Expanding and Factorising Polynomials

Expanding Factorised Polynomials

Expanding factorised polynomials is the same as multiplying them, that is use the distributive law. Expanding the product of three or more polynomials can be done in two ways:

- multiply pairs of polynomials until all brackets are removed, or
- consider all possible combinations of products between the terms in the polynomials.

Example

Expand $(x^2 - 7x + 2)(x - 5)$
$\equiv (x^3 - 7x^2 + 2x) + (-5x^2 + 35x - 10)$
$\equiv x^3 - 12x^2 + 37x - 10$

Example

Expand $(x + 2)(x + 3)(x - 5)$
$\equiv (x^2 + 2x + 3x + 6)(x - 5)$

$\equiv (x^3 + 2x^2 + 3x^2 + 6x)$
$+(-5x^2 - 10x - 15x - 30)$

$\equiv x^3 + x^2 - 19x - 30$

Example

Expand $(x + 3)^2(x + 5)$
$\equiv (x^2 + 6x + 9)(x + 5)$
$\equiv (x^3 + 6x^2 + 9x) + (5x^2 + 30x + 45)$
$\equiv x^3 + 11x^2 + 39x + 45$

Example

Expand $(x^2 - 7x + 2)(x^2 + 3x - 5)$

$\equiv (x^4 - 7x^3 + 2x^2)$
$+(3x^3 - 21x^2 + 6x)$
$+(-5x^2 + 35x - 10)$

$\equiv x^4 - 4x^3 - 24x^2 + 41x - 10$

Example

Expand $(x + 2)(x + 3)(x - 5)(x - 4)$
$\equiv (x^2 + 5x + 6)(x^2 - 9x + 20)$

$\equiv (x^4 + 5x^3 + 6x^2)$
$+(-9x^3 - 45x^2 - 54x)$
$+(20x^2 + 100x + 120)$

$\equiv x^4 - 4x^3 - 19x^2 + 46x + 120$

Factorising Polynomials

We can take a polynomial out as a factor of another by dividing it out. If there is a remainder we can multiply it out to write the polynomial as a product with the remainder added on.

Example

Take $(x + 2)$ out as a factor of $x^2 - 6x + 8$

$$x^2 - 6x + 8$$
$$\equiv (x + 2)\left(\frac{x^2 - 6x + 8}{x + 2}\right)$$
$$\equiv (x + 2)\left(x - 8 + \frac{24}{x + 2}\right)$$
$$\equiv (x + 2)(x - 8) + 24$$

Example

Take $(x^2 + 2)$ out as a factor of $x^3 + 2x^2 - 6x + 8$

$$x^3 + 2x^2 - 6x + 8$$
$$\equiv (x^2 + 2)\left(\frac{x^3 + 2x^2 - 6x + 8}{x^2 + 2}\right)$$
$$\equiv (x^2 + 2)\left(x + 2 + \frac{4}{x^2 + 2}\right)$$
$$\equiv (x^2 + 2)(x + 2) + 4$$

When a polynomial exactly divides another without remainder, the divisor and the quotient are factors of the dividend. We consider a polynomial fully factorised when it is written as the product of linear factors and/or irreducible quadratic factors.

Example

Factorise $x^3 - 4x^2 + x + 6$ given $x - 2$ is a factor.

$$x^3 - 4x^2 + x + 6 \equiv (x - 2)\left(\frac{x^3 - 4x^2 + x + 6}{x - 2}\right)$$
$$\equiv (x - 2)(x^2 - 2x - 3)$$
$$\equiv (x - 2)(x + 1)(x - 3)$$

$$\begin{array}{r|l} & x^2 - 2x - 3 \\ \hline x - 2 & x^3 - 4x^2 + x + 6 \\ & -(x^3 - 2x^2) \\ \hline & -2x^2 + x \\ & -(-2x^2 + 4x) \\ \hline & -3x + 6 \\ & -(-3x + 6) \\ \hline & 0 \end{array}$$

Sum and Difference of Two Powers

The difference of powers, $x^n - a^n$, has $(x-a)$ as a factor.
The sum of powers, $x^n + a^n$, has $(x+a)$ as a factor when n is odd.

