Polynomial Equations

Number of Real Solution

A polynomial with real coefficients of up to degree n will have up to n real solutions.

Null Factor Law

If $a \times b = 0$ then $a = 0$ and/or $b = 0$.

For the polynomial equation $P(x) = 0$, if you can factorise $P(x)$, you can use the null factor law to solve each factor of $P(x)$ separately equal to 0.
Specifically, if they are linear factors, you solve the linear equations.
Remember, you can use the factor theorem and the rational root theorem to factorise polynomials.

Example VCAA 2003 Exam 1 Question 20

Let $p(x) = (x^2 + a)(x + b)(x - c)$, where a, b and c are three distinct positive real numbers.
Determine the number of real solutions to the equation $p(x) = 0$
For $p(x) = 0$, the solutions can be found by solving $x^2 + a = 0$, $x + b = 0$, and $x - c = 0$.

The solutions to the equations are $x = \pm\sqrt{-a}, -b, c$.
Since a, b, c are positive real numbers $\sqrt{-a}$ is not a real number but $-b$ and c are.
Therefore, the number of real solutions to the equation $p(x) = 0$ is exactly 2.

Example VCAA 2000 Exam 1 Question 3b

Find the exact values of all the roots of the equation $(x - 1)(x - 3)(x + 2) + 4 = 0$.

$(x - 1)(x - 3)(x + 2) + 4 = 0$

$\Rightarrow (x^2 - 4x + 3)(x + 2) + 4 = 0$

$\Rightarrow x^3 + 2x^2 - 4x^2 - 8x + 3x + 6 + 4 = 0$

$\Rightarrow x^3 - 2x^2 - 5x + 10 = 0$

$\Rightarrow (x - 2)(x^2 - 5) = 0$

$\Rightarrow x - 2 = 0, \qquad x^2 - 5 = 0$

$\Rightarrow x = 2, \qquad x = \pm\sqrt{5}$

$\times$	$\boldsymbol{x^2}$	$\mathbf{-5}$
$\boldsymbol{x}$	$+x^3$	$-5x$
$\mathbf{-2}$	$-2x^2$	$+10$

$\therefore x^3 - 2x^2 - 5x + 10 = (x - 2)(x^2 - 5)$

Example Modified VCAA 2015 Exam 1 Question 4

Consider the function $f: [-3, 2] \to R, f(x) = \frac{1}{2}(x^3 + 3x^2 - 4)$.
The solutions to the equation $f(x) = 0$ are

$\frac{1}{2}(x^3 + 3x^2 - 4) = 0$

$\Rightarrow \frac{1}{2}(x - 1)(x^2 + 4x + 4) = 0$

$\Rightarrow \frac{1}{2}(x - 1)(x + 2)^2 = 0$

$\Rightarrow x - 1 = 0, \qquad x + 2 = 0$

$\Rightarrow x = 1, \qquad x = -2$

$\times$	$\boldsymbol{x^2}$	$\boldsymbol{+4x}$	$\mathbf{+4}$
$\boldsymbol{x}$	$+x^3$	$+4x^2$	$+4x$
$\mathbf{-1}$	$-x^2$	$-4x$	-4

$x^3 + 3x^2 - 4 = (x - 1)(x^2 + 4x + 4)$

$\therefore f(x) = \frac{1}{2}(x - 1)(x^2 + 4x + 4)$

Quadratic Formula

Quadratic equations can be solved in three ways: by factorising and using the null factor law, by completing the square and solving as a power equation, or by using the quadratic formula.

For a quadratic equation in the form $ax^2 + bx + c = 0$, the solutions are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

It is helpful to consider the quadratic formula as:

$x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a} =$ axis of symmetry $\pm \dfrac{1}{2}$distance between solutions

Example

$2x^2 - 5x - 12 = 0$

$$x = \frac{--5 \pm \sqrt{5^2 - 4(2)(-12)}}{2(2)} = \frac{5 \pm \sqrt{25 + 96}}{4} = \frac{5 \pm \sqrt{121}}{4} = \frac{5 \pm 11}{4}$$

$x = \dfrac{16}{4} = 4$ or $x = -\dfrac{6}{4} = -\dfrac{3}{2}$

Quadratic Equations using Substitutions

Like factoring, some equations can be written as a quadratic equation and thus solved using quadratic methods.

