pr(B'|A) = \frac{3}{7} \div \frac{29}{35} = \frac{3}{7} \times \frac{35}{29} = \frac{15}{29}$$

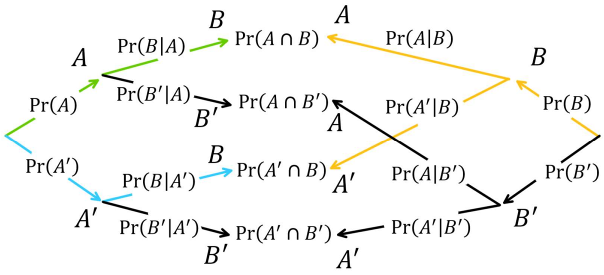
	B	B'	Total
A	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{29}{35}$
A'			
Total			1



Law of Total Probability

The probability of B can be determined using
$\Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B)$, where
$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$ and
$\Pr(A' \cap B) = \Pr(A') \times \Pr(B|A')$

Therefore, the law of total probability is
$\Pr(B) = \Pr(A) \times \Pr(B|A) + \Pr(A') \times \Pr(B|A')$



Bayes' Theorem

Often we are asked to change use $\Pr(B|A)$ to determine $\Pr(A|B)$. This can be done by utilising the multiplication rule and the law of total probability. The best way to do this is using the double ended tree diagram to determine $\Pr(A \cap B)$ and $\Pr(B)$ NOT by memorising the formula.

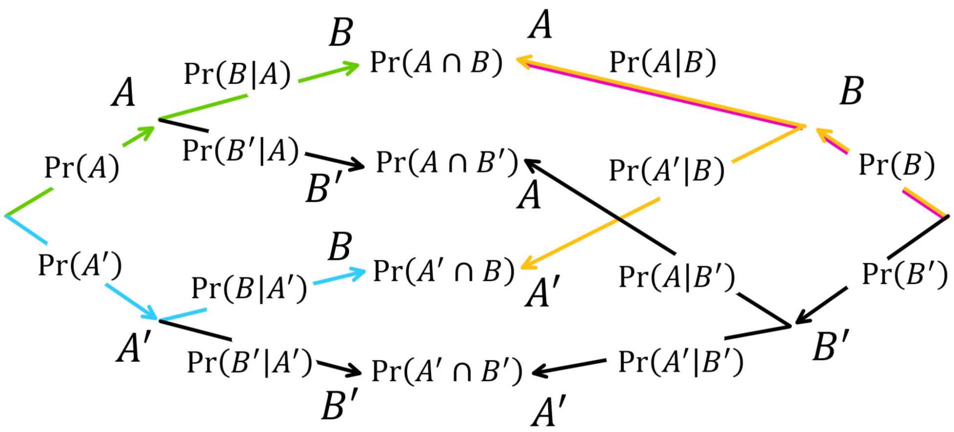
$$\Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$$

$$\Pr(A) \times \Pr(B|A) = \Pr(B) \times \Pr(A|B)$$

$$\Pr(A) \times \Pr(B|A) = (\Pr(A \cap B) + \Pr(A' \cap B)) \times \Pr(A|B)$$

$$\Pr(A) \times \Pr(B|A) = (\Pr(A) \times \Pr(B|A) + \Pr(A') \times \Pr(B|A')) \times \Pr(A|B)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A) \times \Pr(B|A)}{\Pr(B)} = \frac{\Pr(A) \times \Pr(B|A)}{(\Pr(A) \times \Pr(B|A) + \Pr(A') \times \Pr(B|A'))}$$



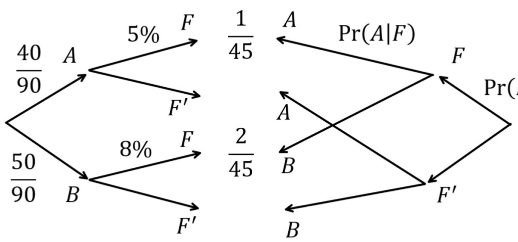
Example VCAA 2016 Exam 1 Question 7

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B.
5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty.
In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

The probability that the selected motor is faulty is
$\Pr(F) = \Pr(F \cap A) + \Pr(F \cap B)$

$$= \Pr(A) \times \Pr(F|A) + \Pr(B) \times \Pr(F|B)$$

$$= \frac{40}{90} \times \frac{5}{100} + \frac{50}{90} \times \frac{8}{100} = \frac{1}{45} + \frac{2}{45} = \frac{1}{15}$$



The selected motor is found to be faulty. The probability that it was assembled on Line A is

$$\Pr(A|F) = \frac{\Pr(F \cap A)}{\Pr(F)} = \frac{1/45}{1/15} = \frac{1}{3}$$