Manipulating Quadratic Expressions

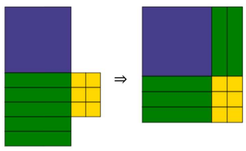
Factorising Monic Quadratics using a Product and Sum

A monic quadratic can be factorised by looking for two numbers that multiply together to form the constant, and add together to form the coefficient of x. Consider the magnitude and sign of the constant and the coefficient of x, this will inform which of the factors is positive / negative.

$$\begin{aligned} &x^2 + bx + c \\ &\equiv x^2 + (p+q)x + pq \\ &\equiv x^2 + px + qx + pq \\ &\equiv x(x+p) + q(x+p) \\ &\equiv (x+p)(x+q) \end{aligned}$$

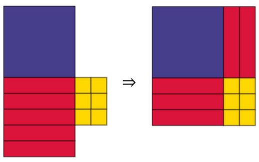
Example

$$\begin{aligned} &x^2 + 5x + 6 \\ &\equiv x^2 + 3x + 2x + 6 \\ &\equiv x(x+3) + 2(x+3) \\ &\equiv (x+3)(x+2) \end{aligned}$$



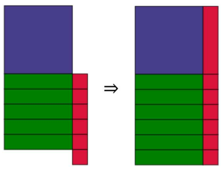
Example

$$\begin{aligned} &x^2 - 5x + 6 \\ &\equiv x^2 - 3x - 2x + 6 \\ &\equiv x(x-3) - 2(x-3) \\ &\equiv (x-3)(x-2) \end{aligned}$$



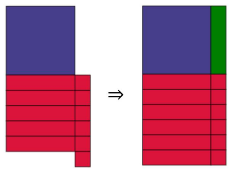
Example

$$\begin{aligned} &x^2 + 5x - 6 \\ &\equiv x^2 + 6x - x - 6 \\ &\equiv x(x+6) - (x+6) \\ &\equiv (x+6)(x-1) \end{aligned}$$



Example

$$\begin{aligned} &x^2 - 5x - 6 \\ &\equiv x^2 - 6x + x - 6 \\ &\equiv x(x-6) + (x-6) \\ &\equiv (x-6)(x+1) \end{aligned}$$



