Polynomial Operations

Substitution

Replace the variable, x, in the brackets of $P(x)$ with the number or expression you are substituting. Then replace each occurrence of x in the polynomial with the number or expression, in brackets. Writing the substituted number in brackets helps to avoid errors with negative number substitutions.

Example

$P(x) = x^3 + 3x^2 - 2x + 1$ when $x = -1, x = \frac{1}{2}, x = \sqrt{3}, x = y^2, x = t + 1$

$$P(-1) = (-1)^3 + 3(-1)^2 - 2(-1) + 1 = -1 + 3 + 2 + 1 = 5$$

$$P\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1 = \frac{1^3}{2^3} + 3 \times \frac{1^2}{2^2} - 1 + 1 = \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$$

$$P(\sqrt{3}) = (\sqrt{3})^3 + 3(\sqrt{3})^3 - 2\sqrt{3} + 1 = \sqrt{3}^2\sqrt{3} + 3(3) - 2\sqrt{3} + 1 = 9 + 1 + 3\sqrt{3} - 2\sqrt{3} = 10 + \sqrt{3}$$

$$P(y^2) = (y^2)^3 + 3(y^2)^2 - 2(y^2) + 1 = y^6 + 3y^4 - 2y^3 + 1$$

$$P(t + 1) = (t + 1)^3 + 3(t + 1)^2 - 2(t + 1) + 1$$

Equivalent Polynomials and Equating Coefficients

Two polynomials are equivalent if every term of both polynomials share the same coefficients.

This is not equation true for only some values of x, but all values of x. So we use the identically equal to symbol ($\equiv$) instead of an equals sign. This also indicates that we are not solving the equation for x.

If coefficients of the polynomials are unknown, we can equate them to determine their value.
This can be a useful way to write a polynomial in an equivalent form.

Example

$4x^3 - bx^2 - 12 \equiv ax^3 + 8x + cx - d$

$a = 4, \qquad -b = 8 \Rightarrow b = -8, \qquad c = 0, \qquad -d = -12 \Rightarrow d = 12$

Example

$(2a + 1)x^2 - (5 - b)x + 12 - c \equiv 1 + 2x + 3x^2$

$(2a + 1)x^2 - (5 - b)x + 12 - c \equiv 3x^2 + 2x + 1$

$2a + 1 = 3 \Rightarrow a = 1, \qquad -(5 - b) = 2 \Rightarrow b = 7, \qquad 12 - c = 1 \Rightarrow c = 11$

Arithmetic of Polynomials

Since polynomials work like numerals, base x, arithmetic with polynomials works the same as base 10. Therefore, the algorithms and notations used for numerical arithmetic can be used for polynomial arithmetic. Remember, the powers of x act like the place value in base x.

Addition (Sum $P(x) + Q(x)$) and Subtraction (Difference $P(x) - Q(x)$)

Add (or subtract) like terms of the two polynomials together to determine the sum (or difference).
Alternatively, set up the column algorithm for addition (or subtraction) and work column by column.
Ensure that like terms are aligned vertically in the same column. Placeholder zeroes can be used.
The degree of the sum or difference polynomial can be no more than the degree of either polynomial.

Example

$3593 + 4865$
$= (3000 + 4000) + (500 + 800) + (90 + 60) + (3 + 5)$
$= 7000 + 1300 + 150 + 8$
$= 8458$

	3000	500	90	3
$+$	4000	800	60	5
	7000 +	1300 +	150 +	8
				8458

Example

$(3x^3 + 5x^2 + 9x + 3) + (4x^3 + 8x^2 + 6x + 5)$
$\equiv (3x^3 + 4x^3) + (5x^2 + 8x^2) + (9x + 6x) + (3 + 5)$
$\equiv 7x^3 + 13x^2 + 15x + 8$

	$+3x^3$	$+5x^2$	$+9x$	$+3$
$+$	$+4x^3$	$+8x^2$	$+6x$	$+5$
	$+7x^3$	$+13x^2$	$+15x$	$+8$

Example

$(3x^3 - 5x^2 + 9x - 3) + (4x^3 + 8x^2 - 6x - 5)$
$\equiv (3x^3 + 4x^3) + (-5x^2 + 8x^2) + \big(9x + (-6x)\big) + \big(-3 + (-5)\big)$
$\equiv 7x^3 + 3x^2 + 3x - 8$

	$+3x^3$	$-5x^2$	$+9x$	-3
$+$	$+4x^3$	$+8x^2$	$-6x$	-5
	$+7x^3$	$+3x^2$	$+3x$	-8

Example

$(3x^3 + 9x - 3) + (4x^3 + 8x^2)$
$\equiv (3x^3 + 4x^3) + 8x^2 + 9x - 3$
$\equiv 7x^3 + 8x^2 + 9x - 3$

