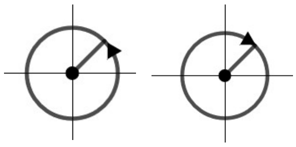
Periodicity and Symmetry

Periodicity of Circular Functions

Coterminal angles point to the same position on the unit circle, so the values of sine, cosine, and tangent of coterminal angles are the same. Since coterminal angles occur periodically (they are all 2π apart), we say the circular functions are periodic with a period of 2π.



$\sin(\theta + 2\pi) = \sin(\theta)$ $\cos(\theta + 2\pi) = \cos(\theta)$ $\tan(\theta + 2\pi) = \tan(\theta)$

$\sin(\theta + 4\pi) = \sin(\theta)$ $\cos(\theta + 4\pi) = \cos(\theta)$ $\tan(\theta + 4\pi) = \tan(\theta)$

$\sin(\theta - 2\pi) = \sin(\theta)$ $\cos(\theta - 2\pi) = \cos(\theta)$ $\tan(\theta - 2\pi) = \tan(\theta)$

Example

$\cos(3\pi) = \cos(3\pi - 2\pi) = \cos(\pi)$, $\quad \cos(-\pi) = \cos(-\pi + 2\pi) = \cos(\pi)$

Example

$\cos(4\pi) = \cos(4\pi - 2\pi) = \cos(2\pi)$, $\quad \cos(4\pi) = \cos(4\pi - 4\pi) = \cos(0)$

Example

$$\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{7\pi}{3} - 2\pi\right) = \sin\left(\frac{\pi}{3}\right), \quad \sin\left(-\frac{5\pi}{3}\right) = \sin\left(-\frac{5\pi}{3} + 2\pi\right) = \sin\left(\frac{\pi}{3}\right)$$

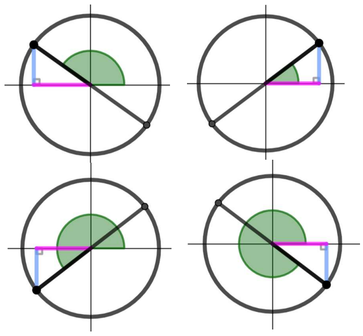
Example

$$\sin\left(\frac{23\pi}{6}\right) = \sin\left(\frac{23\pi}{6} - 2\pi\right) = \sin\left(\frac{11\pi}{6}\right), \quad \sin\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6} + 2\pi\right) = \sin\left(\frac{11\pi}{6}\right)$$

Tangent's True Period

The values of tangent repeat more periodically than 2π. Since tangent is the gradient of the radius, wherever the gradients of the radius are the same, the values of the tangent will be the same.

Extending the radius to the diameter we can see that the gradients are the same in quadrants 1 and 3 as well as quadrants 2 and 4, that is where the angles are a straight angle apart. So tangent values repeats every π.



$\tan(\theta + \pi) = \tan(\theta)$ $\tan(\theta + 2\pi) = \tan(\theta)$ $\tan(\theta + 3\pi) = \tan(\theta)$ $\tan(\theta - \pi) = \tan(\theta)$

Example

$$\tan\left(\frac{7\pi}{6}\right) = \tan\left(\frac{7\pi}{6} - \pi\right) = \tan\left(\frac{\pi}{6}\right)$$

Example

$$\tan\left(\frac{7\pi}{3}\right) = \tan\left(\frac{7\pi}{3} - 2\pi\right) = \tan\left(\frac{\pi}{3}\right)$$

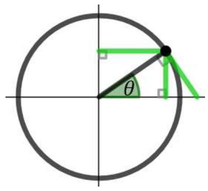
Example

$$\tan\left(\frac{23\pi}{6}\right) = \tan\left(\frac{23\pi}{6} - \pi\right) = \tan\left(\frac{17\pi}{6}\right), \quad \tan\left(\frac{23\pi}{6}\right) = \tan\left(\frac{23\pi}{6} - 2\pi\right) = \tan\left(\frac{11\pi}{6}\right)$$

$$\tan\left(\frac{23\pi}{6}\right) = \tan\left(\frac{23\pi}{6} - 3\pi\right) = \tan\left(\frac{5\pi}{6}\right), \quad \tan\left(\frac{23\pi}{6}\right) = \tan\left(\frac{23\pi}{6} - 4\pi\right) = \tan\left(-\frac{\pi}{6}\right)$$