Example

$x^2 - 6x + 8$

$\equiv (x-2)(x-4)$

$x^2 - 6x + 8$

Negative sum of -6

Positive product of $+8$

$-6 \times 0 = 0$ ✗

$-5 \times (-1) = 5$ ✗

$-4 \times (-2) = 8$ ✓

Example

$x^2 + 6x + 8$

$\equiv (x+2)(x+4)$

$x^2 + 6x + 8$

Positive sum of $+6$

Positive product of $+8$

$6 \times 0 = 0$ ✗

$5 \times 1 = 5$ ✗

$4 \times 2 = 8$ ✓

Example

$x^2 + x - 12$

$\equiv (x-3)(x+4)$

$x^2 + 1x - 12$

Positive sum of $+1$

Negative product of -12

$1 \times 0 = 1$ ✗

$2 \times (-1) = -2$ ✗

$3 \times (-2) = -6$ ✗

$4 \times (-3) = -12$ ✓

Example

$x^2 - x - 12$

$\equiv (x+3)(x-4)$

$x^2 - 1x - 12$

Negative sum of -1

Negative product of -12

$-1 \times 0 = 0$ ✗

$-2 \times 1 = -2$ ✗

$-3 \times 2 = -6$ ✗

$-4 \times 3 = -12$ ✓

Example

$x^2 + 6x + 9$

$\equiv (x+3)(x+3)$

$\equiv (x+3)^2$

$x^2 + 6x + 9$

Positive sum of $+6$

Positive product of $+9$

$6 \times 0 = 0$ ✗

$5 \times 1 = 5$ ✗

$4 \times 2 = 8$ ✗

$3 \times 3 = 9$ ✓

Example

$x^2 - 9x$

$\equiv (x+0)(x-9)$

$\equiv x(x-9)$

$x^2 - 9x + \mathbf{0}$

Negative sum of -9

Product of 0

$-9 \times 0 = 0$ ✓

Example

$x^2 - 9$

$\equiv (x-3)(x+3)$

$x^2 + \mathbf{0}x - 9$

Sum of 0

Negative product of -9

$0 \times 0 = 0$ ✗

$-1 \times 1 = -1$ ✗

$-2 \times 2 = -4$ ✗

$-3 \times 3 = -9$ ✓

Example

$2x^2 - 10x - 12$

$\equiv 2(x^2 - 5x - 6)$

$\equiv 2(x-6)(x+1)$

$x^2 - 5x - 6$

Negative sum of -5

Negative product of -6

$-5 \times 0 = 0$ ✗

$-6 \times 1 = -6$ ✗

$-7 \times 2 = -14$ ✓

Factorising Non-Monic Quadratics

To factorise non-monic quadratics where the coefficient of x^2, a, is not a common factor, we can manipulate the expression by multiplying the quadratic by the coefficient of x^2, a, to write the quadratic in terms of ax instead of x.

We can then factorise using the usual product and sum method instead putting ax in both brackets and dividing by a to compensate for earlier multiplying by a.
You can quickly check you haven't forgotten by expanding the x^2 or constant term.

We can abbreviate the working somewhat by focusing on the fact that only the constant looks like it is multiplied by the coefficient of x^2, a.

Example

$6x^2 + 23x + 20$

$\equiv \frac{1}{6}\left((6x)^2 + 23(6x) + 120\right)$

$\equiv \frac{1}{6}\left((6x)^2 + 15(6x) + 8(6x) + 120\right)$

$\equiv \frac{1}{6}(6x + 15)(6x + 8)$

$\equiv \frac{1}{6}\left(3(2x + 5)\right)\left(2(3x + 4)\right)$

$\equiv (2x + 5)(3x + 4)$

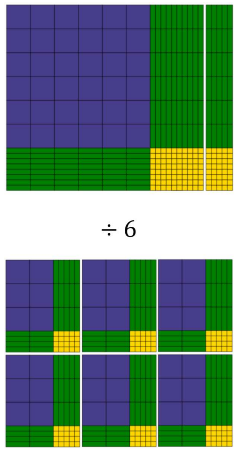
Shortened Example

$6x^2 + 23x + 20 \qquad 6 \times 20 = 120$

Positive sum of $+8$ and positive product of $120 = 15 \times 8$

$\equiv \frac{1}{6}(6x + 15)(6x + 8)$

$\equiv (2x + 5)(3x + 4)$



Example

$2x^2 - 5x - 12$

$\equiv \frac{1}{2}\left((2x)^2 - 5(2x) - 24\right)$

$\equiv \frac{1}{2}\left((2x)^2 - 8(2x) + 3(2x) - 24\right)$

$\equiv \frac{1}{2}(2x - 8)(2x + 3)$

$\equiv \frac{1}{2}\left(2(x - 4)\right)(2x + 3)$

$\equiv (x - 4)(2x + 3)$

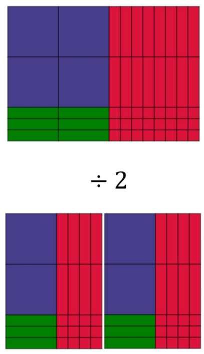
Shortened Example

$2x^2 - 5x - 12 \qquad 2 \times -12 = -24$

Negative sum of -5 and negative product of $-24 = -8 \times 3$

$\equiv \frac{1}{2}(2x - 8)(2x + 3)$

$\equiv (x - 4)(2x + 3)$



Taking a Binomial Expression out as a Factor

In the same way that we can take a number or power of a variable out as a factor, we can also take a binomial out as a factor. This requires the use of polynomial division.

Example

Take $(x + 2)$ out as a factor of $x^2 - 6x + 8$

$$x^2 - 6x + 8 \equiv (x + 2)\left(\frac{x^2 - 6x + 8}{x + 2}\right)$$

$\times$	x	-8	
x	x^2	$-8x$	$+24$
$+2$	$+2x$	-16	

$\equiv (x + 2)\left((x - 8) + \frac{24}{x + 2}\right)$

or $(x + 2)(x - 8) + 24$

Example

Take $(x - 2)$ out as a factor of $x^2 - 6x + 8$

$$x^2 - 6x + 8 \equiv (x - 2)\left(\frac{x^2 - 6x + 8}{x - 2}\right)$$

$\times$	x	-4	
x	x^2	$-4x$	$+0$
-2	$-2x$	-8	

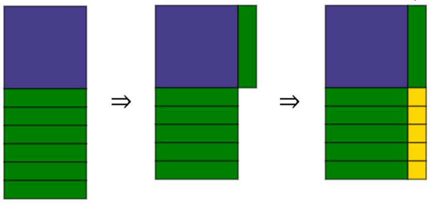
$\equiv (x - 2)(x - 4)$

Completing the Rectangle

The x^2 and x terms of a monic quadratic can always be arranged into a rectangle.
If we take this rectangle and change the length and width, we will be missing the constant part.
We can complete the rectangle by adding the units necessary to fill the missing rectangle.