	$+3x^3$		$+9x$	-3
$+$	$+4x^3$	$+8x^2$		
	$+7x^3$	$+8x^2$	$+9x$	-3

Example

$3593 - 4865$
$= (3000 - 4000) + (500 - 800) + (90 - 60) + (3 - 5)$
$= -1000 - 300 + 30 - 2$
$= -1272$

	3000	500	90	3
$-$	4000	800	60	5
	− 1000 −	300 +	30 −	2
				−1272

Example

$(3x^3 + 5x^2 + 9x + 3) - (4x^3 + 8x^2 + 6x + 5)$
$\equiv (3x^3 - 4x^3) + (5x^2 - 8x^2) + (9x - 6x) + (3 - 5)$
$\equiv -x^3 - 3x^2 + 3x - 2$

	$+3x^3$	$+5x^2$	$+9x$	$+3$
$-$	$+4x^3$	$+8x^2$	$+6x$	$+5$
	$-x^3$	$-3x^2$	$+3x$	-2

Example

$(3x^3 - 5x^2 + 9x - 3) - (4x^3 + 8x^2 - 6x - 5)$
$\equiv (3x^3 - 4x^3) + (-5x^2 - 8x^2) + \big(9x - (-6x)\big) + \big(-3 - (-5)\big)$
$\equiv -x^3 - 13x^2 + 15x + 2$

	$+3x^3$	$-5x^2$	$+9x$	-3
$-$	$+4x^3$	$+8x^2$	$-6x$	-5
	$-x^3$	$-13x^2$	$+15x$	$+2$

Example

$(3x^3 + 9x - 3) - (4x^3 + 8x^2)$
$\equiv (3x^3 - 4x^3) + (0x^2 - 8x^2) + 9x - 3$
$\equiv -x^3 + 8x^2 + 9x - 3$

	$+3x^3$	$+0x^2$	$+9x$	-3
$-$	$+4x^3$	$+8x^2$		
	$-x^3$	$-8x^2$	$+9x$	-3

Example

$Q(x) + (x^3 - 3x^2 + 4x + 8) = 5x^3 + 8x^2 - 5$. Find $Q(x)$.
$Q(x) = (5x^3 + 8x^2 + 4x - 5) - (x^3 - 3x^2 + 4x + 8)$
$Q(x) = (5x^3 - x^3) + \big(8x^2 - (-3x^2)\big) + (4x - 4x) + (-5 - 8)$
$Q(x) = 4x^3 + 11x^2 - 13$

	$+5x^3$	$+8x^2$	$+4x$	-5
$-$	$+\ x^3$	$-3x^2$	$+4x$	$+8$
	$+4x^3$	$+11x^2$	$+0x$	-13

Multiplication (Product $P(x) \times Q(x)$) and Distribution ($P(x)Q(x)$)

Multiplication of polynomials is the same as the distributive law and expanding brackets.
Multiply each term of the multiplicand, $P(x)$, by each term of the multiplier, $Q(x)$. Add the products.
The degree of the product polynomial is the sum of the degrees of the polynomials P and Q.
Keep in mind: $x^m \times x^n = x^{m+n}$

Standard Algorithm

Multiply the every term in the second row, by every term in the first row. Put the products for each term in the second row in a new row. Align each product in the column with the same power of x.
Use column addition to add the products.

Example

593×65

		500	90	3
$\times$			60	5
		2500	450	15
$+$	30 000	5400	180	
	30 000 +	7900 +	630 +	15
				38 545

Example

$(5x^2 + 9x + 3) \times (6x + 5)$

		$+5x^2$	$+9x$	$+3$
$\times$			$+6x$	$+5$
		$+25x^2$	$+45x$	$+15$
$+$	$+30x^3$	$+54x^2$	$+18x$	
	$+30x^3$	$+79x^2$	$+63x$	$+15$

Grid Method / Area Model

Write the multiplicand and multiplier polynomials on the sides of a rectangle.
Multiply each term in the "length" and "width" of the rectangle to fill the grid, then add the like terms.
Like terms will be on diagonals if the polynomials were written out in descending exponent order.

