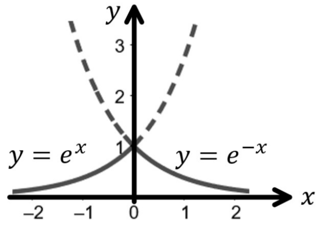
Density Functions for the Normal Distribution

Probability Density Function for the Standard Normal Distribution

The shape of the normal distribution is not an elementary function, it is a composite function.

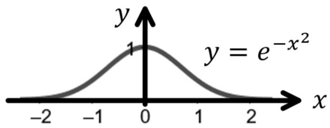
The shape to the left and right of the mean is roughly the decreasing section of an exponential curve.



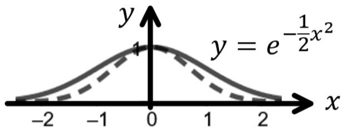
So, we could compose the exponential function e^x with a function that is negative for all values except at the mean where it is 0 and has a local maximum. A function that meets these requirements is $-x^2$.

The curve of the composite function $y = e^{-x^2}$ is a bell shaped curve called the Gaussian function, but we want

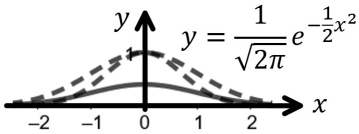
- the variance to be 1, and
- the area bound by the curve and the x-axis to be 1.



By dilating by a scale factor of $\sqrt{2}$ from the y-axis, to get the equation $y = e^{-\left(\frac{x}{\sqrt{2}}\right)^2} = e^{-\frac{1}{2}x^2}$ both the area bound by the curve and the x-axis and the variance are $\sqrt{2\pi}$.



Therefore, if we dilate $y = e^{-\frac{1}{2}x^2}$ by a scale factor of $\frac{1}{\sqrt{2\pi}}$ from the x-axis, the area bound by the curve and the x-axis and the variance will be 1.



Therefore, the equation for the standard normal distribution $Z\sim\mathrm{N}(0,1)$ is $\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$.

Integral for the Area Bound by the Curve and the x-axis

$$\int_{-\infty}^{\infty}\left(e^{-x^2}\right)\mathrm{d}x = \sqrt{\pi} \;\Rightarrow\; \int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}x^2}\right)\mathrm{d}x = \sqrt{2\pi} \;\Rightarrow\; \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right)\mathrm{d}x = 1$$

Integral for the Variance where $\mu = 0$

$$\int_{-\infty}^{\infty}\left(x^2e^{-x^2}\right)\mathrm{d}x = \frac{\sqrt{\pi}}{2} \;\Rightarrow\; \int_{-\infty}^{\infty}\left(x^2e^{-\frac{1}{2}x^2}\right)\mathrm{d}x = \sqrt{2\pi} \;\Rightarrow\; \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}x^2e^{-\frac{1}{2}x^2}\right)\mathrm{d}x = 1$$

Transforming the Probability Density Function for the Normal Distribution

Since z, is the standardised value for value x, where the mean is μ, and standard deviation is σ, we can substitute $z = \frac{x-\mu}{\sigma}$, to consider the normal distribution for any μ and σ.

This substitution translates the graph μ units right and dilates by a scale factor of σ^2 from the y-axis. To ensure the area bound by the curve and the x-axis remains as 1 and the variance as σ^2, we need to dilate by a scale factor of $\frac{1}{\sigma}$ from the x-axis.

Therefore, the equation for the general normal distribution $X\sim\mathrm{N}(\mu,\sigma^2)$ is $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

Integral for the Area Bound by the Curve and the x-axis

$$\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right)\mathrm{d}x = \sigma \;\Rightarrow\; \int_{-\infty}^{\infty}\left(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right)\mathrm{d}x = 1$$

Integral for the Variance

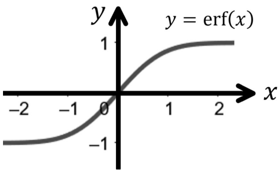
$$\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}x^2e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right)\mathrm{d}x = \sigma^3 \;\Rightarrow\; \int_{-\infty}^{\infty}\left(\frac{1}{\sigma\sqrt{2\pi}}x^2e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right)\mathrm{d}x = \sigma^2$$

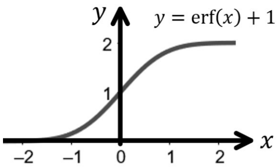
Cumulative Density Function for the Normal Distribution and The Error Function

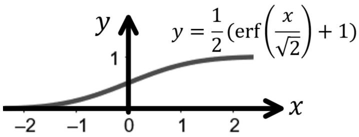
The antiderivative of the Gaussian function and normal distribution is related to the error function.

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \left(e^{-t^2}\right) \mathrm{d}t \qquad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \left(e^{-\frac{1}{2}t^2}\right) \mathrm{d}t \qquad F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \left(e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) \mathrm{d}t$$

$$\Phi(z) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + 1\right) \qquad F(x) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right) + 1\right)$$







The integrals cannot be expressed in terms of elementary functions, only approximated or written in terms of an infinitely long polynomial.

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\left(z - \frac{1}{3}z^3 + \frac{1}{10}z^5 - \frac{1}{42}z^7 + \frac{1}{216}z^9 + \cdots\right) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n z^{2n+1}}{n!\,(2n+1)}\right)$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\left(\frac{z}{2} - \frac{1}{12}z^3 + \frac{1}{80}z^5 - \frac{1}{672}z^7 + \frac{1}{6912}z^9 + \cdots\right) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n z^{2n+1}}{2^n n!\,(2n+1)}\right)$$