Trigonometric Equations

Solving Trigonometric Equations

1) Express the equation as $\sin(ax+b)=c,\ \cos(ax+b)=c,\ \tan(ax+b)=c$.
2) Determine whether to use a special triangle or the unit circle to solve the equation.

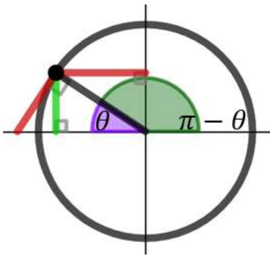
For sine or cosine equations,
- use the unit circle if it is equal to $0, 1, -1$,
- use the half-equilateral triangle if it is $\frac{1}{2}, \frac{\sqrt{3}}{2}$,
- use the half-square triangle if it is $\frac{1}{\sqrt{2}}$

For tangent equations,
- use the unit circle if it is equal to 0,
- use the half-equilateral triangle if it is $\frac{1}{\sqrt{3}}, \sqrt{3}$,
- use the half-square triangle if it is 1

3) Use
   - the special triangles to determine the acute reference angle, or
   - the unit circle to determine the solutions that are integer multiples of $90°$ or $\frac{\pi}{2}$.
4) Use the graphs or unit circle to determine which quadrants the solutions will lie in.
5) Set up two equations one for each quadrant the solution lies in.
6) Solve each equation for x.
7) Add and subtract the period to find all solutions in the domain.
8) List all valid solutions.

Quadrant 2 (Supplement)

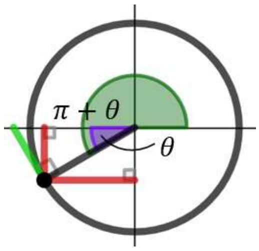
$\sin(\theta)$ +ve, $\cos(\theta)$ −ve, $\tan(\theta)$ −ve



$\pi-\theta,\quad 3\pi-\theta$
$-\pi+\theta,\quad -3\pi+\theta$

Quadrant 3 (Supplement of the Explement)

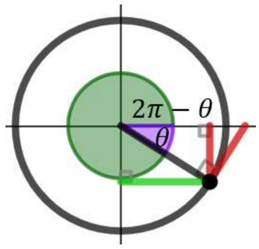
$\sin(\theta)$ −ve, $\cos(\theta)$ −ve, $\tan(\theta)$ +ve



$\pi+\theta,\quad 3\pi-\theta$
$-\pi-\theta,\quad -3\pi-\theta$

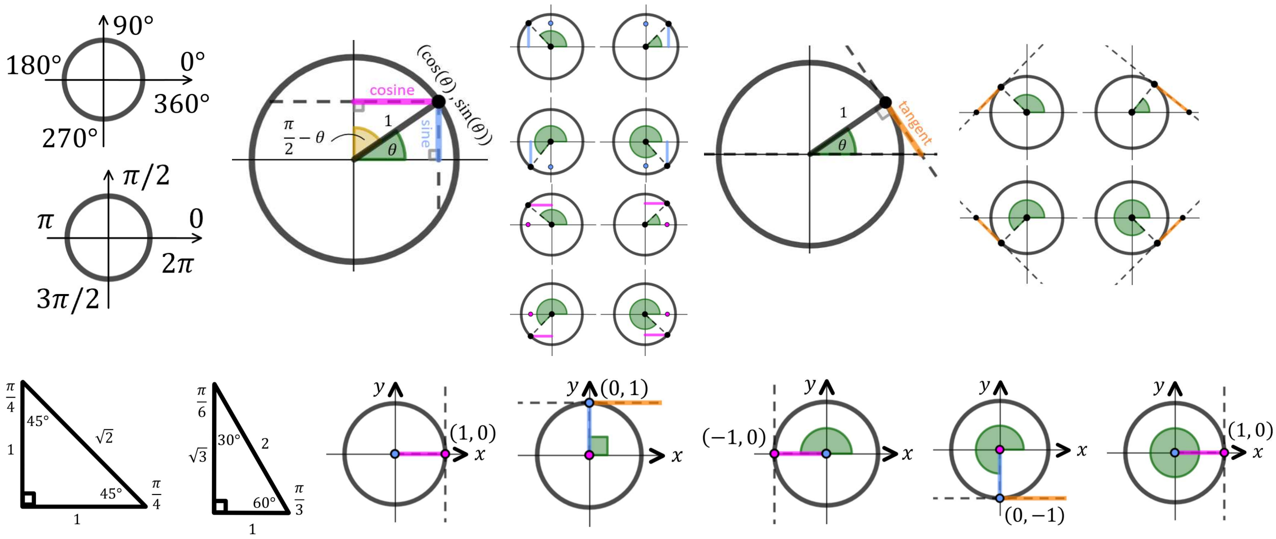
Quadrant 4 (Explement)