Example

$x^4 - 16$
$\equiv x^4 - 2^4$
$\equiv (x-2)\left(\frac{x^4-16}{x-2}\right)$
$\equiv (x-2)(x^3+2x^2+4x+8)$

Example

$x^3 - 27$
$\equiv x^3 - 3^3$
$\equiv (x-3)\left(\frac{x^3-27}{x-3}\right)$
$\equiv (x-3)(x^2+3x+9)$

Example

$x^3 + 27$
$\equiv x^3 + 3^3$
$\equiv (x+3)\left(\frac{x^3+27}{x+3}\right)$
$\equiv (x+3)(x^2-3x+9)$

Factorising Quadratic-like Expressions using a Substitution

We can factorise expressions in the form $a\big(f(x)\big)^2 + b\big(f(x)\big) + c$ using quadratic methods by first substituting a variable for $f(x)$. Where necessary, more than one substitution can be made.

Example

$x^4 + 7x^2 + 10$
$\equiv (x^2)^2 + 7(x^2) + 10$
Let $a = x^2$
$\equiv a^2 - a + 12$
$\equiv (a+3)(a-4)$
$\equiv (x^2+3)(x^2-4)$
$\equiv (x^2+3)(x-2)(x+2)$

Example

$(2x+1)^2 + 5(2x+1) + 6$
Let $y = 2x+1$
$\equiv y^2 + 5y + 6$
$\equiv (y+2)(y+3)$
$\equiv (2x+1+2)(2x+1+3)$
$\equiv (2x+3)(2x+4)$
$\equiv 2(2x+3)(x+2)$

Example

$(x^2+5x+6)^2 + 5(x^2+5x+6) + 6$
Let $p = x^2+5x+6$
$\equiv p^2 + 5p + 6$
$\equiv (p+2)(p+3)$
$\equiv (x^2+5x+6+2)(x^2+5x+6+3)$
$\equiv (x^2+5x+8)(x^2+5x+9)$

Equating Coefficients to Convert between Forms

If there is a form we want to write our polynomial in, we can equate the coefficients of our polynomial with that form with unknown coefficients on the new form after expanding and grouping the like terms of both polynomials. This is a brute force method of changing form.

Example

$(x-2)(x-3) \equiv a(x-1)^2 + b(x-1) + c$
$x^2 - 5x + 6 \equiv ax^2 - 2ax + a + bx - b + c$
$\equiv ax^2 + (-2a+b)x + (a-b+c)$

$\therefore (x-2)(x-3) \equiv (x-1)^2 - 3(x-1) + 2$

Equating Coefficients

x^2: $a = 1$
x: $-2a + b = -5$, since $a = 1$
$\Rightarrow -2 + b = -5 \Rightarrow b = -3$
x^0: $a - b + c = 6$, since $a = 1$ and $b = -3$
$\Rightarrow 1 + 3 + c = 6 \Rightarrow c = 2$

Example VCAA 2001 Exam 1 Question 2a

Let $f: R \to R, f(x) = 2x^2 + 12x + 10$.
$f(x)$ in the form $a(x+b)^2 + c$ is

$2x^2 + 12x + 10 \equiv a(x+b)^2 + c$
$2x^2 + 12x + 10 \equiv a(x^2 + 2bx + b^2) + c$
$2x^2 + 12x + 10 \equiv ax^2 + 2abx + (ab^2 + c)$

Equating Coefficients

x^2: $a = 2$
x: $2ab = 12 \Rightarrow 2(2)b = 12 \Rightarrow b = 3$
x^0: $ab^2 + c = 10 \Rightarrow (2)(3)^2 + c = 10 \Rightarrow c = -8$

$\therefore f(x) = 2(x+3)^2 - 8$

Example

Factorise $x^3 - 6x^2 + 11x - 6$ given that $x - 1$ is a factor.

$(x-1)(ax^2 + bx + c) \equiv x^3 - 6x^2 + 11x - 6$
$ax^3 - ax^2 + bx^2 - bx + cx - c \equiv x^3 - 6x^2 + 11x - 6$
$ax^3 + (-a+b)x^2 + (-b+c)x - c \equiv x^3 - 6x^2 + 11x - 6$

Equating Coefficients

x^3: $a = 1$
x^0: $-c = -6 \Rightarrow c = 6$

x^2: $-a + b = -6$
$\Rightarrow -1 + b = -6 \Rightarrow b = -5$

$\therefore x^3 - 6x^2 + 11x - 6 \equiv (x-1)(x^2 - 5x + 6) \equiv (x-1)(x+2)(x+3)$