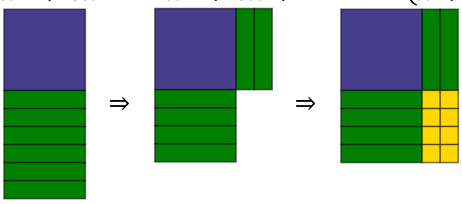
Example

$x^2 + 6x \Rightarrow x^2 + 6x + 1 \times 5 \equiv (x + 1)(x + 5)$



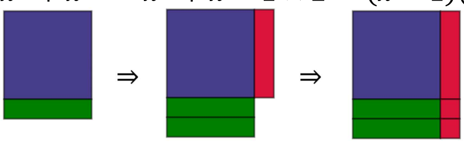
Example

$x^2 + 6x \Rightarrow x^2 + 6x + 2 \times 4 \equiv (x + 2)(x + 4)$



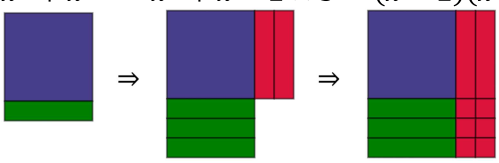
Example

$x^2 + x \Rightarrow x^2 + x - 1 \times 2 \equiv (x - 1)(x + 2)$



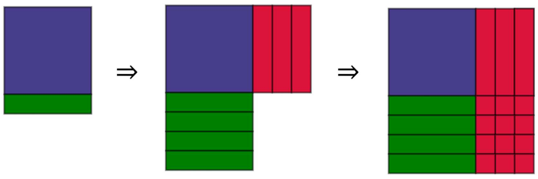
Example

$x^2 + x \Rightarrow x^2 + x - 2 \times 3 \equiv (x - 2)(x + 3)$



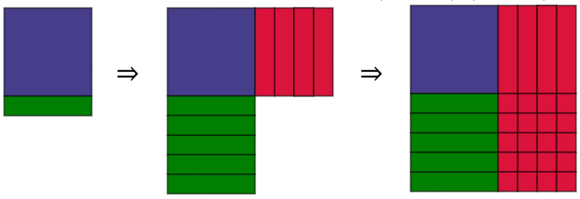
Example

$x^2 + x \Rightarrow x^2 + x - 3 \times 4 \equiv (x - 3)(x + 4)$



Example

$x^2 + x \Rightarrow x^2 + x - 5 \times 6 \equiv (x - 4)(x + 5)$

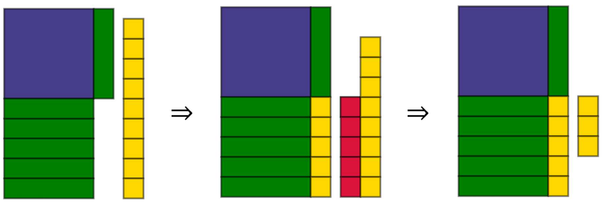


Changing the form of a Quadratic using Completing the Rectangle

Adding the units changes the expression, so to compensate we need to subtract the same amount (or add the same amount to the other side if it is an equation).
Generalising this, we get: $x^2 + bx = (x + n)(x + b - n) - n(b - n)$