Example

3593×865

$= 2\,400\,000$	$= 2\,400\,000$
$+(400\,000 + 180\,000)$	$+ \quad 580\,000$
$+(72\,000 + 30\,000 + 15\,000)$	$+ \quad 117\,000$
$+(2400 + 5400 + 2500)$	$+ \quad 10\,300$
$+(180 + 450)$	$+ \quad 630$
$+15$	$+ \quad 15$

$= 3\,107\,945$

$\times$	**3000**	**+500**	**+90**	**+3**
800	2 400 000	400 000	72 000	2400
+60	180 000	30 000	5400	180
+5	15 000	2500	450	15

Example

$(3x^3 + 5x^2 + 9x + 3) \times (8x^2 + 6x + 5)$

$\equiv 24x^5$	$\equiv \quad 24x^5$
$+(40x^4 + 18x^4)$	$+ \quad 58x^4$
$+(72x^3 + 30x^3 + 15x^3)$	$+117x^3$
$+(24x^2 + 54x^2 + 25x^2)$	$+103x^2$
$+(18x + 45x)$	$+ \quad 63x$
$+15$	$+ \quad 15$

$\equiv 24x^5 + 58x^4 + 117x^3 + 103x^2 + 63x + 15$

$\times$	$3x^3$	$+5x^2$	$+9x$	$+3$
$8x^2$	$+24x^5$	$+40x^4$	$+72x^3$	$+24x^2$
$+6x$	$+18x^4$	$+30x^3$	$+54x^2$	$+18x$
$+5$	$+15x^3$	$+25x^3$	$+45x$	$+15$

Example

$(3x^3 - 5x^2 + 9x - 3) \times (8x^2 - 6x - 5)$

$\equiv 24x^5 - 58x^4 + 87x^3 - 53x^2 - 27x + 15$

$\times$	$3x^3$	$-5x^2$	$+9x$	-3
$8x^2$	$+24x^5$	$-40x^4$	$+72x^3$	$-24x^2$
$-6x$	$-18x^4$	$+30x^3$	$-54x^2$	$+18x$
-5	$-15x^3$	$+25x^3$	$-45x$	$+15$

Division (Quotient and Remainder $P(x) \div D(x)$)

Subtract multiples of the divisor, $D(x)$, from the dividend, $P(x)$, without using negative powers of x.
The degree of the quotient polynomial is the difference of the degrees of the polynomials P and D.
Keep in mind: $x^m \div x^n = x^{m-n}$ and $x^m \div x^m = x^0 = 1$

Numerical division can be written as a mixed fraction. Polynomial division can also be written this way.
That is, dividend $\div$ divisor $= \dfrac{\text{dividend}}{\text{divisor}} = \text{quotient} + \dfrac{\text{remainder}}{\text{divisor}}$

Standard Algorithm

Divide the leading term of the dividend by the leading term of the divisor and write the quotient. Subtract the product of the quotient and the divisor from the dividend to get the remainder.

$$\begin{array}{r|l} & \text{quotient} \\ \hline \text{divisor} & \text{dividend} \\ & -\text{quotient} \times \text{divisor} \\ \hline & \text{remainder} \end{array}$$

Example
$49 \div 12$

$$\begin{array}{r|r} & 4 \\ \hline 12 & 49 \\ & -48 \\ \hline & 1 \end{array}$$

$\dfrac{49}{12} = 4\dfrac{1}{12}$

Example
$(4x + 9) \div (x + 2)$

$$\begin{array}{r|r} & 4 \\ \hline x+2 & 4x+9 \\ & -(4x+8) \\ \hline & +1 \end{array}$$

$\dfrac{4x+9}{x+2} \equiv 4 + \dfrac{1}{x+2}$

Example
$(8x - 9) \div (2x + 3)$

$$\begin{array}{r|r} & 4 \\ \hline 2x+3 & 8x-9 \\ & -(8x+12) \\ \hline & -21 \end{array}$$

$\dfrac{8x-9}{2x+3} \equiv 4 - \dfrac{21}{2x+3}$

Example
$(4x^2 + 9x + 3) \div (x^2 + 2x + 1)$

$$\begin{array}{r|r} & 4 \\ \hline x^2+2x+1 & 4x^2+9x+3 \\ & -(4x^2+8x+4) \\ \hline & +x-1 \end{array}$$

$\dfrac{4x^2+9x+3}{x^2+2x+1} \equiv 4 + \dfrac{x-1}{x^2+2x+1}$

Step by Step Example

$(4x + 9) \div (x + 2)$

$$\begin{array}{r|r} \hline x+2 & 4x+9 \end{array}$$

Set up the divisor on the left and the dividend on the right.

$$\begin{array}{r|r} & 4 \\ \hline x+2 & 4x+9 \end{array}$$

$4x \div x = 4$

Divide the leading term of the dividend by the leading term of the divisor and write the quotient.

$$\begin{array}{r|r} & 4 \\ \hline x+2 & 4x+9 \\ & (4x+8) \\ \hline \end{array}$$

$4 \times (x+2) = 4x + 8$

Multiply the quotient by the divisor. Align it under the dividend.

$$\begin{array}{r|r} & 4 \\ \hline x+2 & 4x+9 \\ & -(4x+8) \\ \hline & +1 \end{array}$$

$(4x + 9) - (4x + 8) = +1$

Subtract the product of the quotient and the divisor from the dividend to get the remainder.

