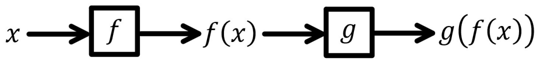
Composite Functions

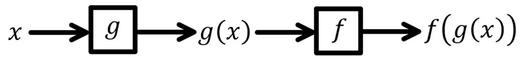
Composite Functions

Composing two functions is a chaining process in which the output of the inner function becomes the input of the outer function. That is, a function of another function. The rule for the composite function can be found by substituting the inside function into the outside function.

Order of Composition and Notation

f composition g	Apply g then f	$f(g(x))$	f of g of x	$f \circ g(x)$
g composition f	Apply f then g	$g(f(x))$	g of f of x	$g \circ f(x)$





Note that $f(g(x)) \neq g(f(x))$

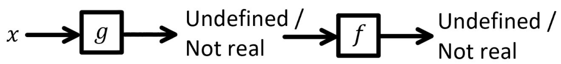
Example Modified VCAA 2017 Exam 2 Question 4

Let f and g be functions such that $f(2) = 5, f(3) = 4, g(2) = 5, g(3) = 2$ and $g(4) = 1$.
The value of $f(g(3)) = f(2) = 5$. The value of $g(f(3)) = g(4) = 1$.

Domain of a Composite Function

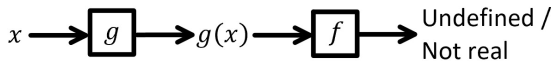
Let domain $f = d_f$, domain $g = d_g$, domain $f \circ g = d_{f \circ g}$ range $f = r_f$, range $g = r_g$, range $f \circ g = r_{f \circ g}$ where f is the outside function and g is the inside function and $f \circ g$ is the composite function.

If the inside function is not defined or real, then the composed function is not defined or not real.



Therefore, the domain of the composite function is, at most, the domain of the inside function.
$x \in d_g \Rightarrow d_{f \circ g} \subseteq d_g$

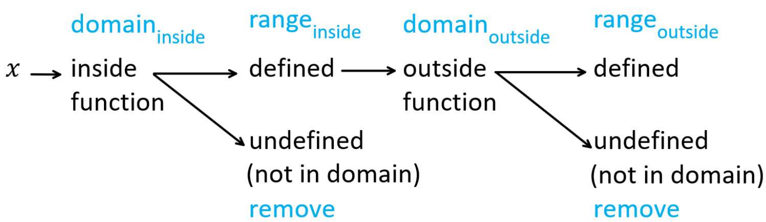
If the inside function is defined but the result is not in the domain of the outside function, then composed function is not defined or real.



Therefore, the range of the inside function needs to be within the domain of the outside function.
$g(x) \in d_f \quad \Rightarrow \quad r_g \subseteq d_f$

Restricting the Domain

If we required the maximal domain that a composite function can be defined for, we need the subset of the domain of the inside function such that the range of the inside function is a subset of the domain of the outside function. In other words, we need to exclude any x-value from the domain of the inside function that makes the inside function's value not in the domain of the outside function.
$d_{f \circ g} = d_g \setminus \{x : r_g \not\subseteq d_f\}$



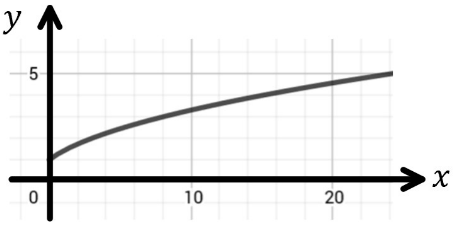
Range of a Composite Function

The values that can be output by a composite function are defined by the range of the outside function. However, if the range of the inside function is strictly a subset of the domain of the outside function, then there are values in the range of the outside function that can no longer be produced. Keep in mind: after restricting the domain, the range of the inside function r_g will also be restricted.