Symmetry of Circular Functions

Quadrant 1

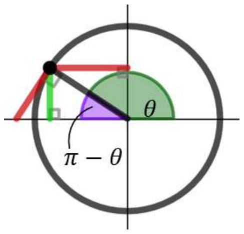


Not only can we relate the sine, cosine, and tangent values of coterminal angles to their counterpart with in the first revolution, 0 to 2π, we can relate the sine, cosine, and tangent values of obtuse and reflex angles in quadrants 2, 3, and 4 with acute angles in quadrant 1 by considering a right-angled triangle formed in the quadrant and the sign of sine, cosine, and tangent in each quadrant. That is, the right-angled triangle in any quadrant is symmetric to one in quadrant 1.

The symmetric relationships formed hold for all angles, not just angles in that quadrant.

Supplement of Angles

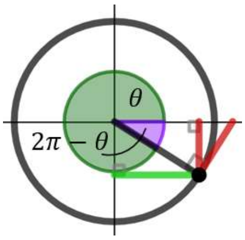
Obtuse angles are supplementary (add to π) to acute angles.



$$\sin(\theta) = +\sin(\pi - \theta)$$
$$\cos(\theta) = -\cos(\pi - \theta)$$
$$\tan(\theta) = -\tan(\pi - \theta)$$

Explement of Angles

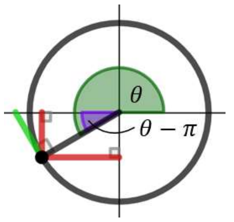
Reflex angles greater than a 3/4 turn are explementary (add to 2π) to acute angles. The acute explement is coterminal with the negative of the reflex angle, $-\theta$.



$$\sin(\theta) = -\sin(2\pi - \theta) = -\sin(-\theta)$$
$$\cos(\theta) = +\cos(2\pi - \theta) = +\cos(-\theta)$$
$$\tan(\theta) = -\tan(2\pi - \theta) = -\tan(-\theta)$$

Supplement of Explement of Angles

Reflex angles less than a 3/4 turn are explementary to obtuse angles which are supplementary to acute angles: $\pi - (2\pi - \theta) = \theta - \pi$, which is coterminal with $\pi + \theta$



$$\sin(\theta) = -\sin(\theta - \pi) = -\sin(\pi + \theta)$$
$$\cos(\theta) = -\cos(\theta - \pi) = -\cos(\pi + \theta)$$
$$\tan(\theta) = +\tan(\theta - \pi) = +\tan(\pi + \theta)$$

Example

$$\pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$
$$\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right)$$
$$\tan\left(\frac{2\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right)$$

Example

$$2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

$$\sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$$
$$\cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$$
$$\tan\left(\frac{5\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right)$$

Example

$$\frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

$$\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$$
$$\cos\left(\frac{4\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right)$$
$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right)$$

Example

$$\pi - \left(-\frac{7\pi}{6}\right) = \frac{13\pi}{6} \Rightarrow \frac{13\pi}{6} - 2\pi = \frac{\pi}{6}, \qquad \tan\left(-\frac{7\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right)$$

Example

$$\pi + \left(-\frac{2\pi}{3}\right) = \frac{\pi}{3}, \qquad \cos\left(-\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right)$$

Example

$$-\frac{\pi}{4} \Rightarrow 2\pi - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$$
$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right)$$

Negative Angles and Function Symmetry

Negative angle symmetry lets us describe the circular functions in terms of odd and even functions.

Odd Function

$$\sin(-\theta) = -\sin(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$

Even Function

$$\cos(-\theta) = \cos(\theta)$$