Systems of Simultaneous Linear Equations

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A set of linear equations that must be solved together to find the values of all unknowns. To solve for all unknowns, the number of unknowns must be the same as the number of unique equations.

For example, when there are two linear equations that share two variables, they can be solved together to find the values of both variables that satisfy both equations simultaneously.

Graphical Method for Two Linear Equations with Two Variables

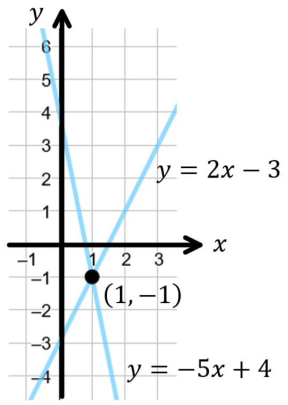
Graphically, the solution to a pair of simultaneous linear equations is the coordinate where the graphs of both lines intersect. So, we can solve the equations simultaneously by sketching the lines of both equations and read off the coordinate of the intersection point.

Example

$y = 2x - 3$
$y = -5x + 4$

The graphs intersect at $(1, -1)$.

$\therefore x = 1, y = -1$



Substitution Method for Two Linear Equations with Two Variables

Since the two equations share the same value for their unknowns, we can use substitution between the equations involving the variables. This may include:

- equating a variable from both equations, that is,
  - make $y = y$ if the equations are both $y =$ an expression, or
  - make $x = x$ if the equations are both $x =$ expression
- substitute one equation into the other when one equation is $y =$ or $x =$

The idea is to reduce the equations to one unknown, find its value, then use it to find the value of the other unknown by substituting it into one of the original equations.

Example

$y = 2x - 3$ and $y = -5x + 4$

$y = -5x + 4,$ substitute $y = 2x - 3$
$\Rightarrow (2x - 3) = -5x + 4$
$\Rightarrow 7x - 3 = 4$
$\Rightarrow 7x = 7$
$\Rightarrow x = 1$

$y = 2(1) - 3$
$\Rightarrow y = 2 - 3$
$\Rightarrow y = -1$

$\therefore x = 1, y = -1$

Example

$y = 4x - 1$ and $2x - 3y = 8$

$2x - 3y = 8,$ substitute $y = 4x - 1$
$\Rightarrow 2x - 3(4x - 1) = 8$
$\Rightarrow 2x - 12x + 3 = 8$
$\Rightarrow -10x + 3 = 8$
$\Rightarrow -10x = 5$
$\Rightarrow x = -\frac{1}{2}$

$y = 4\left(-\frac{1}{2}\right) - 1$
$\Rightarrow y = -2 - 1$
$\Rightarrow y = -3$

$\therefore x = -\frac{1}{2}, y = -3$

Elimination Method for Two Linear Equations with Two Variables

Adding the equal amounts to both sides of an equation preserves equality. This includes adding equal expressions and values, even if writing them differently on both sides.
Specifically, if we add (or subtract) multiples of an equation, we can reduce the number of variables down to 1. That is, we can eliminate one of the variables to find the value of the other variable first.

Example

$x + y = 10$ ①
$x - y = 4$ ②

Eliminate y

$x + y = 10$ $+$ ②
$\Rightarrow x + y + (x - y) = 10 + (4)$
$\Rightarrow 2x = 14$
$\Rightarrow x = 7$

Substitute $x = 7$ into ①
$7 + y = 10$
$\Rightarrow y = 3$
$\therefore x = 7, y = 3$

Example

$2x + 3y = 6$ ③
$2x - 7y = -54$ ④

Eliminate x

$2x + 3y = 6$ $-$ ④
$\Rightarrow 2x + 3y - (2x - 7y) = 6 - (-54)$
$\Rightarrow 10y = 60$
$\Rightarrow y = 6$

Substitute $y = 6$ into ③
$2x + 3(6) = 6$
$\Rightarrow 2x + 18 = 6$
$\Rightarrow 2x = -12$
$\Rightarrow x = -6$
$\therefore x = -6, y = 6$

Example

$y = 2x - 3$ ⑤
$y = -5x + 4$ ⑥

Eliminate y

$y = 2x - 3$ $-$ ⑥
$y - (y) = 2x - 3 - (-5x + 4)$
$\Rightarrow 0 = 7x - 7$
$\Rightarrow 7x = 7$
$\Rightarrow x = 1$

Substitute $x = 1$ into ⑤
$y = 2(1) - 3$
$\Rightarrow y = 2 - 3$
$\Rightarrow y = -1$
$\therefore x = 1, y = -1$

Example

$y = 4x - 1$ ⑦
$2x - 3y = 8$ ⑧

Eliminate y

$2x - 3y = 8$ $+ 3 \times$ ⑦
$\Rightarrow 2x - 3y + 3(y) = 8 + 3(4x - 1)$
$\Rightarrow 2x = 8 + 12x - 3$
$\Rightarrow -10x = 5$
$\Rightarrow x = -\frac{1}{2}$

Substitute $x = -\frac{1}{2}$ into ⑦
$y = 4\left(-\frac{1}{2}\right) - 1$
$\Rightarrow y = -2 - 1$
$\Rightarrow y = -3$
$\therefore x = -\frac{1}{2}, y = -3$

