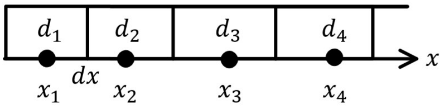
Continuous Random Variables

Continuous Random Variable

A continuous random variable can assume any continuous value in a given domain.

Probability Density Function (pdf)

For a continuous random variable, we define a function (the probability density function or pdf) that describes that density of the probability (probability/x) at particular positions rather than the mass of the probability at those positions.



Density is the rate of change of the mass to its length (area or volume), that is kg/m (kg/A or kg/V). To obtain the total mass we add up the products of the density and the length (area or volume), that is kg/m × m.

This is also true of the probability density function to get the sum of the probability mass. Since the pdf is a continuous function, instead of adding a discrete number of products of density and length, we find the limiting sum of the product of the density and a small change in x. That is, the integral of the pdf function with respect to x.

$$\text{Total Mass (Probability)} = \lim_{dx\to 0}\sum_{x=a}^{b} d(x)\mathrm{d}x = \int_a^b d(x)\mathrm{d}x$$

Since the pdf is not a probability, the function can be greater than 1, but it still must be non-negative and have an integral over all real values equal to 1.

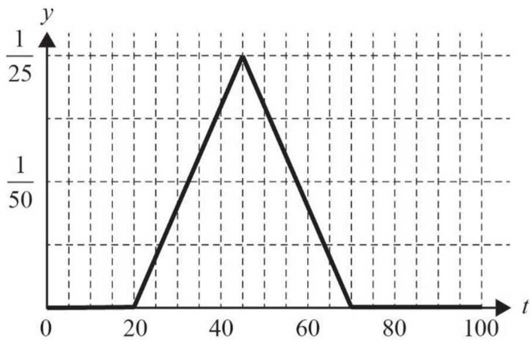
$$d(x) \geq 0 \qquad \int_{-\infty}^{\infty} d(x)\mathrm{d}x = 1$$

Probability density functions are often described using hybrid functions (where the pdf is equal to 0 for part of the domain) to ensure the integral is 0 over all the real values.

Example VCAA 2017 Exam 2 Question 3

The continuous random variable T, has a probability density function f.

$$f(t) = \begin{cases} \dfrac{1}{625}(t-20) & 20 \leq t < 45 \\ \dfrac{1}{625}(70-t) & 45 \leq t \leq 70 \\ 0, & \text{elsewhere} \end{cases}$$



Example VCAA 2010 Exam 1 Question 7a

The continuous random variable X has a distribution with probability density function given by

$$f(x) = \begin{cases} ax(5-x) & \text{if } 0 \leq x \leq 5 \\ 0 & \text{if } x < 0 \text{ or if } x > 5 \end{cases}$$, where a is a positive constant.

The value of a will need to make the area under f between $0 \leq x \leq 5$ equal 1.

$$\int_0^5 ax(5-x)\mathrm{d}x = a\int_0^5 (5x - x^2)\mathrm{d}x = a\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = a\left(\frac{5(5)^2}{2} - \frac{(5)^3}{3}\right) = a\left(\frac{125}{2} - \frac{125}{3}\right) = 1$$

$$\Rightarrow \frac{375a - 250a}{6} = 1 \;\Rightarrow\; 125a = 6 \;\Rightarrow\; a = \frac{6}{125}$$

Calculating Probabilities using Continuous Random Variables

Probabilities are determined by the area under the curve of the probability density function, therefore, the probability of getting **exactly** a value (e.g. 5 not 5.0001 or 4.9999) is practically impossible. Therefore, we say it has a probability of 0.

$$\Pr(X = a) = \int_a^a f(x)\,\mathrm{d}x$$
$$= F(a) - F(a) = 0$$

Since the area under a point is 0, it doesn't matter if the probability is between and including or not including the value. That is, if the probability uses $<$ or $\leq$ they will have the same probability.

$$\Pr(a < X < b) = \Pr(a \leq X < b) = \Pr(a < X \leq b) = \Pr(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x$$

Be Careful

Always check the domain to see if working out the probability is even possible.
If your answer is > 1 or < 0, then some values included are outside of the domain.

Probabilities For Hybrid Functions

To determine the probabilities across the domains that include different parts of the hybrid, the integral must be broken up into separate parts.

$$\Pr(x_1 < x < x_2) = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x$$
$$= \int_{x_1}^{b} f(x)\,\mathrm{d}x + \int_{b}^{x_2} f(x)\,\mathrm{d}x$$

Using a CAS calculator, you can enter a hybrid function and integrate over the whole domain.

Example VCAA 2011 Exam 2 Question 2aii

In a chocolate factory the material for making each chocolate is sent to one of two machines, machine A or machine B. The time, Y seconds, taken to produce a chocolate by machine B, has the following probability density function.

$$f(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y}{16} & 0 \leq y \leq 4 \\ 0.25e^{-0.5(y-4)} & y > 4 \end{cases}$$

$$\Pr(3 \leq Y \leq 5) = \int_3^5 f(y)\,\mathrm{d}y = \int_3^4 \frac{y}{16}\,\mathrm{d}y + \int_4^5 0.25e^{-0.5(y-4)}\,\mathrm{d}y = \frac{23}{32} - \frac{1}{2}e^{-0.5} \approx 0.4155$$

Calculating Values that Give a Probability

To find a particular value that gives a particular probability, we can solve the equation of the definite integral and the given probability.

Example VCAA 2006 Exam 1 Question 6b

The probability density function of a continuous random variable X is given by f. If $\Pr(X \geq a) = \frac{5}{8}$, the value of a is

$$f(x) = \begin{cases} \dfrac{x}{12} & 1 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^5 \frac{x}{12}\,\mathrm{d}x = \frac{5}{8} \;\Rightarrow\; \frac{1}{12}\left[\frac{x^2}{2}\right]_a^5 = \frac{5}{8} \;\Rightarrow\; \frac{1}{2}(25 - a^2) = \frac{15}{2} \;\Rightarrow\; 25 - a^2 = 15 \;\Rightarrow\; a^2 = 10$$

$$\Rightarrow a = \pm\sqrt{10}$$

Since $1 \leq x \leq 5$, then $a \neq -\sqrt{10}$. Therefore, $a = \sqrt{10}$.