Functional Relations

Functional Relations

A relationship that describes a property, symmetry, periodicity, or algebraic equivalence of a specific function. Use features such as index and logarithm laws, and the symmetry and periodicity of circular functions. If possible solutions are provided, substitute them into the functional equation and see if the equation holds.

Example

Show that $f(x) = 3x$ satisfies the function relation $f(x+y) = f(x) + f(y)$
$\text{LHS} = f(x+y) = 3(x+y) = 3x + 3y = f(x) + f(y) = \text{RHS}$

Example

Show that $f(x) = x^2$ does not satisfy the function relation $f(x+y) = f(x) + f(y)$
$\text{LHS} = f(x+y) = (x+y)^2 = x^2 + 2xy + y^2 = f(x) + f(y) + 2xy \neq \text{RHS}$

Symmetry

Some functions have reflectional or rotational symmetry. Functions that have reflectional symmetry about the y-axis are called even functions, while functions that have rotational symmetry about the original are called odd functions. These can be written as functional relations.

Odd Functions

$f(x) + f(-x) = 0, \quad f(x) = -f(-x)$
$f(x) = x^n, n = 2k+1, k \in \mathbb{Z}$
$f(x) = \sin(x), \quad f(x) = \tan(x)$

Even Functions

$f(x) - f(-x) = 0, \quad f(x) = f(-x)$
$f(x) = x^n, n = 2k, k \in \mathbb{Z}$
$f(x) = \cos(x)$

Circular functions also have other symmetries due to their periodic nature.
These can be written as functional relations.

Rotational Symmetry

$f(\pi - x) = -f(x), \quad f(x) = \cos(x), f(x) = \tan(x)$
$f(x + \pi) = -f(x), \quad f(x) = \sin(x), f(x) = \cos(x)$
$f(2\pi - x) = -f(x), \quad f(x) = \sin(x), f(x) = \tan(x)$

Reflectional Symmetry

$f(\pi - x) = f(x), \quad f(x) = \sin(x)$
$f(2\pi - x) = f(x), \quad f(x) = \cos(x)$

Periodicity

Circular functions are periodic so they repeat values. Sine and cosine have a period of 2π while tangent has a period of π. These can be written as functional relations:

$f(x + 2\pi) = f(x), \quad f(x) = \sin(x), f(x) = \cos(x) \qquad f(x+\pi) = f(x), \quad f(x) = \tan(x)$

Exponent and Logarithm Laws

The exponent and logarithm laws apply generally so we can write them as functional relations. Some apply with the same index others apply with the same base.

Exponent Law

$f(x+y) = f(x) \times f(y), \quad f(x) = e^{kx}$

$f(x-y) = \dfrac{f(x)}{f(y)}, \quad f(x) = e^{kx}$

$\left(f(x)\right)^n = f(nx), \quad f(x) = e^{kx}$

$f(xy) = f(x) \times f(y), \quad f(x) = x^n$

$f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}, \quad f(x) = x^n$

Logarithm Law

$f(xy) = f(x) + f(y), \quad f(x) = k\log_e(x)$

$f\left(\dfrac{x}{y}\right) = f(x) - f(y), \quad f(x) = k\log_e(x)$

$f(x^n) = nf(x), \quad f(x) = \log_e(kx)$