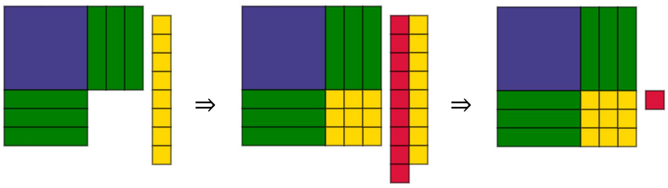
Example

$x^2 + 6x + 8$
$\equiv (x + 1)(x + 5) - 5 + 8$
$\equiv (x + 1)(x + 5) + 3$



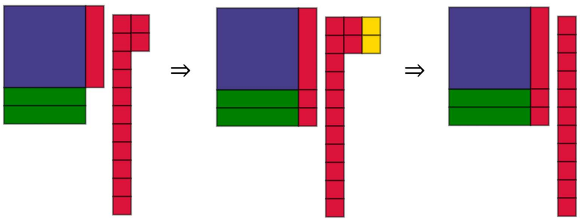
Example

$x^2 + 6x + 8$
$\equiv (x + 3)(x + 3) - 9 + 8$
$\equiv (x + 3)(x + 3) - 1$



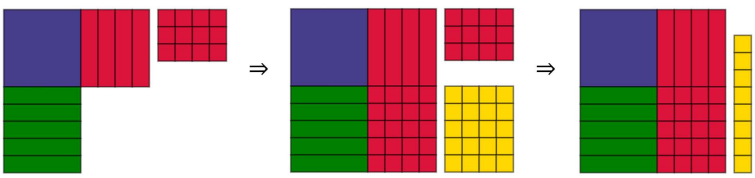
Example

$x^2 + x - 12$
$\equiv (x - 1)(x + 2) + 10 - 12$
$\equiv (x - 1)(x + 2) - 10$

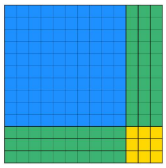


Example

$x^2 + x - 12$
$\equiv (x - 4)(x + 5) + 20 - 12$
$\equiv (x - 4)(x + 5) + 8$



Perfect Squares



Square numbers can be shown as a square by arranging Dienes blocks.
For example, $169 = (10 + 3)^2 = 13^2$ shown adjacent.

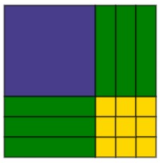
Likewise, the square of an expression can be shown as a square using algebra tiles.

$$a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b)$$

$$a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b)$$

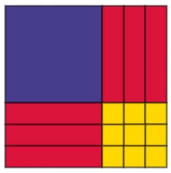
Example

$x^2 + 6x + 9$
$= x^2 + 2(3)x + (+3)^2$
$= (x + 3)^2$



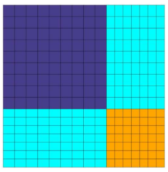
Example

$x^2 - 6x + 9$
$= x^2 + 2(-3)x + (-3)^2$
$= (x - 3)^2$



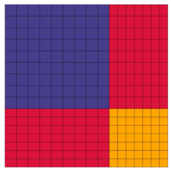
Example

$81x^2 + 126xy + 49y^2$
$= (9x)^2 + 2(9x)(7y) + (7y)^2$
$= (9x + 7y)^2$



Example

$81x^2 - 126xy + 49y^2$
$= (9x)^2 - 2(9x)(7y) + (7y)^2$
$= (9x - 7y)^2$



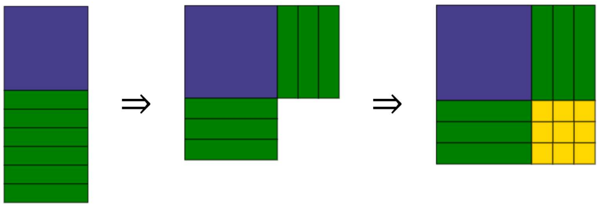
Completing the Square for Monic Quadratics

The x^2 and x terms of a monic quadratic can always be arranged into a rectangle.
If we take this rectangle and try to make a square, we will be missing the constant part.
We can complete the square by adding the units necessary to fill the missing square.

Example

$x^2 + 6x = x^2 + 2(3)x$

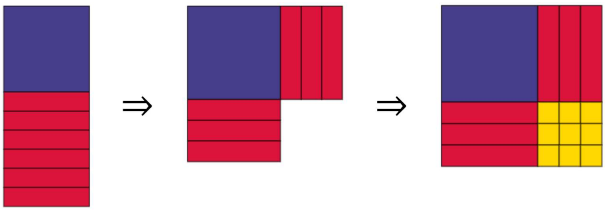
$\Rightarrow x^2 + 2(3)x + (+3)^2$
$= (x + 3)^2$



Example

$x^2 - 6x = x^2 + 2(-3)x$

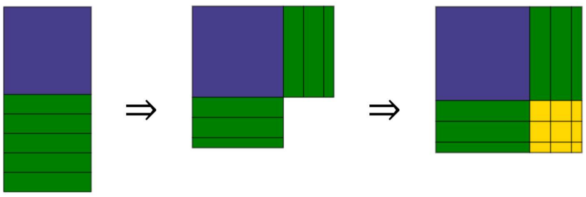
$\Rightarrow x^2 + 2(-3)x + (+3)^2$
$= (x - 3)^2$



