Logarithms

Logarithm

A logarithm is a quantity representing the exponent to which a fixed number (the base) must be raised to produce a given power. That is, the logarithm is the exponent of a power (exponential) with a given base.

index / exponent / logarithm

$$2^4 = 16$$

base power

The notation for a logarithm is read similar to that for sine, cosine, and tangent of an angle.
That is, $\log_2(16)$ is read as "the logarithm, base 2, of 16".

index / exponent / logarithm

$$\log_2(16) = 4$$

base power

While calling logarithms exponents would be convenient, the concept of a logarithm was derived 150 years before the direct connection between logarithms and exponents was identified and so the name stuck.

Evaluating Logarithms

To evaluate the logarithm $\log_a(y)$ where $y = a^x$, ask yourself "what power of the base, a, is y?" The answer is the xth power: therefore the logarithm (exponent) is x. That is, $\log_a(y) = x$, where $y = a^x$.

Example
$\log_3(9)$: What power of 3 is 9?
The second power of 3, $3^2 = 9$.
Therefore, $\log_3(9) = 2$.

Example
$\log_3\left(\frac{1}{9}\right)$: What power of 3 is $\frac{1}{9}$?
The negative second power of 3, $3^{-2} = \frac{1}{9}$.
Therefore, $\log_3\left(\frac{1}{9}\right) = -2$.

Example
$\log_9(3)$: What power of 9 is 3?
The half power of 9, $9^{\frac{1}{2}} = 3$.
Therefore, $\log_9(3) = \frac{1}{2}$.

Example
$\log_2(1)$: What power of 2 is 1?
The zeroth power of 2, $2^0 = 1$.
Therefore, $\log_2(1) = 0$.

Example
$\log_2(0)$: What power of 2 is 0?
No power of 2 is 0.
Therefore, $\log_2(0)$ is undefined.

Example
$\log_2(-8)$: What power of 2 is -8?
All real powers of 2 are positive.
Therefore, $\log_2(-8)$ is not a real number.

Estimating the Value of Logarithms

Most logarithms are not rational numbers. Therefore, it is beneficial to be able to approximate the value of a logarithm by considering powers of the base that are nearby.

Example
Between what two rational numbers is $\log_2(10)$:
Between what powers of 2 is 10?
$2^3 = 8$ and $2^4 = 16$
So, $3 < \log_2(10) < 4$

Example
Between what two rational numbers is $\log_3(50)$:
Between what powers of 3 is 50?
$3^3 = 27$ and $3^4 = 81$
So, $3 < \log_3(50) < 4$

Example
Between what rational numbers is $\log_2(50)$:
Between what powers of 2 is 50?
$2^5 = 32$ and $2^6 = 64$
So, $5 < \log_2(50) < 6$

Example
Between what two rational numbers is $\log_2\left(\frac{1}{10}\right)$:
Between what powers of 2 is $\frac{1}{10}$?
$2^{-4} = \frac{1}{16}$ and $2^{-3} = \frac{1}{8}$
So, $-4 < \log_2\left(\frac{1}{10}\right) < -3$

Example
Between what two rational numbers is $\log_9(5)$:
Between what powers of 9 is 5?
$9^{\frac{1}{2}} = 3$ and $9^1 = 9$
So, $\frac{1}{2} < \log_9(5) < 1$

Example
Between what two rational numbers is $\log_8\left(\frac{1}{3}\right)$:
Between what powers of 8 is $\frac{1}{3}$?
$8^{-\frac{1}{3}} = \frac{1}{2}$ and $8^{-1} = \frac{1}{8}$
So, $-1 < \log_8\left(\frac{1}{3}\right) < -\frac{1}{3}$

Applying Exponentials as Operations

Usually with exponentials we apply an exponent to a number to obtain its powers, such as $2 = 2 \Rightarrow 2^3 = 8$.

However, we can also make the number we start with the exponent by introducing a base.

Example
$3 + 4 = 7$
two to the power of both sides
$\Rightarrow 2^{3+4} = 2^7$

Example
$7 - 4 = 3$
two to the power of both sides
$\Rightarrow 2^{7-4} = 2^3$

Example
$x = 2$
three to the power of both sides
$\Rightarrow 3^x = 9$

Applying Logarithms as Operations

We can apply a logarithm to a number to find out, for a given base, what exponent gives it as a power.

Example
$8 = 8$
logarithm base 2 of both sides
$\Rightarrow \log_2(8) = 3$

Example
$40 = 40$
logarithm base 2 of both sides
$\Rightarrow \log_2(40) = 5.3219\ldots$

Example
$x = 4$
logarithm base 2 of both sides
$\Rightarrow \log_2(x) = 2$

Exponentials and Logarithms as Inverses

The exponential finds the power given an exponent, the logarithm finds the exponent given a power.
That is, exponentials, a^x, and logarithms, $\log_a(y)$, are inverse operations in the same way that addition and subtraction, multiplication and division, and powers and roots are.

Logarithm of an Exponential: $\log_a(a^x) = x$

The logarithm asks, "what is the exponent with this base whose power is this number?".
If we apply a logarithm to an exponential with the same base, you get the exponent.
For example, $\log_2(2^4)$ asks $2^\square = 2^4$, $\square = 4 = \log_2(2^4)$

Example
$2^3 = 8$
logarithm base 2 of both sides
$\Rightarrow \log_2(2^3) = \log_2(8)$
$\Rightarrow 3 = \log_2(8)$

Example
$3^4 = 81$
logarithm base 3 of both sides
$\Rightarrow \log_3(3^4) = \log_3(81)$
$\Rightarrow 4 = \log_3(81)$

Example
$2^{5.3219\ldots} = 40$
logarithm base 2 of both sides
$\Rightarrow \log_2(2^{5.3219\ldots}) = \log_2(40)$
$\Rightarrow 5.3219\ldots = \log_2(40)$

Exponential of a Logarithm: $a^{\log_a(x)} = x$

The exponential asks, "what is the power of this base whose exponent this number?".
If we exponentiate a logarithm using the same base you get the power.
For example, $2^{\log_2(16)}$ asks $\log_2(\square) = \log_2(16)$, $\square = 16 = 2^{\log_2(16)}$

Example
$\log_3(81) = 4$
three to the power of both sides
$\Rightarrow 3^{\log_3(81)} = 3^4$
$\Rightarrow 81 = 3^4$

Example
$\log_6(12) + \log_6(18) = 3$
six to the power of both sides
$\Rightarrow 6^{\log_6(12)+\log_6(18)} = 6^3$
$\Rightarrow 6^{\log_6(12)} \times 6^{\log_6(18)} = 216$
$\Rightarrow 12 \times 18 = 216$

Example
$5\log_6(36) = 10$
six to the power of both sides
$\Rightarrow 6^{5\log_6(36)} = 6^{10}$
$\Rightarrow \left(6^{\log_6(36)}\right)^5 = 6^{10}$
$\Rightarrow (6^2)^5 = 6^{10}$