Remainder, Factor, and Rational Root Theorems

Remainder Theorem

The remainder of the polynomial division $P(x) \div (qx - p)$ is equal to $P\left(\frac{p}{q}\right)$.

Where $\frac{p}{q}$ is the solution to the equation $qx - p = 0$.

Example

The remainder when $P(x) = x^2 - x + 2$ is divided by $3x - 2$ is

$3x - 2 = 0 \quad \Rightarrow \quad x = \frac{2}{3}, \qquad P\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right) + 2 = \frac{16}{9}$. Therefore, the remainder is $\frac{16}{9}$.

Example

The remainder when $P(x) = x^2 - x + 2$ is divided by $x - 4$ is

$x - 4 = 0 \quad \Rightarrow \quad x = 4, \qquad P(4) = (4)^2 - (4) + 2 = 14$. Therefore, the remainder is 14.

Example

The remainder when $P(x) = 3x^2 + ax + 8$ is divided by $x - 2$ is 10, where $a \in R \setminus \{0\}$. The value of a is

$x - 2 = 0$
$\Rightarrow x = 2$

$P(2) = 3(2)^2 + a(2) + 8$
$10 = 12 + 2a + 8$
$2a = -10$
$a = -5$

Example

The remainder when $P(x) = x^2 + px + q$ is divided by $x + 3$ is 7 and 5 when divided by $x + 2$, where $p, q \in R \setminus \{0\}$. The values of p and q are

$x + 3 = 0 \quad \Rightarrow \quad x = -3$
$P(-3) = (-3)^2 + p(-3) + q$
$7 = 9 - 3p + q$
$-3p + q = -2 \quad (\mathbf{1})$

$x + 2 = 0 \quad \Rightarrow \quad x = -2$
$P(-2) = (-2)^2 + p(-2) + q$
$5 = 4 - 2p + q$
$-2p + q = 1 \quad (\mathbf{2})$

$(\mathbf{1}) - (\mathbf{2})$
$(-3p + q) - (-2p + q)$
$= (-2) - (1)$
$\Rightarrow -p = -3$
$\Rightarrow p = 3$

$-3(3) + q = -2$
$\Rightarrow -9 + q = -2$
$\Rightarrow q = 7$

Remainder Theorem Proof

Multiply both sides of the division by the divisor $(qx - p)$ to write the equation as a product.

$$\frac{P(x)}{qx - p} = Q(x) + \frac{r}{qx - p} \quad \Rightarrow \quad P(x) = (qx - p)\left(Q(x) + \frac{r}{qx - p}\right) = (qx - p)Q(x) + r$$

When the divisor is equal to zero the polynomial is equal to the remainder.

That is, $qx - p = 0$ when $x = \frac{p}{q}$ and $P\left(\frac{p}{q}\right) = 0 \times Q\left(\frac{p}{q}\right) + r = r$. Therefore, $P\left(\frac{p}{q}\right) = r$

Factor Theorem

$(qx - p)$ is a factor of the polynomial $P(x)$ if and only if $P\left(\frac{p}{q}\right) = 0$.

Example

Determine if $(2x + 1)$ and $(x + 4)$ are factors of $P(x) = 2x^2 + 7x + 3$.

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

$P\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^2 + 7\left(-\frac{1}{2}\right) + 3 = 0$

Therefore, $(2x + 1)$ is a factor of $P(x)$.

$x + 4 = 0 \Rightarrow x = -4$

$P(-4) = 2(-4)^2 + 7(-4) + 3 = 7$

Therefore, $(x + 4)$ is **not** a factor of $P(x)$.

Example VCAA 2013 Exam 2 Question 3

If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R \setminus \{0\}$, then the value of a is

Let $P(x) = 7x^3 + 9x^2 - 5ax$

If $x + a$ is to be a factor, then,
$x + a = 0 \Rightarrow x = -a,$
$P(-a) = 0$

$P(-a) = 7(-a)^3 + 9(-a)^2 - 5a(-a)$
$\Rightarrow 0 = -7a^3 + 9a^2 + 5a^2$
$\Rightarrow 0 = -7a^3 + 14a^2$
$\Rightarrow 7a^2(2 - a) = 0$
$\Rightarrow a = 2$ since $a \in R \setminus \{0\}$

Factor Theorem Proof

$(qx - p)$ is a factor of $P(x)$ if $P(x) \div (qx - p)$ has a remainder of zero.