$\sin(\theta)$ −ve, $\cos(\theta)$ +ve, $\tan(\theta)$ −ve



$2\pi-\theta,\quad 4\pi-\theta$
$-\theta,\quad -2\pi-\theta$

Special Triangles and The Unit Circle



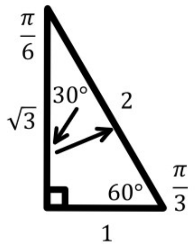
Example VCAA 2014 Exam 1 Question 3

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \le x \le \pi$.

$$\cos(2x) = -\frac{\sqrt{3}}{2}$$

Cosine is negative in the second and third quadrants

$$\Rightarrow 2x = \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right), \qquad 2x = \pi + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$



$$\Rightarrow 2x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}, \qquad 2x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\Rightarrow x = \frac{5\pi}{12}, \qquad x = \frac{7\pi}{12}$$

The period of the solutions is $\frac{2\pi}{2} = \pi$

$$x = \frac{5\pi}{12} - \pi < 0, \qquad x = \frac{7\pi}{12} - \pi < 0$$

$$x = \frac{5\pi}{12} + \pi > \pi, \qquad x = \frac{7\pi}{12} + \pi > \pi$$

$$\therefore x = \frac{5\pi}{12}, \frac{7\pi}{12}$$

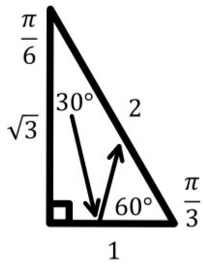
Example VCAA 2011 Exam 1 Question 3b

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, \pi]$.

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$$

Sine is positive in the first and second quadrants

$$\Rightarrow 2x + \frac{\pi}{3} = \sin^{-1}\left(\frac{1}{2}\right), \qquad 2x + \frac{\pi}{3} = \pi - \sin^{-1}\left(\frac{1}{2}\right)$$



$$\Rightarrow 2x + \frac{\pi}{3} = \frac{\pi}{6}, \qquad 2x + \frac{\pi}{3} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\Rightarrow 2x = -\frac{\pi}{6}, \qquad 2x = \frac{3\pi}{6} = \frac{\pi}{2}$$

$$\Rightarrow x = -\frac{\pi}{12}, \qquad x = \frac{\pi}{4}$$

The period of the solutions is $\frac{2\pi}{2} = \pi$

$$x = -\frac{\pi}{12} + \pi = \frac{11\pi}{12}, \qquad x = \frac{\pi}{4} + \pi > \pi$$

$$x = \frac{\pi}{4} - \pi < 0$$

$$\therefore x = \frac{\pi}{4}, \frac{11\pi}{12}$$

Example VCAA 2017 NHT Exam 1 Question 3b

Solve $2\sin^2(x) + 3\sin(x) - 2 = 0$, where $0 \le x \le 2\pi$.

$$2\sin^2(x) + 3\sin(x) - 2 = 0$$

Let $a = \sin(x)$, $2a^2 + 3a - 2 = 0$

$$a = \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm \sqrt{9 + 16}}{4} = \frac{-3 \pm 5}{4}, \qquad a = \frac{-3+5}{4} = \frac{2}{4} = \frac{1}{2}, a = \frac{-3-5}{4} = -\frac{8}{4} = -2$$

$\sin(x) = \frac{1}{2}$, $\qquad \sin(x) = -2 \Rightarrow$ has no real solutions

Sine is positive in the first and second quadrants

$$x = \sin^{-1}\left(\frac{1}{2}\right), \qquad x = \pi - \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{6}, \qquad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Equations Involving Sine and Cosine

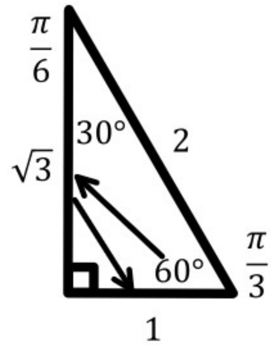
Equations that equation sine and cosine can be solved by dividing both sides of the equation by the cosine and using the property that $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$.

Example VCAA 2000 Exam 1 Question 5

Find the exact solutions of the equation $\sin(2x) = 3\cos(2x)$, $-\pi \leq x \leq \pi$.