Example

$x^2 + 5x = x^2 + 2(2.5)x$

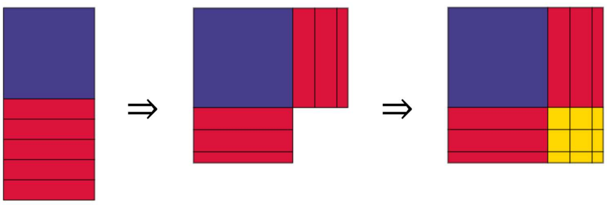
$\Rightarrow x^2 + 2(2.5)x + (+2.5)^2$
$= (x + 2.5)^2$



Example

$x^2 - 5x = x^2 + 2(-2.5)x$

$\Rightarrow x^2 + 2(-2.5)x + (+2.5)^2$
$= (x - 2.5)^2$



Changing the form of a Quadratic using Completing the Square

Adding the units changes the expression, so to compensate we need to subtract the same amount (or add the same amount to the other side if it is an equation). Generalising this, we get the difference of two squares:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Completing the square has three useful purposes:

- finding the turning point of a quadratic
- writing the quadratic with only one instance of x to help solve quadratic equations directly
- factorising quadratics with rational or irrational roots

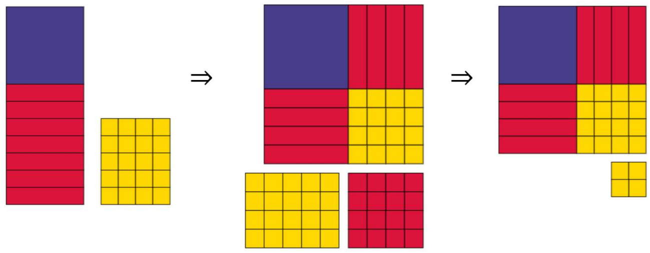
Example

$x^2 - 8x + 20$

$= \left(x - \frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + 20$

$= (x-4)^2 - 16 + 20$

$= (x-4)^2 + 4$



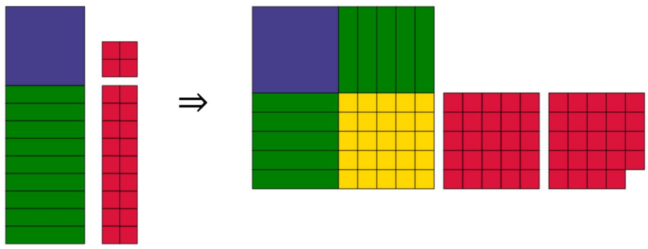
Example

$x^2 + 10x - 24$

$= \left(x + \frac{10}{2}\right)^2 - \left(\frac{10}{2}\right)^2 - 24$

$= (x+5)^2 - 25 - 24$

$= (x+5)^2 - 49$



Example

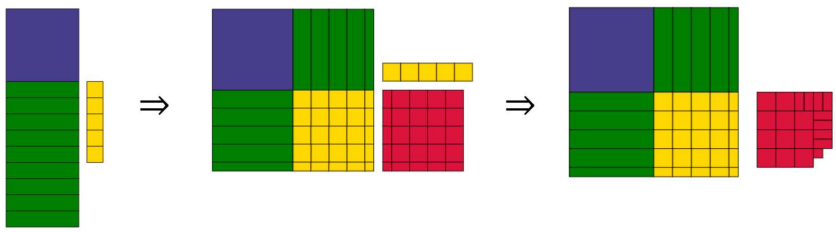
$x^2 + 9x + 5$

$= \left(x + \frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 5$

$= \left(x + \frac{9}{2}\right)^2 - \frac{81}{4} - \frac{20}{4}$

$= \left(x + \frac{9}{2}\right)^2 - \frac{61}{4}$

or $(x + 4.5)^2 - 15.25$



Completing the Square for Non-Monic Quadratics

When the quadratic is not monic, one way to complete the square is to first take the coefficient of x^2, a, out as a factor of the x^2 and x terms. You can take it out of the constant as well, but you don't have to.

