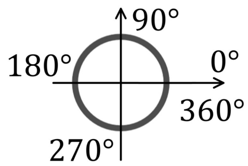
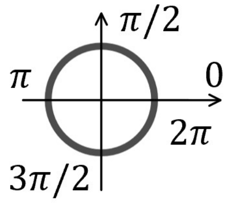
Evaluating Circular Functions with Exact Values

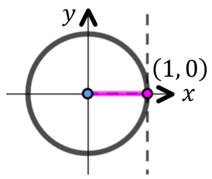
Unit Circle for Exact Values of Sine, Cosine, and Tangent



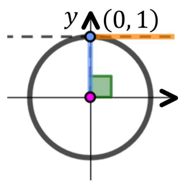


Using the unit circle definitions of sine and cosine, and their quotient for tangent, we can deduce the values of the functions of sine, cosine, and tangent using a circle with radius 1. If we look at the angles $0°, 90°, 180°, 270°$, and, $360°$ or $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π in radians respectively then we can deduce the exact values.

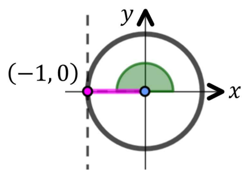
Remember:
The sine of the angle is the y value of a point on the unit circle.
The cosine of the angle is the x value of a point on the unit circle.
The tangent of the angle is the length of the tangent from the point on the circle to the x-axis.



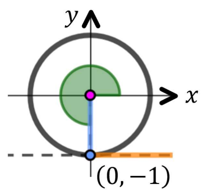
When $\theta = 0°$ / $\theta = 0$
$\sin(0) = 0$ as the y value when $\theta = 0$ is 0.
$\cos(0) = 1$ as the x value when $\theta = 0$ is 1.
$\tan(0) = 0$ as the length of the tangent is 0.



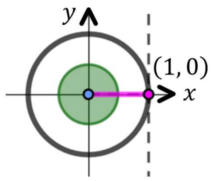
When $\theta = 90°$ / $\theta = \frac{\pi}{2}$
$\sin\left(\frac{\pi}{2}\right) = 1$ as the y value when $\theta = \frac{\pi}{2}$ is 1.
$\cos\left(\frac{\pi}{2}\right) = 0$ as the x value when $\theta = \frac{\pi}{2}$ is 0.
$\tan\left(\frac{\pi}{2}\right)$ is undefined as the tangent at $\frac{\pi}{2}$ never touches the x-axis.



When $\theta = 180°$ / $\theta = \pi$
$\sin(\pi) = 1$ as the y value when $\theta = \pi$ is 0.
$\cos(\pi) = -1$ as the x value when $\theta = \pi$ is -1.
$\tan(\pi) = 0$ as the length of the tangent is 0.



When $\theta = 270°$ / $\theta = \frac{3\pi}{2}$
$\sin\left(\frac{3\pi}{2}\right) = -1$ as the y value when $\theta = \frac{3\pi}{2}$ is -1.
$\cos\left(\frac{3\pi}{2}\right) = 0$ as the x value when $\theta = \frac{3\pi}{2}$ is 0.
$\tan\left(\frac{3\pi}{2}\right)$ is undefined as the tangent at $\frac{3\pi}{2}$ never touches the x-axis.



When $\theta = 360°$ / $\theta = 2\pi$
$\sin(2\pi) = 1$ as the y value when $\theta = \pi$ is 0.
$\cos(2\pi) = 1$ as the x value when $\theta = 2\pi$ is 0.
$\tan(2\pi) = 0$ as the length of the tangent is 0.

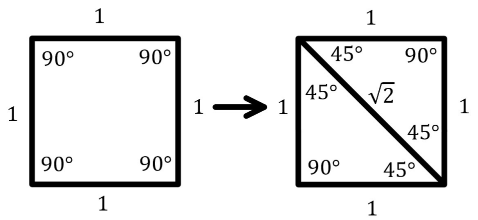
Special Triangles for Exact Values of Sine, Cosine, and Tangent

There are particular right-angled triangles that can be constructed such that all angles and side lengths are known exactly (and simply). This allows us to know the exact value of sine, cosine, and tangent of particular angles. We focus on two in particular.

