Proof of Linear Combinations of Discrete Random Variables

Expected Value / Mean

Expected Value of Linear Transformation of a Random Variable

$$E(aX+b)=\sum(ax+b)\,p(x)$$

$$=\sum\big(ax\,p(x)+b\,p(x)\big)$$

The sum of two things is the same as the sum of each part added together. $(a+b)=a+b$

$$=\sum ax\,p(x)+\sum b\,p(x)$$

The product of elements that are not being summed act as a scalar and can be factored out.
$ax_1+ax_2=a(x_1+x_2)$

$$=a\sum x\,p(x)+b\sum p(x)$$

The sum of all probabilities in a distribution is 1.

$$=aE(X)+b$$

Expected Value of Sum of Random Variables

$$E(X+Y)=\sum_x\sum_y(x+y)\,p(x,y)\,,\qquad p(x,y)=\Pr(X=x\cap Y=y)$$

Addition is commutative so it can be done in any order. $a+b=b+a$

$$=\sum_x\sum_y x\,p(x,y)+\sum_y\sum_x y\,p(x,y)$$

$$=\sum_x x\sum_y p(x,y)+\sum_y y\sum_x p(x,y)$$

$$=\sum_x x\sum_y p(x)p(y|x)+\sum_y y\sum_x p(y)p(x|y)$$

$$=\sum_x x\,p(x)\sum_y p(y|x)+\sum_y y\,p(y)\sum_x p(x|y)$$

The sum of all probabilities in a distribution is 1.

$$=\sum_x x\,p(x)+\sum_y y\,p(y)$$

$$=E(X)+E(Y)$$

Variance

Variance of Linear Transformation of a Random Variable

$$\mathrm{var}(aX+b) = E(aX+b)^2 - [E(aX+b)]^2$$

$$= E\left(a^2X^2 + 2abX + b^2\right) - [aE(X)+b]^2$$

$$= E\left(a^2X^2\right) + E(2abX) + E\left(b^2\right) - \left(a^2[E(X)]^2 + 2abE(X) + b^2\right)$$

$$= a^2E\left(X^2\right) + 2abE(X) + b^2 - a^2[E(X)]^2 - 2abE(X) - b^2$$

$$= a^2E\left(X^2\right) - a^2[E(X)]^2$$

$$= a^2\left(E\left(X^2\right) - [E(X)]^2\right)$$

$$= a^2\mathrm{var}(X)$$

Variance of Sum of Independent Random Variables

$$\mathrm{var}(X+Y) = \sum_x \sum_y (x+y)^2\, p(x,y) - \big(E(X+Y)\big)^2$$

For independent events $p(x,y) = p(x)p(y)$

$$= \sum_x \sum_y \left(x^2 + 2xy + y^2\right) p(x)\, p(y) - \left([E(X)]^2 + 2E(X)E(Y) + [E(Y)]^2\right)$$

$$= \sum_x \sum_y x^2\, p(x)p(y) + \sum_x \sum_y 2xy\, p(x)p(y) + \sum_y \sum_x y^2\, p(x)p(y)$$
$$-[E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2$$

$$= \sum_x x^2\, p(x) - [E(X)]^2 + 2\sum_x x\, p(x) \sum_y y\, p(y) - 2E(X)E(Y) + \sum_y y^2\, p(y) - [E(Y)]^2$$

$$= E\left(X^2\right) - [E(X)]^2 + 2E(X)E(Y) - 2E(X)E(Y) + E\left(Y^2\right) - [E(Y)]^2$$

$$= E\left(X^2\right) - [E(X)]^2 + E\left(Y^2\right) - [E(Y)]^2$$

$$= \mathrm{var}(X) + \mathrm{var}(Y)$$