Since $P\left(\frac{p}{q}\right)$ is equal to the remainder, if $P\left(\frac{p}{q}\right) = 0$, then $(qx - p)$ must be a factor.

Roots, Factors, and Horizontal Axis Intercepts

If $x = a$ is a root of the polynomial $P(x)$, then $P(a) = 0$. Therefore, by the factor theorem, $(x - a)$ is a factor. Also, for the graph of $y = P(x)$, since $P(a) = 0$, $x = a$ is a horizontal, x-axis intercept.

Using the Factor Theorem to Factorise a Polynomial

1) Test values of a by substituting them into the polynomial $P(x)$ until $P(x)$ equals 0
2) Since a is a root, conclude $(x - a)$ is a factor of $P(x)$
3) Find $Q(x) = \frac{P(x)}{x - a}$ using a division method
4) Continue factorising $Q(x)$, either by factor theorem or other factoring techniques.

Example

Factorise $P(x) = x^3 + 2x^2 - 5x - 6$

$P(1) = -8 \neq 0$ $\quad$ $P(-1) = 0$

Therefore, $(x + 1)$ is a factor of $P(x)$

$\times$	x^2	$+x$	-6
x	$+x^3$	$+x^2$	$-6x$
$+1$	$+x^2$	$+x$	-6

Therefore, $(x^2 + x - 6)$ is also a factor.

Therefore, $P(x) = (x + 1)(x^2 + x - 6) = (x + 1)(x - 2)(x + 3)$

Rational Root Theorem

We can use the rational root theorem to reduce the number of roots to check by eliminating any root is not possible and focusing on those that are possible.

For a polynomial $P(x)$ of degree n with integer coefficients, a_i, when the rational roots of $P(x)$ are written as $x = \frac{p}{q}$ for p and q are with a highest common factor of 1, then

- q is an integer factor of the leading coefficient, a_n, and
- p is an integer factor of the constant term, a_0.

$$P(x) = a_n x^n + \cdots + a_0$$

$$x = \frac{p}{q} \Rightarrow P\left(\frac{p}{q}\right) = 0$$

$\therefore (qx - p)$ is a factor of $P(x)$

Example

Factorise $P(x) = 3x^3 - 5x^2 + 5x - 2$

Factors of -2: $\pm 1, \pm 2$
Factors of 3: $\pm 1, \pm 3$,

Possible rational roots: $\pm 1, \pm 2, \pm\frac{1}{3}\pm, \frac{2}{3}$

$P(1) = 1 \neq 0$ $\quad P(-1) = -15 \neq 0$ $\quad P(2) = 12 \neq 0$ $\quad P(-2) = -56 \neq 0$

$P\left(\frac{1}{3}\right) = -\frac{7}{9} \neq 0$ $\quad P\left(-\frac{1}{3}\right) = -\frac{13}{3} \neq 0$ $\quad P\left(\frac{2}{3}\right) = 0$ ✔ $\quad P\left(-\frac{2}{3}\right) = -\frac{76}{9} \neq 0$

Therefore, $(3x - 2)$ is the only rational factor.
The other factor is $(x^2 - x + 1)$.

Therefore, $P(x) = (2x - 3)(x^2 - x + 1)$.

$\times$	$\boldsymbol{x^2}$	$\boldsymbol{-x}$	$\boldsymbol{+1}$
$\boldsymbol{3x}$	$+3x^3$	$-3x^2$	$+3x$
$\boldsymbol{-2}$	$-2x^2$	$+2x$	-2

Proof of the Rational Root Theorem

For $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots + a_1 x + a_0$, suppose $P\left(\frac{p}{q}\right) = 0$ where p and q are integers with a highest common factor of 1, such that $\frac{p}{q}$ is a fraction that cannot simplify.

If $x = \frac{p}{q}$ is a root of $P(x)$, then $(qx - p)$ is a factor of $P(x)$ by the remainder and factor theorems.
Therefore, $P(x) = (qx - p)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0)$

Expanding this,

- the only way to get the x^n term is $(qx)(b_{n-1}x^{n-1}) = a_n x^n$, therefore q must be a factor of a_n.
- the only way to get a constant term is $(-p)(b_0) = a_0$, therefore p must be a factor of a_0.