If the remainder divided by the leading term of the divisor is not a negative power of x, (e.g. x^{-1}) repeat using the remainder as the dividend. Subtract from the entire dividend.
If the dividend is missing a term, put add it in with a coefficient of 0 as a placeholder.

Example
$\dfrac{1249}{12} = 104\dfrac{1}{12}$

$$\begin{array}{r|r} & 104 \\ \hline 12 & 1249 \\ & -1200 \\ \hline & 49 \\ & -48 \\ \hline & 1 \end{array}$$

Example
$\dfrac{x^3 + 2x^2 + 4x + 9}{x + 2} \equiv x^2 + 2 + \dfrac{1}{x+2}$

$$\begin{array}{r|r} & x^2 \quad\quad +4 \\ \hline x+2 & x^3+2x^2+4x+9 \\ & -(x^3+2x^2) \quad\quad\quad \\ \hline & +4x+9 \\ & -(4x+8) \\ \hline & +1 \end{array}$$

Example
$\dfrac{4x^3 + 6x^2 + 2}{2x - 1} \equiv 2x^2 + 4x + 2 + \dfrac{4}{2x-1}$

$$\begin{array}{r|r} & 2x^2+4x+2 \\ \hline 2x-1 & 4x^3+6x^2+0x+2 \\ & -(4x^3-2x^2) \quad\quad\quad \\ \hline & +8x^2+0x+2 \\ & -(8x^2-4x) \quad \\ \hline & +4x+2 \\ & -(4x-2) \\ \hline & +4 \end{array}$$

Grid Method / Area Model

Using the grid method for division, fill in what you know and use that and multiplication to fill in more.

Divide the leading term of the dividend by the leading term of the divisor and write the quotient. Subtract the product of the quotient and the divisor from the dividend to get the remainder.

$\times$	**quotient**	
divisor leading term	dividend leading term	remainder
divisor term 2	quotient $\times$ divisor term 2	

Example

$49 \div 12$

$\times$	$+4$	
10	40	$+1$
+2	$+8$	

$$\frac{49}{12} = 4\frac{1}{12}$$

Example

$(4x + 9) \div (x + 2)$

$\times$	$+4$	
x	$4x$	$+1$
$+2$	$+8$	

$$\frac{4x+9}{x+2} \equiv 4 + \frac{1}{x+2}$$

Example

$(8x - 9) \div (2x + 3)$

$\times$	$+4$	
$2x$	$8x$	-21
$+3$	$+12$	

$$\frac{8x-9}{2x+3} \equiv 4 - \frac{21}{2x+3}$$

Example

$(4x^2 + 9x + 3) \div (x^2 + 2x + 1)$

$\times$	$+4$	
x^2	$4x^2$	$+x$
$+2x$	$+8x$	-1
$+1$	$+4$	

$$\frac{4x^2+9x+3}{x^2+2x+1} \equiv 4 + \frac{x-1}{x^2+2x+1}$$

Step by Step Example

$(4x + 9) \div (x + 2)$

$\times$		
x	$4x$	
$+2$		

Set up the divisor on the side and leading term of the dividend in the grid.

$\times$	$+4$	
x	$4x$	
$+2$		

$4x \div x = +4$

Divide the leading term of the dividend by the leading term of the divisor and write the quotient.

$\times$	$+4$	
x	$4x$	
$+2$	$+8$	

$4 \times 2 = +8$

Multiply the quotient by the other terms of the divisor to fill the column.

$\times$	$+4$	
x	$4x$	$+1$
$+2$	$+8$	

$(4x + 9) - (4x + 8) = +1$

Subtract the product of the quotient and the divisor from the dividend to get the remainder.

If the remainder divided by the leading term of the divisor is not a negative power of x, (e.g. x^{-1}) repeat using the remainder as the dividend.

