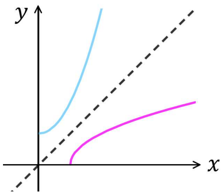
Inverse Relations and Functions

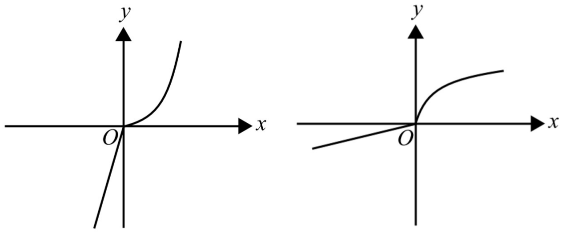
Relations and their Inverses



The inverse of a relation is found by interchanging the coordinates of each ordered pair in the relation. By interchanging the coordinates, we cause the graph of a relation to reflect in the line $y = x$. That is, the graph of a relation and its inverse are reflections in the line $y = x$.

Example VCAA 2002 Exam 1 Question 7

The graph of the function g with rule $y = g(x)$ is shown below. The graph of the inverse function of g is also shown. (A one-to-one scale has been used on both graphs.)



Features of Graphs and their Inverses

Original	(x, y)	Domain	Horizontal Asymptote	$x > 0$	$x < 0$	Above $y = x$	x-intercept
Inverse	(y, x)	Range	Vertical Asymptote	$y > 0$	$y < 0$	Below $y = x$	y-intercept

Inverse Functions

An inverse function, is a function that maps values in the range back to their value in the domain. All relations and functions have inverses but they are not all functions (but they are all relations). If the inverse of a relation is a function, then the relation has an inverse function.

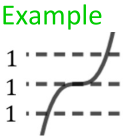
If the inverse of a function is also a function, then it is called a one-to-one function as every x-value is connected to only one y-value and every y-value is connected to only one x-value.

Horizontal Line Test

Tests how many x-values are have the same y-value by sliding a horiztonal line across the entire graph and checking the number of intersections the graph has with the horizontal line at every y-value.

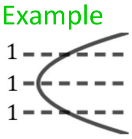
A graph has an inverse function if for all values of y, the horizontal line crosses the graph at most once. If any one horizontal line crosses the graph more than once, it does not have an inverse function.

Example



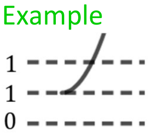
Function has an inverse function

Example



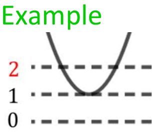
Not a function but has an inverse function

Example



Function that has an inverse function

Example



Function that has no inverse function

Example

Function that has no inverse function

Notation for Inverse Functions

For $f: d_f \to \mathbb{R}$, where $f(x) =$ a function of x
$f^{-1}: d_{f^{-1}} \to \mathbb{R}$, where $f^{-1}(x) =$ the inverse function of x
Be careful that you don't confuse the inverse f^{-1} with the derivative f'.

Domain of the Inverse Function

The domain of the original function will become the range of the inverse function as x and y are swapped. Likewise, the range of the original function will becomes the domain of the inverse function.

Example VCAA 2017 NHT Exam 1 Question 8a

The rule for a function f is given by $f(x) = \sqrt{2x + 3} - 1$, where f is defined on its maximal domain.
The range of f is $y \geq -1$, so the maximal domain of f^{-1} is $x \geq -1$

Determining the Rule of the Inverse Function

To determine the rule of the inverse function we swap the x and y in the rule of the function. However, it is important that you do not proceed directly from $y = f(x)$ to $x = f(y)$. This is not correct working.
You need to indicate that new working is starting. This can be as simple as stating that "for inverse, swap x and y" and then "make y the subject". Then make the connection that "y is the inverse of $f(x)$".

Function	Inverse
$f(x) = e^x$	$f^{-1}(x) = \log_e(x)$
$f(x) = \sin(x)$	$f^{-1}(x) = \sin^{-1}(x)$
$f(x) = \cos(x)$	$f^{-1}(x) = \cos^{-1}(x)$
$f(x) = \tan(x)$	$f^{-1}(x) = \tan^{-1}(x)$
$f(x) = x^n$	$f^{-1}(x) = \sqrt[n]{x}$

Do not write $f^{-1}(x) = x = f(y)$, this is not correct notation.

If the function does not contain y, you must introduce it first before swapping x and y.

Example VCAA 2017 NHT Exam 1 Question 8a

The rule for a function f is given by $f(x) = \sqrt{2x+3} - 1$, where f is defined on its maximal domain. Find the rule of the inverse function.

Let $y = \sqrt{2x+3} - 1$
For inverse swap x and y
$x = \sqrt{2y+3} - 1$

Make y the subject
$y = \frac{1}{2}(x+1)^2 - \frac{3}{2}$

y is the inverse of $f(x)$
$\therefore f^{-1}(x) = \frac{1}{2}(x+1)^2 - \frac{3}{2}$

Example VCAA 2006 Sample Exam 1 Question 1a

For the function $f: (-1, \infty) \to R, f(x) = 2\log_e(x+1)$, the rule of the inverse function f^{-1} is

Let $y = 2\log_e(x+1)$
For inverse swap x and y
$x = 2\log_e(y+1)$

Make y the subject
$y = e^{\frac{x}{2}} - 1$

y is the inverse of $f(x)$.
$\therefore f^{-1}(x) = e^{\frac{x}{2}} - 1$

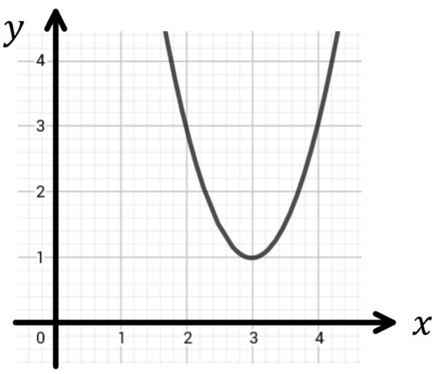
Restricting the Domain

If a graph does not pass the horizontal line test, we can restrict the domain so that is becomes one-to-one so that its inverse will be a function.

