Cumulative Density Functions

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Since the area under a curve is the probability less than x of a probability density function, the anti-derivative of the probability density function must describe the cumulative value of the probability to x.

For $f(x)$ the continuous random variable for $a < x < b$, the cumulative density function can be found using one of the following:

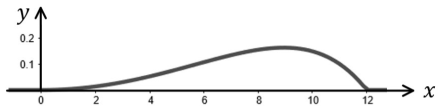
$\Pr(X \leq x) = \int f(x)\mathrm{d}x + c$, where $F(a) = 0$ and $F(b) = 1$ $\quad \Pr(X \leq x) = \int_a^x f(x)\mathrm{d}x = \int_b^x f(x)\mathrm{d}x + 1$

The probability between any two values is then:
$\Pr(a < X < b) = \Pr(X < b) - \Pr(X < a) = F(b) - F(a)$

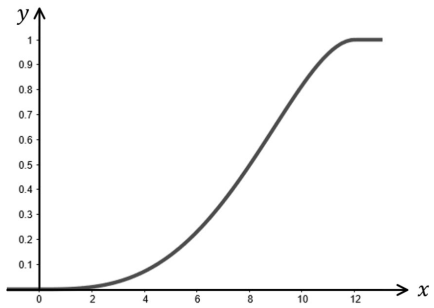
Example Modified VCAA 2017 NHT Exam 2 Question 3

The use-time of Grade B whiteboard pens is described by the probability density function f where x is the use-time in hours. The cumulative density function is given by $F(x)$.

$$f(x) = \begin{cases} \frac{x}{576}(12 - x)\left(e^{\frac{x}{6}} - 1\right) & 0 \leq x \leq 12 \\ 0 & \text{otherwise} \end{cases}$$



$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{576}\left[\frac{x^3}{3} - 6x^2 - 6(12 - x)^2 e^{\frac{x}{6}} + 864\right] & 0 < x < 12 \\ 1 & x \geq 12 \end{cases}$$

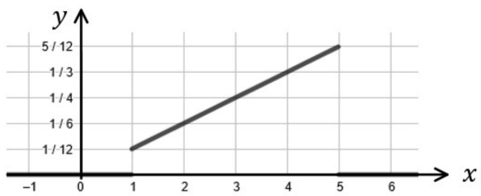


Example Modified VCAA 2006 Exam 1 Question 6b

The probability density function of a continuous random variable X is given by $f(x) = \begin{cases} \frac{x}{12} & 1 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$

The cumulative density function of this continuous random variable X is

$$F(x) = \int_1^x \frac{t}{12}\mathrm{d}t = \frac{1}{12}\left[\frac{t^2}{2}\right]_1^x = \frac{1}{24}(x^2 - 1) = \frac{x^2}{24} - \frac{1}{24}$$

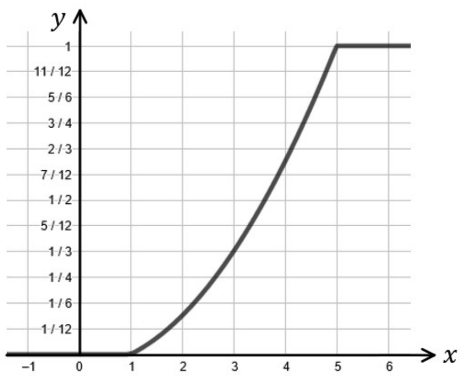


Alternatively,

$$F(x) = \int_5^x \frac{t}{12}\mathrm{d}t + 1 = \frac{1}{12}\left[\frac{t^2}{2}\right]_5^x + 1 = \frac{1}{24}(x^2 - 25) + 1$$

$$= \frac{x^2}{24} - \frac{25}{24} + \frac{24}{24} = \frac{x^2}{24} - \frac{1}{24}$$

$$\Pr(X \leq x) = \begin{cases} 0 & x \leq 1 \\ \frac{x^2}{24} - \frac{1}{24} & 1 < x < 5 \\ 1 & x \geq 5 \end{cases}$$



$$\Pr(2 < x \leq 3) = \Pr(x \leq 3) - \Pr(x < 2) = F(3) - F(2) = \left(\frac{3^2}{24} - \frac{1}{24}\right) - \left(\frac{2^2}{24} - \frac{1}{24}\right) = \frac{8}{24} - \frac{3}{24} = \frac{5}{24}$$