Example

$1249 \div 12$

$\times$	**100**	$+4$	
10	1000	$+40$	$+1$
+2	$+200$	$+8$	

$$\frac{1249}{12} = 104\frac{1}{12}$$

Example

$(x^3 + 2x^2 + 4x + 9) \div (x + 2)$

$\times$	x^2	$+4$	
x	x^3	$+4x$	$+1$
$+2$	$+2x^2$	$+8$	

$$\frac{x^3 + 2x^2 + 4x + 9}{x+2} \equiv x^2 + 4 + \frac{1}{x+2}$$

Example

$(4x^3 + 6x^2 + 2) \div (2x - 1)$

$\times$	$2x^2$	$+4x$	$+4$	
$2x$	$4x^2$	$+8x^2$	$+4x$	$+4$
-1	$-2x^2$	$-4x$	-2	

$$\frac{4x^3 + 6x^2 + 2}{2x - 1}$$

$$\equiv 2x^2 + 4x + 2 + \frac{4}{2x-1}$$

Powers of Binomials / Binomial Expansion

The power of a power function is relatively simple: $(ax^m)^n \equiv a^n x^{m\times n}$
However, powers of sums such as binomials (or other polynomials) are not so simple.

Exponents do **not** distribute over addition. They indicate repeated multiplication of the sum.
To perform the repeated multiplication, we can: multiply pairs of polynomials until all brackets are removed, or consider all possible combinations of products between the terms in the polynomials.

$(x+a)^n \neq x^n + a^n$ $\qquad (x+a)^n \equiv (x+a)(x+a)\cdots(x+a)$ (n times)

Example

Expand $(a+b)^3$
$\equiv (a+b)(a+b)(a+b)$
$\equiv (a^2+ab+ab+b^2)(a+b)$

$\equiv (a^3+a^2b+a^2b+ab^2)$
$+(a^2b+ab^2+ab^2+b^3)$

$\equiv a^3+3a^2b+3ab^2+b^3$

Example

Expand $(a+b)^4$
$\equiv (a+b)(a+b)(a+b)(a+b)$
$\equiv (a^2+2ab+b^2)(a^2+2ab+b^2)$

$\equiv (a^4+2a^3b+a^2b^2)$
$+(2a^3b+4a^2b^2+2ab^3)$
$+(a^2b^2+2ab^3+b^4)$

$\equiv a^4+4a^3b+6a^2b^2+4ab^3+b^4$

Example

Expand $(a+b)^3 \equiv (a+b)(a+b)(a+b)$

1 way of getting the a^3 term: aaa
$(a+b)(a+b)(a+b)$

3 ways of getting the a^2b term: $aab+aba+baa$
$(a+b)(a+b)(a+b)$;
$(a+b)(a+b)(a+b)$;
$(a+b)(a+b)(a+b)$

$\therefore (a+b)^3 \equiv a^3+3a^2b+3ab^2+b^3$

3 ways of getting the ab^2 term: $abb+bab+bba$
$(a+b)(a+b)(a+b)$;
$(a+b)(a+b)(a+b)$;
$(a+b)(a+b)(a+b)$

1 way of getting the b^3 term: bbb
$(a+b)(a+b)(a+b)$

Pascal's Triangle

Each number in Pascal's triangle is found by adding the two numbers diagonally above it. Empty spaces are zeroes.

Compare the values in Pascal's triangle to the coefficients in these expansions.

The exponents of a decrease left to right and the exponents of b increase left to right, but always add to the original exponent of the power.

Row 0:							1						
Row 1:						1		1					
Row 2:					1		2		1				
Row 3:				1		3		3		1			
Row 4:			1		4		6		4		1		
Row 5:		1		5		10		10		5		1	
Row 6:	1		6		15		20		15		6		1

$(a+b)^0 \equiv 1$
$(a+b)^1 \equiv a+b$
$(a+b)^2 \equiv a^2+2ab+b^2$
$(a+b)^3 \equiv a^3+3a^2b+3ab^2+b^3$
$(a+b)^4 \equiv a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$(a+b)^5 \equiv a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
$(a+b)^6 \equiv a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+5ab^5+b^6$

Example VCAA 2003 Exam 1 Question 19

The coefficient of x^2 in the expansion of $(2x-3)^5$ is equal to

x^2 uses the 4th number in the 5th row as a coefficient:
Row 5: 1 5 10 **10** 5 1

$10(2x)^2(-3)^3 = 10 \times 4 \times -27 \times x^2 = -1080x^2$
The coefficient of x^2 is equal to -1080

Example

Expand $(3x-2y)^3$ (Row 3: 1331)
$\equiv 1(3x)^3(-2y)^0$
$+3(3x)^2(-2y)^1$
$+3(3x)^1(-2y)^2$
$+1(3x)^0(-2y)^3$

$\equiv 27x^3-54x^2y+36xy^2-8y^3$