If new $r_g = d_f$, then $r_{f \circ g} = r_f$ as the inputs to the outside function are unchanged.
If new $r_g \subset d_f$, then $r_{f \circ g} = r_f \setminus \{f(x) : x \in d_f \cap x \notin \text{new } r_g\}$ [the range of f excluding the values of $f(x)$ where x is in the domain of f but not in the range of g] as the values that are in d_f that are not in new r_g can no longer be input into f (as new r_g is the set of inputs).

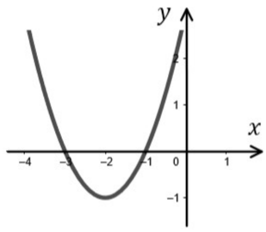
Example VCAA 2017 Exam 1 Question 7abc

Let $f: [0, \infty) \rightarrow R, f(x) = \sqrt{x+1}$.



$f(0) = \sqrt{0+1} = 1, \qquad \lim_{x \to \infty} f(x) = \infty$

From the graph we can see $f(x)$ is strictly increasing
∴ The range of f is $r_f = [1, \infty)$



Let $g: (-\infty, c] \rightarrow R, g(x) = x^2 + 4x + 3$, where $c < 0$.

$g(x) = (x+2)^2 - 1$
∴ The range of $(x+2)^2 - 1$ is $[-1, \infty)$

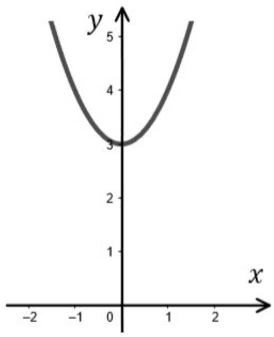
For $f\big(g(x)\big) = \sqrt{g(x) + 1} = \sqrt{(x+2)^2}$:
<u>Domain:</u>
Check if $r_g \subseteq d_f$: $\quad [-1, \infty) \nsubseteq [0, \infty)$
Therefore, the range of g must be restricted to $[0, \infty)$.

Exclude $\{x: -1 \leq g(x) < 0\}: x \in (-3, -1)$
∴ Domain of $f \circ g = d_g \setminus (-3, -1) = (-\infty, -3] \cup [-1, \infty)$

Since the required domain is $(-\infty, c]$, then $c = -3$

<u>Range:</u>
restricted $r_g = [0, \infty), \qquad d_f = [0, \infty)$
f will still get all inputs in its domain, therefore its range will be unchanged.
∴ Range of $r_{f \circ g} = r_g = [1, \infty)$

Let $h: R \rightarrow R, h(x) = x^2 + 3$. For $f\big(h(x)\big) = \sqrt{h(x) + 1} = \sqrt{x^2 + 4}$:



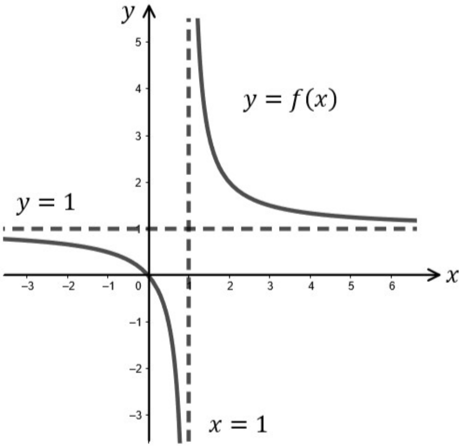
<u>Domain:</u>
Check if $r_h \subseteq d_f$: $\quad [3, \infty) \subset [0, \infty)$ ∴ Domain of $f \circ h = d_h = R$