Example

$(2x + 1)^2 + 5(2x + 1) + 6 = 0$

Let $y = 2x + 1$

$$y = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2} = \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm \sqrt{1}}{2} = \frac{-5 \pm 1}{2}$$

$y = 2x + 1 = -\dfrac{4}{2} = -2,$ $y = 2x + 1 = -\dfrac{6}{2} = -3$

$2x = -3,$ $2x = -4$

$x = -\dfrac{3}{2},$ $x = -2$

Example

$x^4 + x^2 - 2 = 0$

Let $a = x^2$

$a^2 + a - 2 = 0$

$a = \dfrac{-1 \pm 3}{2} = -2, 1$

$a = x^2 = -2, 1$

$x = \pm\sqrt{-2}, \pm\sqrt{1}$

reject $\sqrt{-2}$, $x = \pm 1$

Discriminant and the Number of Solutions of a Quadratic Equation

The part of the quadratic formula under the square root is the discriminant. That is, $\Delta = b^2 - 4ac$. It is called the discriminant because it discriminates between the number of solutions of a quadratic equation. If the discriminant if a square number (or fraction), then the solutions are rational.

Number of Real Solutions	Condition	Reason
2	$\Delta > 0$	$\pm\sqrt{\Delta}$ creates two unique values
1	$\Delta = 0$	± 0 does not change the value
0	$\Delta < 0$	The square root of a negative is not a real value.

Example VCAA 2017 Exam 2 Question 7

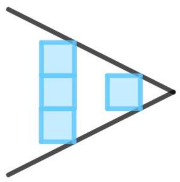
The equation $(p - 1)x^2 + 4x = 5 - p$ has no real roots when the discriminant is negative.

$(p - 1)x^2 + 4x = 5 - p \Rightarrow (p - 1)x^2 + 4x + p - 5 = 0$

$\Delta = (4)^2 - 4(p - 1)(p - 5) = 16 - 4(p^2 - 6p + 5) = p^2 - 6p + 5 - 4 = p^2 - 6p + 1 < 0$

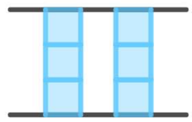
Inequalities Symbols

Greater Than



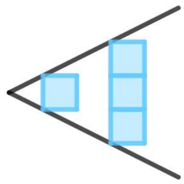
$3 > 1$: 3 is greater than 1

Equals



$3 = 3$: 3 is equal to 3

Less Than



$1 < 3$: 1 is less than 3

By combining the symbols for:

- greater than $>$ and equals $=$ we create the symbol for greater than or equal to $\geq$
- less than $<$ and equals $=$ we create the symbol for and less than or equal to $\leq$

Writing Inequalities from Worded Descriptions

We can write inequalities using variables to describe the possible values that can or should be take.

$x < 0$
x less than 0
x is negative

$x \leq 0$
x less than or equal to 0

$x > 0$
x greater than 0
x is positive

$x \geq 0$
x greater than or equal to 0

$0 < x < 1$
0 is less than x which is less than 1. Alternatively, x is between 0 and 1.

Single Variable Linear Inequalities

Solving an Inequality

To solve a single variable inequality, we solve for the pronumeral as we would a linear equation. That is, we determine the end values that satisfy the inequality and the direction in which the other values that satisfy the equation exist. If there are two operators, whatever you do to one part of the inequation, you do to ALL parts of the inequation.

Example

$x + 5 \leq 2$

$x \leq -3$

Example

$3x > 6$

$x > 2$

Example

$2x - 1 \geq 7$

$2x \geq 8$

$x \geq 4$

Example

$5 < 2x + 3 \leq 11$

$2 < 2x \leq 8$

$1 < x \leq 4$

Multiplying and Dividing by a Negative Number

Consider the inequality $-1 < 2$. If we multiply both sides by -1, we would get $1 < -2$. However this statement does not makes sense as 1 is not less than -2. To correct for this error we flip the direction of the inequality sign to change the inequality to $1 > -2$.

In short, when multiplying or dividing by a negative number, you MUST reverse the direction of the sign.