Example VCAA 2001 Exam 1 Question 2a

Let $f: R \to R, f(x) = 2x^2 + 12x + 10$. $f(x)$ in the form $a(x+b)^2 + c$ is

$2x^2 + 12x + 10$

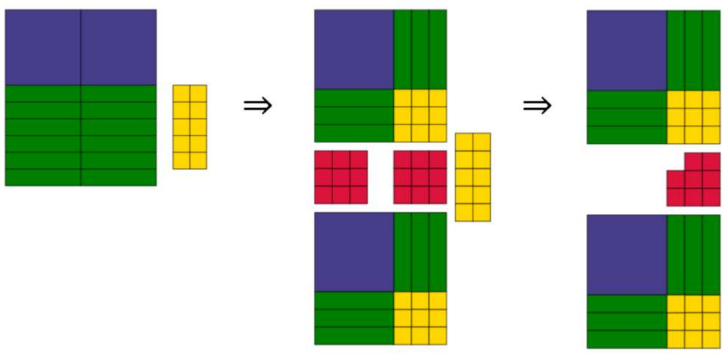
$= 2(x^2 + 6x) + 10$

$= 2\left(\left(x + \frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2\right) + 10$

$= 2\left((x+3)^2 - 9\right) + 10$

$= 2(x+3)^2 - 18 + 10$

$= 2(x+3)^2 - 8$



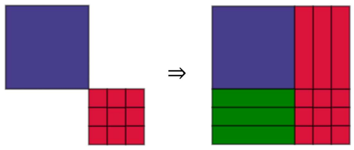
Difference of Two Squares

Expressions involving the difference of squares can be shown as a rectangle by completing the rectangle with an equal amount of positive and negative x terms. The rectangle may look like a square using algebra tiles but remember the positive length adds to the side length where as the negative subtracts from it.

$a^2 - b^2 = (a - b)(a + b)$

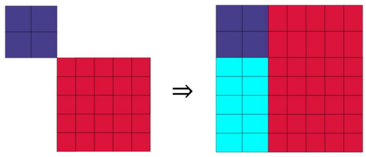
Example

$x^2 - 9$
$= x^2 - 3^2$
$= x^2 + 3x - 3x - 3^2$
$= (x + 3)(x - 3)$



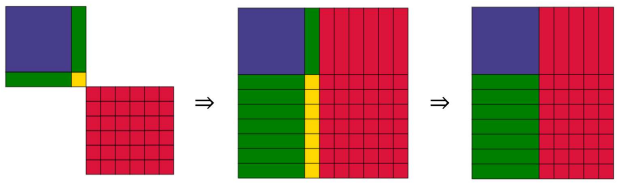
Example

$4x^2 - 25y^2$
$= (2x)^2 - (5y)^2$
$= (2x)^2 + (2x)(5y) - (2x)(5y) - (5y)^2$
$= (2x + 5y)(2x - 5y)$



Example

$(x + 1)^2 - 36$
$= (x + 1)^2 - 6^2$
$= (x + 1)^2 + 6(x + 1) - 6(x + 1) - 6^2$
$= \big((x + 1) + 6\big)\big((x + 1) - 6\big)$
$= (x + 7)(x - 5)$



Example

$$\left(x + \frac{3}{2}\right)^2 - 16 = \left(x + \frac{3}{2}\right)^2 - 4^2$$

$$= \left(\left(x + \frac{3}{2}\right) + 4\right)\left(\left(x + \frac{3}{2}\right) - 4\right)$$

$$= \left(x + \frac{3}{2} + \frac{8}{2}\right)\left(x + \frac{3}{2} - \frac{8}{2}\right)$$

$$= \left(x + \frac{11}{2}\right)\left(x - \frac{5}{2}\right)$$

Example

$5x^2 - 7$
$= (\sqrt{5}x) - \sqrt{7}^2$
$= (\sqrt{5}x - \sqrt{7})(\sqrt{5}x + \sqrt{7})$

Example

$(x - 1)^2 - 3 = (x - 1)^2 - \sqrt{3}^2$
$= \left((x - 1) + \sqrt{3}\right)\left((x - 1) - \sqrt{3}\right)$
$= (x - 1 + \sqrt{3})(x - 1 - \sqrt{3})$

Example

$5x^2 - 10$
$= 5(x^2 - 2)$
$= 5\left(x^2 - \sqrt{2}^2\right)$
$= 5(x - \sqrt{2})(x + \sqrt{2})$