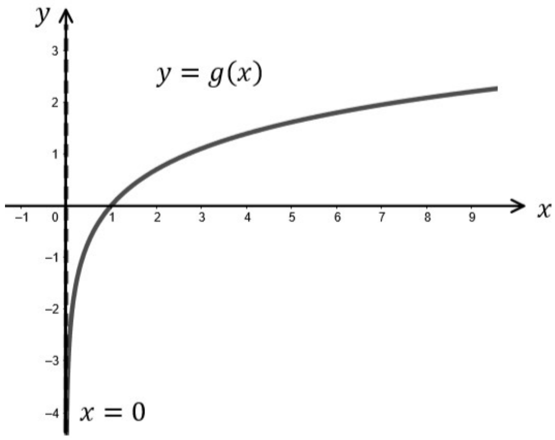
<u>Range:</u>
restricted $r_h = [3, \infty), \qquad d_f = [0, \infty)$
f will no longer get inputs from 0 to 3, therefore its range will be missing
$\big[f(0), f(3)\big) = [1, 2)$ ∴ Range of $f \circ h = r_f \setminus [1, 2) = [2, \infty)$

Example

Let $f: \mathbb{R} \setminus \{1\} \to \mathbb{R}, f(x) = \dfrac{1}{x-1} + 1$ and $g: \mathbb{R}^+ \to \mathbb{R}, g(x) = \log_e(x)$.

The rules, domains, and ranges of $f\big(g(x)\big)$ and $g\big(f(x)\big)$ are





domain of $f = \mathbb{R} \setminus \{1\}$, range of $f = \mathbb{R} \setminus \{1\}$ domain of $g = \mathbb{R}^+$, range of $g = \mathbb{R}$

f Composition g

$$f\big(g(x)\big) = f(\log_e(x)) = \frac{1}{\log_e(x) - 1} + 1$$

<u>Domain:</u>

Check if $r_g \subseteq d_f$: $\mathbb{R} \nsubseteq \mathbb{R} \setminus \{1\}$

Therefore, the range of g must be restricted to $\mathbb{R} \setminus \{1\}$.

Exclude $\{x: g(x) = 1\}$: $\log_e(x) = 1 \Rightarrow x = e$.

∴ Domain of $f \circ g = d_g \setminus \{e\} = \mathbb{R}^+ \setminus \{e\}$

<u>Range:</u>

restricted $r_g = \mathbb{R} \setminus \{1\}$, $d_f = \mathbb{R} \setminus \{1\}$

f will still get all inputs in its domain, therefore its range will be unchanged.

∴ Range of $f \circ g = r_f \setminus \{1\} = \mathbb{R} \setminus \{1\}$

g Composition f

$$g\big(f(x)\big) = g\left(\frac{1}{x-1} + 1\right) = \log_e\left(\frac{1}{x-1} + 1\right)$$

<u>Domain:</u>

Check if $r_f \subseteq d_g$: $\mathbb{R} \setminus \{1\} \nsubseteq \mathbb{R}^+$

Therefore, the range of f must be restricted to $\mathbb{R}^+ \setminus \{1\}$.

Exclude $\{x: f(x) \leq 0\}$: $\dfrac{1}{x-1} + 1 \leq 0 \Rightarrow 0 \leq x < 1$

∴ Domain of $g \circ f = d_f \setminus [0, 1) = \mathbb{R} \setminus [0,1]$

<u>Range:</u>

restricted $r_f = \mathbb{R}^+ \setminus \{1\}$, $d_g = \mathbb{R}^+$

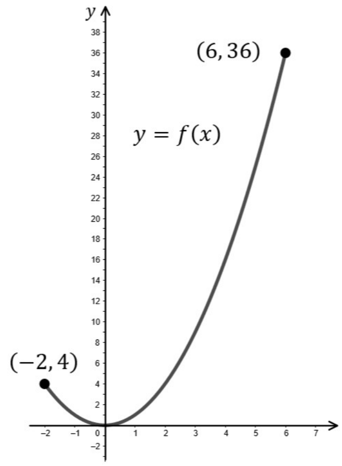
g will no longer get an input of 1, therefore its range will be missing $\log_e(1) = 0$.

