Proof of Linear Combinations of Continuous Random Variables

Expected Value / Mean

Expected Value of Linear Transformation of a Random Variable

$$E(aX+b)=\int_{-\infty}^{\infty}(ax+b)f(x)\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}\big(axf(x)+bf(x)\big)\mathrm{d}x$$

$$=\int_{-\infty}^{\infty}axf(x)\mathrm{d}x+\int_{-\infty}^{\infty}bf(x)\mathrm{d}x$$

$$=a\int_{-\infty}^{\infty}xf(x)\mathrm{d}x+b\int_{-\infty}^{\infty}f(x)\mathrm{d}x$$

The area under the probability distribution function over the whole domain is 1.

$$=aE(X)+b$$

Expected Value of Sum of Random Variables

$$E(X+Y)=\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}(x+y)f(x,y)\mathrm{d}y\,\mathrm{d}x$$

Integrations of different variables can be done in any order

$$=\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}xf(x,y)\mathrm{d}y\,\mathrm{d}x+\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}yf(x,y)\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}xf(x)f(x,y)\mathrm{d}y\,\mathrm{d}x+\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}yf(y)f(x,y)\mathrm{d}x\,\mathrm{d}y$$

Integrating with respect to one variable means that other variables are treated as constants.

$$=\int_{x=-\infty}^{x=\infty}xf(x)\int_{y=-\infty}^{y=\infty}f(x,y)\mathrm{d}y\,\mathrm{d}x+\int_{y=-\infty}^{y=\infty}yf(y)\int_{x=-\infty}^{x=\infty}f(x,y)\mathrm{d}x\,\mathrm{d}y$$

The area under the probability distribution function over the whole domain is 1.

$$=\int_{x=-\infty}^{x=\infty}xf(x)\mathrm{d}x+\int_{y=-\infty}^{y=\infty}yf(y)\mathrm{d}y$$

$$=E(X)+E(Y)$$

Variance

Variance of Linear Transformation of a Random Variable

$$\mathrm{var}(aX+b) = E(aX+b)^2 - [E(aX+b)]^2$$

$$= E(a^2X^2 + 2abX + b^2) - [aE(X)+b]^2$$

$$= E(a^2X^2) + E(2abX) + E(b^2) - (a^2[E(X)]^2 + 2abE(X) + b^2)$$

$$= a^2E(X^2) + 2abE(X) + b^2 - a^2[E(X)]^2 - 2abE(X) - b^2$$

$$= a^2E(X^2) - a^2[E(X)]^2$$

$$= a^2(E(X^2) - [E(X)]^2)$$

$$= a^2\mathrm{var}(X)$$

Variance of Sum of Independent Random Variables

$$\mathrm{var}(X+Y) = \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}(x+y)^2 f(x,y)\mathrm{d}y\,\mathrm{d}x - \big(E(X+Y)\big)^2$$

For independent events $\int\int f(x,y)\mathrm{d}y\,\mathrm{d}x = \int f(x)\mathrm{d}x \int f(y)\mathrm{d}y$

$$= \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}(x^2+2xy+y^2)f(x)f(y)\mathrm{d}y\,\mathrm{d}x - ([E(X)]^2 + 2E(X)E(Y) + [E(Y)]^2)$$

$$= \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}x^2 f(x)f(y)\mathrm{d}y\,\mathrm{d}x + \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}2xyf(x)f(y)\mathrm{d}y\,\mathrm{d}x$$
$$+ \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}y^2 f(x)f(y)\mathrm{d}x\,\mathrm{d}y - [E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2$$

$$= \int_{x=-\infty}^{x=\infty}x^2 f(x)\int_{y=-\infty}^{y=\infty}f(y)\mathrm{d}y\,\mathrm{d}x - [E(X)]^2$$
$$+2\int_{x=-\infty}^{x=\infty}xf(x)\mathrm{d}x\int_{y=-\infty}^{y=\infty}yf(y)\mathrm{d}y - 2E(X)E(Y)$$
$$+\int_{y=-\infty}^{y=\infty}y^2 f(y)\int_{x=-\infty}^{x=\infty}f(x)\mathrm{d}x\,\mathrm{d}y - [E(Y)]^2$$

$$= \int_{=-\infty}^{\infty}x^2 f(x)\mathrm{d}x - [E(X)]^2 + 2\int_{x=-\infty}^{x=\infty}xf(x)\mathrm{d}x\int_{y=-\infty}^{y=\infty}yf(y)\mathrm{d}y - 2E(X)E(Y)$$
$$+\int_{=-\infty}^{\infty}y^2 f(y)\mathrm{d}y - [E(Y)]^2$$

$$= E(X^2) - [E(X)]^2 + 2E(X)E(Y) - 2E(X)E(Y) + E(Y^2) - [E(Y)]^2$$

$$= E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2$$

$$= \mathrm{var}(X) + \mathrm{var}(Y)$$