Logarithmic Equations

Equations Involving Logarithms

To solve a logarithmic equation, we can attempt to write both sides of the equation as a logarithmic expression with the same base to then equate the arguments. Alternatively, and more consistently, we can take the exponential with the same base of both sides which, as the inverse of the logarithm, leaves the argument and writes the other side as a power of the base which may be simplified.

Convert Constant to Log: $b = b\log_a(a) = \log_a(a^b)$

$\log_a(x) = b = b\log_a(a)$
$\log_a(x) = \log_a(a^b)$
Equal logs with equal bases have equal powers
$x = a^b$

Inverse: $a^{\log_a(x)} = x$

$\log_a(x) = b$
a to the power of both sides
$a^{\log_a(x)} = a^b$
$x = a^b$

Example

Solve the equation $2\log_5(x) = 10$ for x.

$2\log_5(x) = 8$
$\log_5(x) = 4 = 4\log_5(5)$
$\log_5(x) = \log_5(5^4)$
Equal logs with equal bases have equal powers
$x = 5^4 = 625$

$2\log_5(x) = 8$
$\log_5(x) = 4$
5 to the power of both sides
$5^{\log_5(x)} = 5^4$
$x = 5^4 = 625$

$2\log_5(x) = 8$
5 to the power of both sides
$5^{2\log_5(x)} = 5^8$
$\left(5^{\log_5(x)}\right)^2 = 5^8$
$x^2 = 5^8$
$x = 5^4 = 625, \quad x > 0$

Log Laws and Equations

Where possible express the equation using a single logarithmic expression on both sides of the equation using logarithm laws. Remember $x > 0$ for $\log_a(x)$.

Example VCAA 2016 Sample Exam 1 Question 6a / VCAA 2013 Exam 1 Question 5a

Solve the equation $2\log_3(5) - \log_3(2) + \log_3(x) = 2$ for x.

$2\log_3(5) - \log_3(2) + \log_3(x) = 2$
$\log_3(5^2) - \log_3(2) + \log_3(x) = 2\log_3(3)$
$\log_3\left(\frac{25x}{2}\right) = \log_3(9)$
Equal logs with equal bases have equal powers

$\frac{25x}{2} = 9$

$x = \frac{18}{25}$

$2\log_3(5) - \log_3(2) + \log_3(x) = 2$
$\log_3(5^2) - \log_3(2) + \log_3(x) = 2$
$\log_3\left(\frac{25x}{2}\right) = 2$
3 to the power of both sides

$3^{\log_3\left(\frac{25x}{2}\right)} = 3^2$

$\frac{25x}{2} = 9 \Rightarrow x = \frac{18}{25}$

Example VCAA 2009 Exam 1 Question 9

Solve the equation $2\log_e(x) - \log_e(x+3) = \log_e\left(\frac{1}{2}\right)$ for x.

$2\log_e(x) - \log_e(x+3) = \log_e(2^{-1})$
$\Rightarrow \log_e(x^2) - \log_e(x+3) = \log_e(2^{-1})$
$\Rightarrow \log_e(x^2/(x+3)) = \log_e(2^{-1})$
$\Rightarrow \frac{x^2}{x+3} = \frac{1}{2}$

$\Rightarrow 2x^2 = x + 3$
$\Rightarrow 2x^2 - x - 3 = 0$
$\Rightarrow x = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)} = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4}$

$x = \frac{1+5}{4} = \frac{6}{4} = \frac{3}{2}, \quad x = \frac{1-5}{4} = -\frac{4}{4} = -1$, reject as $x > 0$ for $\log_e(x)$, $\quad \therefore x = \frac{3}{2}$