Example

$2x + 3y = 18$ ⑨
$8x + 7y = 22$ ⑩

Eliminate x

$8x + 7y = 22$ $- 4 \times$ ⑨
$\Rightarrow 8x + 7y - 4(2x + 3y) = 22 - 4(18)$
$\Rightarrow 7y - 12y = 22 - 72$
$\Rightarrow -5y = -50$
$\Rightarrow y = 10$

Substitute $y = 10$ into ⑨
$2x + 3(10) = 18$
$\Rightarrow 2x + 30 = 18$
$\Rightarrow 2x = -12$
$\Rightarrow x = -6$
$\therefore x = -6, y = 10$

Example

$5x - 3y = 20$ ①
$6x + 7y = 24$ ②

Eliminate x

$5x - 3y = 20$ $\times 7$
$\Rightarrow 7(5x - 3y) = 7(20)$
$\Rightarrow 35x - 21y = 140$ $+ 3 \times$ ②
$\Rightarrow 35x - 21y + 3(6x + 7y) = 140 + 3(24)$
$\Rightarrow 35x + 18x = 140 + 72$
$\Rightarrow 53x = 212$
$\Rightarrow x = 4$

Substitute $x = 4$ into ①
$5(4) - 3y = 20$
$\Rightarrow 20 - 3y = 20$
$\Rightarrow y = 0$
$\therefore x = 4, y = 0$

Example

$5x - 3y = 20$ ①
$6x + 7y = 24$ ②

Eliminate y

$5x - 3y = 20$ $\times 6$
$\Rightarrow 6(5x - 3y) = 6(20)$
$\Rightarrow 30x - 18y = 120$ $- 5 \times$ ②
$\Rightarrow 30x - 18y - 5(6x + 7y) = 120 + 5(24)$
$\Rightarrow -18y - 35y = 120 - 120$
$\Rightarrow 53y = 0$
$\Rightarrow y = 0$

Substitute $y = 0$ into ①
$5x - 3(0) = 20$
$\Rightarrow 5x = 20$
$\Rightarrow x = 4$
$\therefore x = 4, y = 0$

Systems of More than 2 Simultaneous Linear Equations in More than 2 Variables

Solving simultaneous equations with more than two variables requires you to reduce the number of variables down to two so that you can use the methods for two variables. To do this use different combinations of equations and eliminate variables from them strategically until the same two remain in two equations. Solve for the two remaining variables then substitute to find the eliminated variables.

Example

Find the equation of the parabola that passes through $(-2, 39)$ and has a gradient of 22 at $(5, 46)$.

$y = ax^2 + bx + c, \qquad \frac{dy}{dx} = 2ax + b$

$39 = (-2)^2a + (-2)b + c$
$\Rightarrow 39 = 4a - 2b + c$ ①

$46 = (5)^2a + (5)b + c$
$\Rightarrow 46 = 25a + 5b + c$ ②

$22 = 2a(5) + b$
$\Rightarrow 22 = 10a + b$ ③

$46 = 25a + 5b + c$ $\quad - ①$
$46 - 39 = 25a + 4b + c - (4a - 2b + c)$
$\Rightarrow 7 = 21a + 7b$
$\Rightarrow 1 = 3a + b$ ④

$22 = 10a + b$ $\quad - ④$
$22 - 1 = 10a + b - (3a + b)$
$\Rightarrow 21 = 7a \quad \Rightarrow \quad a = 3$

Substitute $a = 3$ into ④
$1 = 3(3) + b \quad \Rightarrow \quad b = -8$

Substitute $a = 3$ and $b = -8$ into ①
$39 = 4(3) - 2(-8) + c$
$\Rightarrow 39 = 12 + 16 + c \quad \Rightarrow \quad c = 11$

$y = 3x^2 - 8x + 11$

Systems of Simultaneous Linear Equations involving a Single Parameter

Simultaneous equations will need to use a parameter if the number of equations is less than the number of variables. That is for a systems of $n - 1$ linear equations in n variables, $n - 1$ variables solutions will be written in terms of the final variable. If two equations are multiples of each other, discount one of the euquations. We generally use λ as a substitution for the final variable and solve the other equations in terms of it.

Example

$2x + y - 4z = 2$ ①
$x + y + 3z = -1$ ②

$2x + y - 4z = 2$ $\quad - ②$
$2x + y - 4z - (x + y + 3z) = 2 - (-1)$
$x - 7z = 3$

Sub $z = \lambda$ and solve for x
$x - 7\lambda = 3$
$x = 3 + 7\lambda$

Sub $z = \lambda$ and $x = 3 + 7\lambda$ into ② and solve for y
$3 + 7\lambda + y + 3\lambda = -1$
$y = -4 - 10\lambda$

Therefore:
$x = 3 + 7\lambda, \qquad y = -4 - 10\lambda, \qquad z = \lambda$

Example

$2x + 3y = 6$ ①
$4x + 6y = 12$ ②

$4x + 6y = 12$ $\quad 2 \times ① \equiv ②$

Let $y = \lambda$

$4x + 6\lambda = 12$

$4x = 12 - 6\lambda$

$x = 3 - \frac{3}{2}\lambda$

$\therefore x = 3 - \frac{3}{2}\lambda, \qquad y = \lambda$