$\sin(2x) = 3\cos(2x)$

$\Rightarrow \frac{\sin(2x)}{\cos(2x)} = \tan(2x) = \sqrt{3}$



Tangent is positive in the first and third quadrants

$\Rightarrow 2x = \tan^{-1}(\sqrt{3})$, or, $2x = \pi + \tan^{-1}(\sqrt{3})$

$\Rightarrow 2x = \frac{\pi}{3}, \quad 2x = \pi + \frac{\pi}{3}$

$\Rightarrow x = \frac{\pi}{6}, \quad x = \frac{2\pi}{3}$

The period of the solutions is $\frac{2\pi}{2} = \pi$

$\Rightarrow x = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}, \quad x = \frac{2\pi}{3} - \pi = -\frac{\pi}{3}$

$\therefore x = -\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

General Solutions of Trigonometric Equations

Since trigonometric equations have an infinite number of solutions, if we want to describe all of the possible solutions, not just those over a specified domain, we need to use a parameter. That is, since the solutions in each quadrant repeat every 2π we can describe the solution in each possible loop of 2π, forwards or backwards, by adding $2n\pi, n \in \mathbb{Z}$ (that is, where n is an integer) to each quadrant's solution. Some of these can be combined with the π's that are in the solution already to condense the solution, or for cosine and tangent reduce to one equation.

Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
$x = \theta + 2n\pi, n \in \mathbb{Z}$	$x = \pi - \theta + 2n\pi, n \in \mathbb{Z}$ $= (2n+1)\pi - \theta, n \in \mathbb{Z}$	$x = \pi + \theta + 2n\pi, n \in \mathbb{Z}$ $= (2n+1)\pi + \theta, n \in \mathbb{Z}$	$x = 2n\pi - \theta, n \in \mathbb{Z}$

Positive Cosine: combine the first and fourth quadrant solutions $x = 2n\pi \pm \cos^{-1}(c), n \in \mathbb{Z}$

Negative Cosine: combine the second and third quadrant solutions $x = (2n+1)\pi \pm \cos^{-1}(c), n \in \mathbb{Z}$

Positive Tangent: combine the first and third quadrant solutions $x = n\pi + \tan^{-1}(c), n \in \mathbb{Z}$

Negative Tangent: combining the second and fourth solutions $x = n\pi - \tan^{-1}(c), n \in \mathbb{Z}$

Sine's solutions cannot be combined into a single equation.

Example VCAA 2014 Exam 1 Question 3

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \le x \le \pi$.

$$\cos(2x) = -\frac{\sqrt{3}}{2}, \qquad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Cosine is negative in the second and third quadrants

$$\Rightarrow 2x = (2n+1)\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}, n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{(2n+1)\pi}{2} \pm \frac{\pi}{12}, n \in \mathbb{Z}$$

Example VCAA 2011 Exam 1 Question 3b

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, \pi]$.

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}, \qquad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Sine is positive in the first and second quadrants

$$\Rightarrow 2x + \frac{\pi}{3} = \frac{\pi}{6} + 2n\pi, 2x + \frac{\pi}{3} = (2n+1)\pi - \frac{\pi}{6}, n \in \mathbb{Z}$$

$$\Rightarrow 2x = 2n\pi - \frac{\pi}{6}, \qquad 2x = (2n+1)\pi - \frac{\pi}{2}$$

$$\Rightarrow x = n\pi - \frac{\pi}{12}, \qquad x = \frac{(2n+1)\pi}{2} - \frac{\pi}{4}$$

The Sum of Solutions of Trigonometric Equations

For most angles, the sequential solutions of sine and cosine equations alternate adding and subtracting the base angle from a multiple of π. So, the sum of the solutions will be the sum of the multiples of π for sine and the even multiples of π for cosine in the given domain and add (or subtract) the base angle if the domain is not a whole number of cycles.

When sine or cosine is equal to ± 1, there is only one solution per period, not two. So, the sum of solutions will be the sum of the even multiples of π and the number of solutions in the domain $\times\, \pi/2$ or $3\pi/2$.

Example VCAA 2004 Exam 1 Question 8

The sum of the solutions of $\cos\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$, for $0 \le x \le 4\pi$, is, $\qquad 0 \le x \le 4\pi \Rightarrow 0 \le \frac{x}{2} \le 2\pi$

Manually adding solutions

$$\cos\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2} \Rightarrow \text{base angle} = \frac{\pi}{6}$$

$$\frac{x}{2} = \frac{\pi}{6}, \qquad \frac{x}{2} = 2\pi - \frac{\pi}{6}, \qquad \Sigma\left(\frac{x}{2}\right) = \frac{\pi}{6} + 2\pi - \frac{\pi}{6} = 2\pi \Rightarrow \Sigma x = 4\pi$$

Sum of even multiples of π for Cosine

$$\sum\left(\frac{x}{2}\right) = 0 + 2\pi = 2\pi \quad \Rightarrow \quad \Sigma x = 4\pi$$

Example VCAA 2017 Exam 2 Question 12

The sum of the solutions of $\sin(2x) = \dfrac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$. The value of $d \geq 0$ could be

$-\pi \leq x \leq d \Rightarrow -2\pi \leq 2x \leq 2d, \qquad \Sigma x = -\pi \;\; \Rightarrow \;\; \Sigma(2x) = -2\pi$

The sum is a multiple of π so it is likely to have always added and subtracted the base angle in pairs.