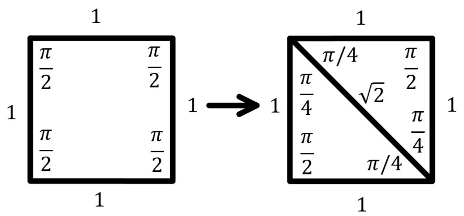
Half a Square / Isosceles Right-Angled Triangle

Start with a square, we could use any side length or a variable but for simplicity we will use 1 unit. We will cut the square in half along a diagonal to form two isosceles right-angled triangles. Using Pythagoras' Theorem, the diagonal length is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

In degrees, a square has a total interior angle of $360°$. Each angle must be equal, so $360 \div 4 = 90°$ in each corner. Cutting the square in half also halves two of the angles, so $90 \div 2 = 45°$.

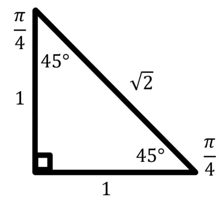


In radians, a square has a total interior angle of 2π. Each angle must be equal, so $2\pi \div 4 = \frac{\pi}{2}$ in each corner. Cutting the square in half also halves two of the angles, so $\frac{\pi}{2} \div 2 = \frac{\pi}{4}$.



Using the ratios for sine, cosine, and tangent to determine their values for $45°$ / $\frac{\pi}{4}$:

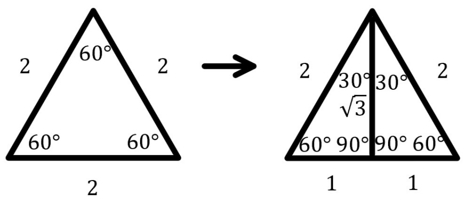
$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \tan\left(\frac{\pi}{4}\right) = \frac{1}{1} = 1$$



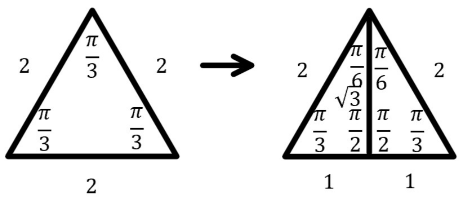
Half an Equilateral Triangle / 30-60-90 Triangle

Start with an equilateral triangle, we could use any side length or a variable but for simplicity we will use 2 units. We will cut the equilateral triangle in half with the perpendicular height to form two right-angled triangles. Using Pythagoras' Theorem, the perpendicular height is $\sqrt{2^2 - 1^2} = \sqrt{3}$.

In degrees, an equilateral triangle has a total interior angle of $180°$.
Each angle must be equal, so $180 \div 3 = 60°$ in each corner. Cutting the triangle in half also halves one of the angles, so $60 \div 2 = 30°$ and adds a right angle, $90°$.



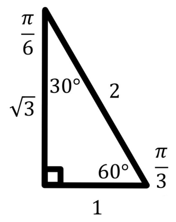
In degrees, an equilateral triangle has a total interior angle of $180°$.
Each angle must be equal, so $\pi \div 3 = \frac{\pi}{3}$ in each corner. Cutting the triangle in half also halves one of the angles, so $\frac{\pi}{3} \div 2 = \frac{\pi}{6}$ and adds a right angle, $\frac{\pi}{2}$.



Using the ratios for sine, cosine, and tangent to determine their values for $30°$ / $\frac{\pi}{6}$ and $60°$ / $\frac{\pi}{3}$:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$$



Evaluating Circular Functions

Since a right-angled triangle can be formed in any quadrant, we can evaluate the sine, cosine, or tangent value of any angle provided we construct a useful right-angled triangle in the relevant quadrant.