$< \leftrightarrow >$ $\quad$ $\geq \leftrightarrow \leq$

Example

$-3x - 8 < -5$

$-3x < 3$

$x > -1$

Example

$5 - \frac{x}{2} \geq -7$

$-\frac{x}{2} \geq -12$

$x \leq 24$

Sketching a Single Variable Linear Inequality

Once we have solved for the pronumeral we can present it graphically.
Linear inequalities of a single variable are sketched over a number line.
End points are where the inequality approaches a particular value.
They are either a closed or a hollow circle depending on the type of sign in the inequality.

Closed Circle if included: $\geq$ or $\leq$ Hollow Circle if not included: $>$ or $<$

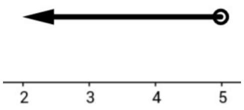
An arrow is drawn in the direction that values are defined in the inequality from the end point

Examples

$7 - 3x > -8$

$-3x > -15$

$x < 5$

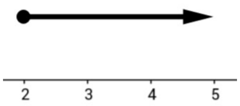


$\dfrac{5x-8}{-2} \leq -1$

$5x - 8 \geq 2$

$5x \geq 10$

$x \geq 2$

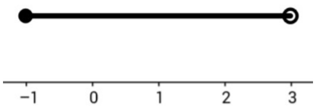


$4 \leq 4(2x+3) < 36$

$1 \leq 2x + 3 < 9$

$-2 \leq 2x < 6$

$-1 \leq x < 3$

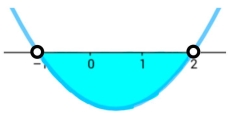


Quadratic and Polynomial Inequalities

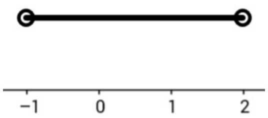
To solve higher degree inequalities, it is easier to move all terms in the inequality to one side of the inequality, factorise the resulting polynomial, and then sketch a quick graph using the x-intercepts to determine where the graph satisfies the inequality. That is, where the graph of the polynomial is above zero, at zero, or below zero. You may be asked to shade the solution on the graph of the polynomial or plot the solutions on a number line.

Example

$x^2 - x < 2$
$x^2 - x - 2 < 0$
$(x-2)(x+1) < 0$

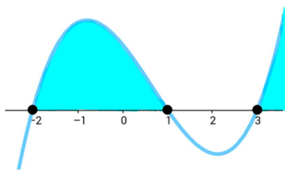


$-1 < x < 2$

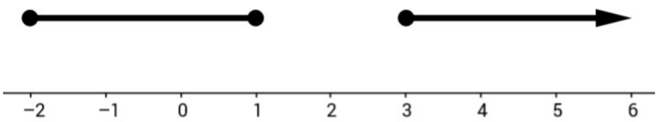


Example

$x^3 - 5x \geq 2x^2 - 6$
$x^3 - 2x^2 - 5x + 6 \geq 0$
$(x-3)(x-1)(x+2) \geq 0$

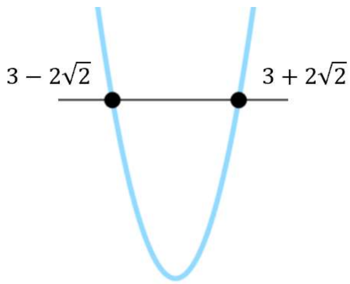


$x \in [-2,1] \cup [3,\infty)$



Example Modified VCAA 2017 Exam 2 Question 7

The values of p that make the equation $(p-1)x^2 + 4x = 5 - p$ have two, one, and no real solutions are



$(p-1)x^2 + 4x = 5 - p \;\;\Rightarrow\;\; (p-1)x^2 + 4x + p - 5 = 0$
$\Delta = (4)^2 - 4(p-1)(p-5) = 16 - 4(p^2 - 6p + 5)$
$\quad = p^2 - 6p + 5 - 4 = p^2 - 6p + 1$
$\quad = (p-3)^2 - 8 = \left(p - 3 + 2\sqrt{2}\right)\left(p - 3 - 2\sqrt{2}\right)$

One real solution when the discriminant is zero: $p = 3 - 2\sqrt{2}$ and $p = 3 + 2\sqrt{2}$
Two real solutions when the discriminant is positive: $p < 3 - 2\sqrt{2}$ and $p > 3 + 2\sqrt{2}$
No real solutions when the discriminant is negative: $3 - 2\sqrt{2} < p < 3 + 2\sqrt{2}$