$\Sigma(2x) = -2\pi - \pi + 0 + \pi = -2\pi, \qquad \therefore \pi < 2d < 2\pi \;\; \Rightarrow \;\; \dfrac{\pi}{2} < d < \pi, \qquad \text{e.g. } \; d = \dfrac{3\pi}{4}$

Example VCAA 2002 Exam 1 Question 2

For the equation $\sin(2x) = 1$, the sum of the solutions in the interval $[0, 4\pi]$ is

Manually adding solutions

$2x = \dfrac{\pi}{2}, \qquad 2x = 2\pi + \dfrac{\pi}{2}, \qquad 2x = 4\pi + \dfrac{\pi}{2}, \qquad 2x = 6\pi + \dfrac{\pi}{2}, \qquad 0 \leq x \leq 4\pi \Rightarrow 0 \leq 2x \leq 8\pi$

$\Sigma(2x) = \dfrac{\pi}{2} + 2\pi + \dfrac{\pi}{2} + 4\pi + \dfrac{\pi}{2} + 6\pi + \dfrac{\pi}{2} = 12\pi + 2\pi = 14\pi \;\; \Rightarrow \;\; \Sigma x = 7\pi$

Sum of even multiples of π and $n \times \pi/2$

$\Sigma(2x) = 0 + 2\pi + 4\pi + 6\pi + 4 \times \dfrac{\pi}{2} = 12\pi + 2\pi = 14\pi \;\; \Rightarrow \;\; \Sigma x = 7\pi$

The Sum of Solutions for Tangent

For tangent, the solutions always add or always subtract the base angle onto a multiple of π, so the sum will be the sum of the multiples of π and the number of solutions in the domain $\times$ base angle added on (or subtracted off if the tangent is negative). Don't forget 0π is an even multiple of π.

Example VCAA 2018 NHT Exam 2 Question 6

The sum of the solutions to the equation $\sqrt{3}\sin(2x) = -3\cos(2x)$ for $x \in [0, 2\pi]$ is equal to

$\sqrt{3}\sin(2x) = -3\cos(2x) \;\; \Rightarrow \;\; \tan(2x) = -\sqrt{3} \;\; \Rightarrow \;\; \text{base angle} = \dfrac{\pi}{3}, \qquad 0 \leq x \leq 2\pi \Rightarrow 0 \leq 2x \leq 4\pi$

Manually adding solutions

$2x = \pi - \dfrac{\pi}{3}, \qquad 2x = 2\pi - \dfrac{\pi}{3}, \qquad 2x = 3\pi - \dfrac{\pi}{3}, \qquad 2x = 4\pi - \dfrac{\pi}{3}$

$\Sigma(2x) = \pi - \dfrac{\pi}{3} + 2\pi - \dfrac{\pi}{3} + 3\pi - \dfrac{\pi}{3} + 4\pi - \dfrac{\pi}{3} = 10\pi - \dfrac{4\pi}{3} = \dfrac{26\pi}{3} \;\; \Rightarrow \;\; \Sigma x = \dfrac{13\pi}{3}$

Sum of multiples of π

$\Sigma(2x) = \pi + 2\pi + 3\pi + 4\pi - 4 \times \dfrac{\pi}{3} = 10\pi - \dfrac{4\pi}{3} = \dfrac{26\pi}{3} \;\; \Rightarrow \;\; \Sigma x = \dfrac{13\pi}{3}$

Example VCAA 2019 Exam 2 Question 19

Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \dfrac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \dfrac{5\pi}{4}$, in terms of α, is , $\qquad 0 < x < \dfrac{5\pi}{4} \Rightarrow 0 < 2x < \dfrac{5\pi}{2}$

Manually adding solutions

$2x = \alpha, \qquad 2x = \pi + \alpha, \qquad 2x = 2\pi + \alpha$

$\Sigma(2x) = \alpha + \pi + \alpha + 2\pi + \alpha = 3\pi + 3\alpha \Rightarrow \Sigma x = \dfrac{3(\pi + \alpha)}{2}$

Sum of multiples of π and $n \times$ base angle

$\Sigma(2x) = 0 + \pi + 2\pi + 3\alpha = 3\pi + 3\alpha \;\; \Rightarrow \;\; \Sigma x = \dfrac{3(\pi + \alpha)}{2}$