Statistics of Discrete Random Variables

Mean from a Frequency Table

The mean is the total score divided by the total frequency.

In a frequency table the total score can be determined by adding the products of the scores and corresponding frequencies.
The mean can then be found by dividing by the total frequency.

$$\mu = \frac{x_1 \times f_1 + x_2 \times f_2 + \cdots + x_n \times f_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum xf}{\sum f}$$

Example

x	0	1	2	3
f	17	28	45	10
xf	0	28	90	30

$$\mu = \frac{\sum xf}{\sum f} = \frac{148}{100} = 1.48$$

We could instead divide each of the frequencies by the total frequency first to get the relative frequencies, then multiply the relative frequencies by the corresponding scores and add them up to get the mean.

$$\mu = x_1 \times \frac{f_1}{\sum f} + x_2 \times \frac{f_2}{\sum f} + \cdots + x_n \times \frac{f_n}{\sum f}$$

$$= \sum x \times \frac{f}{\sum f} = \sum x(rf)$$

Example

x	0	1	2	3
f	17	28	45	10
rf	0.17	0.28	0.45	0.1
$x(rf)$	0	0.28	0.9	0.3

$$\sum f = 100, \qquad \mu = 0 + 0.28 + 0.9 + 0.3 = 1.48$$

Mean / Expected Value / Expectation of a Discrete Random Variable

The same works for the mean of a discrete random variable as probabilities are relative frequencies.

$$\mathrm{E}(X) = x_1 \times \Pr(X = x_1) + x_2 \times \Pr(X = x_2) + \cdots + x_n \times \Pr(X = x_n) = \sum x \Pr(X = x)$$

Expected Value of a Function of a Random Variable

$$\mathrm{E}[f(x)] = f(x_1)\ \Pr(X = x_1) + f(x_2)\ \Pr(X = x_2) + \cdots + f(x_n) \Pr(X = x_n) = \sum f(x) \Pr(X = x)$$

Example VCAA 2012 Exam 1 Question 4a

Let X be the random variable that represents the number of telephone calls that Daniel receives on any given day with probability distribution given by the table below.

x	0	1	2	3
$\Pr(X = x)$	0.2	0.2	0.5	0.1

The mean of X is $0 \times 0.2 + 1 \times 0.2 + 2 \times 0.5 + 3 \times 0.1 = 0 + 0.2 + 1 + 0.3 = 1.5$

Example VCAA 2006 Exam 2 Question 18

The discrete random variable X has the following probability distribution. If the mean of X is 0.3 then

x	-1	0	1
$\Pr(X = x)$	a	b	0.4

$\mu = -1 \times a + 0 \times b + 1 \times 0.4 = 0.3$
$\Rightarrow -a + 0.4 = 0.3$
$\Rightarrow a = 0.1$

$\sum \Pr(X = x) = a + b + 0.4 = 1$
$\Rightarrow 0.1 + b + 0.4 = 1$
$\Rightarrow b = 0.5$

Variance

The variance is the weighted average of the squared deviations from the mean.
That is, the sum of the products of the frequency and their corresponding squared differences between the mean and the scores, divided by the total frequency.

$$\sigma^2 = \frac{(x_1-\mu)^2\times f_1 + (x_2-\mu)^2\times f_2 + \cdots + (x_n-\mu)^2\times f_n}{f_1+f_2+\cdots+f_n} = \frac{\sum (x-\mu)^2\times f}{\sum f}$$

Example

x	0	1	2	3
f	17	28	45	10
$x-\mu$	-1.48	-0.48	0.52	1.52
$(x-\mu)^2$	2.1904	0.2304	0.2704	2.3104
$(x-\mu)^2 f$	0	6.4512	12.168	23.104

$\mu = 1.48$

$$\sigma^2 = \frac{\sum (x-\mu)^2 f}{\sum f} = \frac{41.7232}{100} \approx 0.417$$

We could instead divide each of the frequencies by the total frequency first to get the relative frequencies, then multiply the relative frequencies by the corresponding squared deviations and add them up to get the variance.

$$\sigma^2 = \frac{\sum (x-\mu)^2 f}{\sum f} = \sum (x-\mu)^2 \times \frac{f}{\sum f}$$

$$= \sum (x-\mu)^2 (rf)$$

Example

x	0	1	2	3
f	17	28	45	10
$x-\mu$	-1.48	-0.48	0.52	1.52
$(x-\mu)^2$	2.1904	0.2304	0.2704	2.3104
rf	0.17	0.28	0.45	0.1
$(x-\mu)^2 rf$	0	0.064512	0.12168	0.23104

$\mu = 1.48, \quad \sum f = 100$

$$\sigma^2 = 0 + 0.064512 + 0.12168 + 0.23104 \approx 0.417$$

Variance of a Discrete Random Variables

The same works for the variance of a discrete random variable as probabilities are relative frequencies. We can also simplify the formula to make it easier to calculate with.

$$\mathrm{var}(X) = \mathrm{E}[(X-\mu)^2] = \sum (x-\mu)^2\ \Pr(X=x) = \sum (x-\mu)^2 p(x) = \mathrm{E}\left(X^2 - 2\mu X + \mu^2\right)$$

$$= \mathrm{E}\left(X^2\right) - E(2\mu X) + E\left(\mu^2\right) = \mathrm{E}\left(X^2\right) - 2\mu E(X) + \mu^2 = \mathrm{E}\left(X^2\right) - 2\mu^2 + \mu^2$$

$$= \mathrm{E}\left(X^2\right) - \mu^2 = \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 = \sum x^2\, p(x) - \mu^2 = \sum x^2\, \Pr(X=x) - \mu^2$$

Example VCAA 2009 Exam 1 Question 7b

The random variable X has this probability distribution.
The means of X is

X	0	1	2	3	4
$\Pr(X=x)$	0.1	0.2	0.4	0.2	0.1

$$\mathrm{E}(X) = 0\times 0.1 + 1\times 0.2 + 2\times 0.4 + 3\times 0.2 + 4\times 0.1 = 0 + 0.2 + 0.8 + 0.6 + 0.4 = 2$$

The variance of X is

$$\mathrm{Var}(X) = 0^2\times 0.1 + 1^2\times 0.2 + 2^2\times 0.4 + 3^2\times 0.2 + 4^2\times 0.1 - 2^2$$
$$= 0 + 0.2 + 1.6 + 1.8 + 1.6 - 4 = 1.2$$

Standard Deviation

The square root of the variance.

$$\mathrm{sd}(X) = \sigma = \sqrt{\mathrm{var}(X)}$$