Number of Solutions of Systems of Linear Equations

Number of Solutions

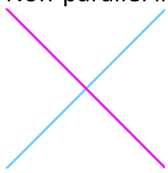
For two equations $y = m_1 x + c_1$ and $y = m_2 x + c_2$ the number of solutions, is defined by the gradient and y-intercept of the two equations.

Unique Solution

There is a single value for each variable that satisfies every equation. This usually occurs when the number of equations is equal to the number of variables.

$m_1 \neq m_2$ / coefficients are not equal

Non-parallel lines

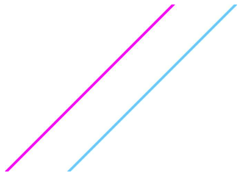


No Solutions

There is no set of values for the variables that satisfies every equation. This usually occurs when two or more equations are multiples of each other in all but one value, or when there are more equations than variables. This can occur when there are more variables than equations.

$m_1 = m_2$, and $c_1 \neq c_2$ / coefficients are equal, constants are not

Parallel lines that do not meet

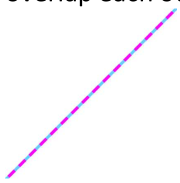


Infinite Solutions

There is an infinite number of values for each variable that satisfies every equation. This usually occurs when two or more equations are multiples of each other. This usually occurs when there are more variables than equations. Solutions are in terms of a parameter.

$m_1 = m_2$, and $c_1 = c_2$ / coefficients and constants are equal

Parallel lines that completely overlap each other



Example VCAA 2007 Exam 2 Question 5

The simultaneous linear equations $mx + 12y = 24$ and $3x + my = m$ have a unique solution for

$$y = -\frac{m}{12}x + 2, \qquad y = -\frac{3}{m}x + 1$$

Unique for $-\frac{m}{12} \neq -\frac{3}{m}$

$\Rightarrow m^2 \neq 36 \quad \Rightarrow \quad m \neq \pm 6$

$\therefore$ unique for $m \in \mathbb{R} \setminus \{-6,6\}$

Example VCAA 2016 Sample Exam 2 Question 12 / 2014 Exam 2 Question 17

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have no solution for

$$y = \frac{a}{3}x - \frac{5}{3}, \qquad y = \frac{3}{a}x + \frac{a-8}{a}$$

Non-unique for $\frac{a}{3} = \frac{3}{a} \quad \Rightarrow \quad a^2 = 9 \quad \Rightarrow \quad a = \pm 3$

For $a = 3$: $\frac{3-8}{3} = -\frac{5}{3}$

$\therefore$ Infinite solutions

For $a = -3$: $\frac{-3-8}{-3} = \frac{11}{3}$

$\therefore$No solutions

$\therefore a = -3$ gives no solutions

Example VCAA 2007 Exam 2 Question 5

The simultaneous linear equations $mx + 12y = 24$ and $3x + my = m$ have a unique solution for

$mx + 12y = 24$ $[\times 3]$
$3x + my = m$ $[\times m]$
$\Rightarrow 3mx + 36y = 72$
$\Rightarrow 3mx + m^2 y = m^2$

Parallel if coefficients are equal:
$\Rightarrow m^2 \neq 36 \quad \Rightarrow \quad m \neq \pm 6$
$\therefore$ unique for $m \in \mathbb{R} \setminus \{-6,6\}$

Example VCAA 2016 Sample Exam 2 Question 12 / 2014 Exam 2 Question 17

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have no solution for

$ax - 3y = 5 \quad [\times 3]$
$3x - ay = 8 - a \quad [\times a]$

$\Rightarrow 3ax - 9y = 15$
$\Rightarrow 3ax - a^2 y = 8a - a^2$

∥ if coefficients are equal:
$\Rightarrow a^2 = 9 \quad \Rightarrow \quad a = \pm 3$

For $a = 3$: $8(3) - (3)^2 = 15$ $\therefore$ Infinite solutions
For $a = -3$: $8(-3) - (-3)^2 \neq 15$ $\therefore$ No solutions
$\therefore a = -3$ gives no solutions