∴ Range of $g \circ f = r_g \setminus \{0\} = \mathbb{R} \setminus \{0\}$

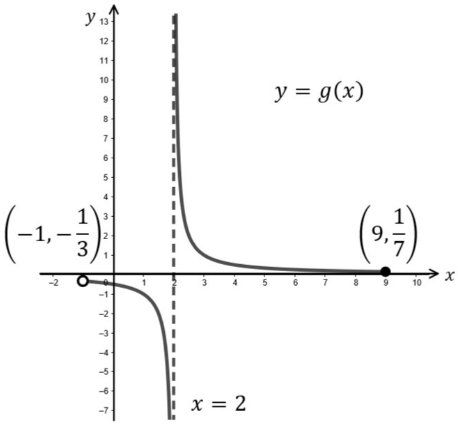
Example

Let $f: [-2, 6] \to R, f(x) = x^2$ and $g: \to (-1, 2) \cup (2, 9], g(x) = \dfrac{1}{x-2}$.

The graphs of $y = f(x)$ and $y = g(x)$ are shown.



Domain: $[-2, 6]$, Range: $[0, 36]$



Domain: $(-1, 9] \setminus \{2\}$, Range: $\mathbb{R} \setminus \left[-\frac{1}{3}, \frac{1}{7}\right)$

For $f(g(x)) = (g(x))^2 = \left(\dfrac{1}{x-2}\right)^2 = \dfrac{1}{(x-2)^2}$

Domain:

Check if $r_g \subseteq d_f$: $\mathbb{R} \setminus \left[-\frac{1}{3}, \frac{1}{7}\right) \nsubseteq [-2, 6]$

Therefore, the range of g must be restricted to $[-2, 6] \setminus \left[-\frac{1}{3}, \frac{1}{7}\right)$.

Exclude $\{x: g(x) < -2 \cap g(x) > 6\}$: $x \in \left(\frac{3}{2}, \frac{13}{6}\right) \setminus \{2\}$

$\therefore$ Domain of $f \circ g = d_g \setminus \left\{\left(\frac{3}{2}, \frac{13}{6}\right) \setminus \{2\}\right\} = (-1, 9] \setminus \left(\frac{3}{2}, \frac{13}{6}\right)$

Range:

restricted $r_g = [-2, 6] \setminus \left[-\frac{1}{3}, \frac{1}{7}\right)$, $\quad d_f = [-2, 6]$

f will no longer get inputs from $-\frac{1}{3}$ to $\frac{1}{7}$, therefore its range will be missing $\left[0, \frac{1}{49}\right)$

$\therefore$ Range of $f \circ g = r_f \setminus \left[0, \frac{1}{49}\right) = \left[\frac{1}{49}, 36\right]$

For $g(f(x)) = \dfrac{1}{f(x) - 2} = \dfrac{1}{x^2 - 2}$

Domain:

Check if $r_f \subseteq d_g$: $[0, 36] \nsubseteq (-1, 9] \setminus \{2\}$

Therefore, the range of f must be restricted to $[0, 9] \setminus \{2\}$.

Exclude $\{x: f(x) > 9 \cap f(x) = 2\}$: $x \in \{-\sqrt{2}, \sqrt{2}\} \cup (3, \infty)$

$\therefore$ Domain of $g \circ f = d_f \setminus \left\{\{-\sqrt{2}, \sqrt{2}\} \cup (3, \infty)\right\} = [-2, 3] \setminus \{-\sqrt{2}, \sqrt{2}\}$

Range:

restricted $r_f = [0, 9] \setminus \{2\}$, $\quad d_f = (-1, 9] \setminus \{2\}$

g will no longer get inputs from -1 to 0, therefore its range will be missing $\left(-\frac{1}{2}, -\frac{1}{3}\right)$

$\therefore$ Range of $g \circ f = r_g \setminus \left(-\frac{1}{2}, -\frac{1}{3}\right) = \mathbb{R} \setminus \left[-\frac{1}{2}, \frac{1}{7}\right)$