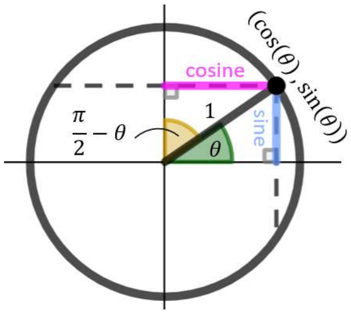
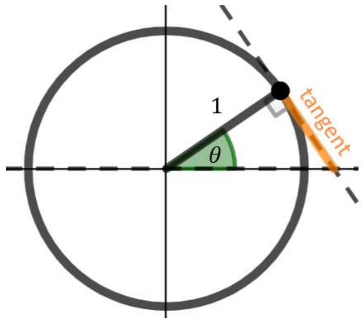
Circular Function Relationships

Defining Sine, Cosine, and Tangent on the Unit Circle

Sine is a mistranslation of the word for 'half-chord' and hence is the vertical height of the triangle.
Cosine is the sine of the complementary angle and hence is the horizontal length of the triangle.
Tangent is the length of the tangent from the point where the tangent is drawn to the x-axis.

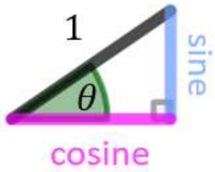




Pythagorean Identity

Since we used a right angle triangle, Pythagoras' Theorem can be applied.
$a^2 + b^2 = c^2 \quad \Rightarrow \quad x^2 + y^2 = 1$

$\Rightarrow \cos^2(\theta) + \sin^2(\theta) = 1$

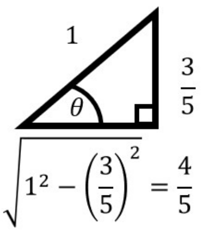


Example

If $\sin(\theta) = \frac{3}{5}$, then

$\cos(\theta) = \pm\sqrt{1 - \left(\frac{3}{5}\right)^2}$

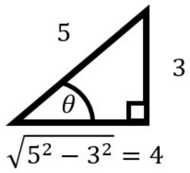
$\cos(\theta) = \pm\frac{4}{5}$



If $\sin(\theta) = \frac{3}{5}$, then

$\cos(\theta) = \pm\frac{\sqrt{5^2 - 3^2}}{5}$

$\cos(\theta) = \pm\frac{4}{5}$



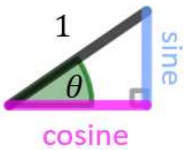
Since $\sin(\theta) > 0$, the angle must be in the first or second quadrant, therefore
$\cos(\theta) = \frac{4}{5}$ when $0 < \theta < \frac{\pi}{2}$, $\quad \cos(\theta) = -\frac{4}{5}$ when $\frac{\pi}{2} < \theta < \pi$.

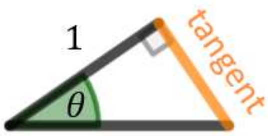
Tangent as the Ratio of Sine and Cosine / Tangent as the Gradient of the Radius

Calculating the length of the tangent can be cumbersome. So we can use an alternative calculation.

Using similar triangles,

$$\frac{\tan(\theta)}{1} = \frac{\sin(\theta)}{\cos(\theta)} \quad \Rightarrow \quad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$





That is, the tangent of an angle is the ratio of the sine of the angle and the cosine of the angle.
Alternatively, we can consider the tangent of the angle as the gradient of the radius.

Example

If $\sin(\theta) = \frac{3}{5}$ and $\cos(\theta) = -\frac{4}{5}$

then $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{3/5}{-4/5} = -\frac{3}{4}$