1. Determine which quadrant the angle lies in.
   Remember for angles: positive goes anti-clockwise, negative goes clockwise.
   a. If the angle is co-terminal to the positive or negative, x- or y-axis, read off the x or y-coordinate of the intercept or consider the tangent.
2. Determine the sign of the circular function in that quadrant.
   Remember: cosine of the angle is the x-value, sine of the angle is the y-value, and the tangent is the length of the tangent to the x-axis and its sign matches the gradient of the radius.
3. Find the acute angle in the quadrant between the radius and the x-axis.
4. Evaluate the circular function of the acute angle. Attach the appropriate sign to the value.

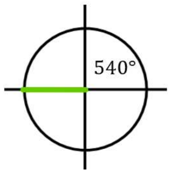
Example

$\cos(540°)$

$540°$ is coterminal with the negative x-axis.

Cosine (x) is the x-coordinate here.

$\cos(540°) = -1$



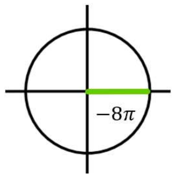
Example

$\tan(-8\pi)$

-8π is coterminal with the positive x-axis.

The length of the tangent here is 0.

$\tan(-8\pi) = 0$



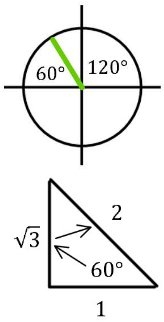
Example

$\sin(120°)$

$120°$ is in the second quadrant.

Sine (y) is positive in the second quadrant. The acute angle between the radius and the x-axis is $60°$

$$\sin(120°) = +\sin(60°) = \frac{\sqrt{3}}{2}$$



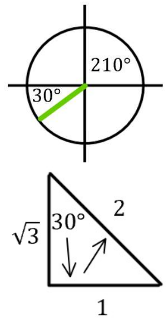
Example

$\sin(210°)$

$210°$ is in the third quadrant.

Sine (y) is negative in the third quadrant. The acute angle between the radius and the x-axis is $30°$

$$\sin(210°) = -\sin(30°) = -\frac{1}{2}$$



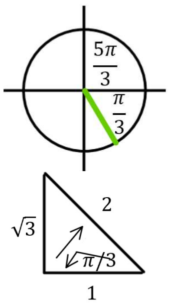
Example

$\cos\left(\frac{5\pi}{3}\right)$

$\frac{5\pi}{3}$ is in the fourth quadrant.

Cosine (x) is positive in the fourth quadrant. The acute angle between the radius and the x-axis is $\frac{\pi}{3}$

$$\cos\left(\frac{5\pi}{3}\right) = +\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$



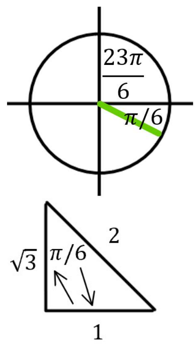
Example

$\tan\left(\frac{23\pi}{6}\right)$

$\frac{23\pi}{6}$ is in the fourth quadrant.

Tangent is negative in the fourth quadrant since the gradient of the radius is negative. The acute angle between the radius and the x-axis is $\frac{\pi}{6}$

$$\tan\left(\frac{23\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$



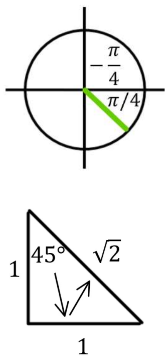
Example

$\sin\left(-\frac{\pi}{4}\right)$

$-\frac{\pi}{4}$ is in the fourth quadrant.

Sine (y) is negative in the fourth quadrant. The acute angle between the radius and the x-axis is $\frac{\pi}{4}$

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$



Example

$\cos(-150°)$

$-150°$ is in the third quadrant.

Cosine (x) is negative in the third quadrant. The acute angle between the radius and the x-axis is $30°$

$$\cos(-150°) = -\cos(30°) = -\frac{\sqrt{3}}{2}$$