For example, $y = x^2$ does not pass the vertical line test so its inverse relation $x = y^2$ is not a function.
If we restrict the domain of $y = x^2$ to $x \geq 0$ then it will have the inverse function $y = \sqrt{x}$.
Likewise, if we restrict the domain of $y = x^2$ to $x \leq 0$ then it will have the inverse function $y = -\sqrt{x}$.

Example VCAA 2005 Exam 1 Question 7

The function $f: [a, \infty) \to R$ with rule $f(x) = 2(x-3)^2 + 1$ will have an inverse function if f is a one-to-one function.



Since f is a parabola, it will be a one-to-one function for a domain left or right of the turning point. Since the domain is written as $[a, \infty)$ the right side is needed.

The x-coordinate of the turning point is $x = 3$.
Therefore, $a \geq 3$ as any value greater than 3 will also restrict the domain so that f is one-to-one.

Inverse Function vs Rule for the Inverse

The inverse function, f^{-1}, requires the rule and the domain to be specified. When the inverse function is asked for, the domain must be given. If only the rule for the inverse function is asked for, the domain does not have to be given. The definition of f should provide a clue to the form of f^{-1}.

Example VCAA 2009 Exam 1 Question 3

Let $f: R \setminus \{0\} \to R$ where $f(x) = \frac{3}{x} - 4$. The inverse function of f, f^{-1}, is

Let $y = \frac{3}{x} - 4$. For inverse swap x and y $\Rightarrow$ $x = \frac{3}{y} - 4$.

Make y the subject $y = \frac{x-4}{3}$. y is the inverse of $f(x)$. $\therefore f^{-1}(x) = \frac{x-4}{3}$.

The range of f is $R \setminus \{-4\}$. The domain of f^{-1} is the range of f. $\therefore d_{f^{-1}} = R \setminus \{-4\}$

The inverse function of f is $f^{-1}: R \setminus \{-4\} \to R$ where $f^{-1}(x) = \frac{x-4}{3}$.

Intersections of Graphs and their Inverses

The original graph and its inverse will intersect on the line $y = x$. These intersections can be found by solving any one of the following equations: $f(x) = f^{-1}(x)$, $f(x) = x$, $f^{-1}(x) = x$

Example VCAA 2017 NHT Exam 1 Question 8b

The rule for a function f is given by $f(x) = \sqrt{2x+3} - 1$, where f is defined on its maximal domain.

The domain and rule of the inverse function are $x \geq -1$ and $f^{-1}(x) = \frac{1}{2}(x+1)^2 - \frac{3}{2}$

Solve $f(x) = f^{-1}(x)$.

The solutions of $f(x) = f^{-1}(x)$ are also solutions of $f(x) = x$ or $f^{-1}(x) = x$

$\sqrt{2x+3} - 1 = \frac{1}{2}(x+1)^2 - \frac{3}{2}$

This will result in a quartic equation, better to avoid this equation as this is a CAS free exam.

$\sqrt{2x+3} - 1 = x$

$\Rightarrow \sqrt{2x+3} = x + 1$

$\Rightarrow 2x + 3 = (x+1)^2$

$\Rightarrow 2x + 3 = x^2 + 2x + 1$

$\Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}, x \geq -\frac{3}{2}$

$\frac{1}{2}(x+1)^2 - \frac{3}{2} = x$

$\Rightarrow x^2 + 2x + 1 - 3 = 2x$

$\Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}, x \geq -\frac{3}{2}$

Composition of Inverse Functions

If you compose a function and its inverse it will always equal x, just check the new domain

$f\left(f^{-1}(x)\right) = x, \quad x \in d_{f^{-1}}$ $\qquad$ $f^{-1}\left(f(x)\right) = x, \quad x \in d_f$

Example Modified VCAA 2008 Exam 1 Question 10

Let $f: R \to R, f(x) = e^{2x} - 1$.

The inverse function of f is f^{-1} is $f^{-1}: (-1, \infty) \to R, f^{-1}(x) = \frac{1}{2}\log_e(x+1)$.

Find $f^{-1}\left(f(x)\right)$ and state its domain.

$f^{-1}(e^{2x} - 1) = \frac{1}{2}\log_e(e^{2x} - 1 + 1) = \frac{1}{2}\log_e(e^{2x}) = \frac{1}{2}(2x) = x$

The range of the inside function is equal to the domain of the outside function: $(-1, \infty) \subseteq (-1, \infty)$.

Therefore, the domain of the composite function is the domain inside function: $x \in R$.

Find $f\left(f^{-1}(x)\right)$ and state its domain.

$f\left(\frac{1}{2}\log_e(x+1)\right) = e^{2\left(\frac{1}{2}\log_e(x+1)\right)} - 1 = e^{\log_e(x+1)} - 1 = x + 1 - 1 = x$

The range of the inside function is equal to the domain of the outside function: $R \subseteq R$.

Therefore, the domain of the composite function is the domain inside function: $x \in (-1, \infty)$.