DIFFERENTIATION

POWER FUNCTIONS

$$y = ax^n$$

$$\frac{dy}{dx} = nax^{x-1}$$

- **Think "Multiply by the power and drop the power by one"**
- $\frac{dy}{dx}$, the derivative, represents the gradient of the tangent to a curve at any x-value.
- Note: $\frac{d}{dx}(2x-1)$ is asking you to differentiate $2x-1$ to find $\frac{dy}{dx}=2$

CHAIN RULE

- Used for embedded functions (a function in a function) → Look for functions within brackets
- Substitute then use chain rule

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

- **Shortcut: Think "differentiate the outside, then multiply by the derivative of the inside"**
- Differentiate the outside leaving the inside alone, then differentiate the inside because it is something other than 'x'"

Eg. Differentiate $y=(3x^2-6x+1)^3$

$3x^2-6x+1$ is embedded within another function, so substitute $u=3x^2-6x+1$ → $y=u^3$

Now we have an equation for y in terms of u and u in terms of x

$\frac{dy}{du}=3u^2$ and $\frac{du}{dx}=6x-6$

Substitute into $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$\frac{dy}{dx}=3(3x^2-6x+1)^2(6x-6)$

$\frac{dy}{dx}=(3x^2-6x+1)^2(18x-18)$

Eg. Differentiate $y=(3x^2-6x+1)^3$ **(shortcut)**

Differentiate the outside but leave the inside $3(3x^2-6x+1)^2$

But because the inside is not just 'x' we have to multiple by the derivative of the inside as well,

$\frac{dy}{dx}=3(3x^2-6x+1)^2\times(6x-6)$

$\frac{dy}{dx}=(3x^2-6x+1)^2(18x-18)$

Same answer but faster

Eg. Differentiate $y=\frac{1}{(2x-2)^2}$

Remove the fraction $y=(2x-2)^{-2}$ → Always convert to power form before differentiating

Chain rule: Times by power, drop power by one, multiply by the derivative of what is inside

$\frac{dy}{dx}=-2(2x-2)^{-3}(2)=-4(2x-2)^{-3}$

PRODUCT RULE

$$\frac{d}{dx}(u \times v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- Substitute each of the functions being multiplied as "u" and "v"

- **Shortcut: Think "Leave it do it, plus do it leave it."**
- i.e. the first function times the derivative of the second function, plus the derivative of the first function times the second function.

Eg. Differentiate $y = (x^2 - 1)(2x + 3)$

Let $u = x^2 - 1$ and $v = 2x + 3$

So $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 2$

$\frac{dy}{dx} = (x^2 - 1)(2) + (2x + 3)(2x)$

$\frac{dy}{dx} = 2x^2 - 2 + 4x^2 + 6x$

$\frac{dy}{dx} = 6x^2 + 6x - 2$

Substitute into $\frac{d}{dx}(u \times v) = u\frac{dv}{dx} + v\frac{du}{dx}$

Eg. Differentiate $y = (x^2 - 1)(2x + 3)$ **(shortcut)**

$\frac{dy}{dx} = (x^2 - 1)(\frac{d}{dx}(2x + 3)) + (\frac{d}{dx}(x^2 - 1))(2x + 3)$

$\frac{dy}{dx} = (x^2 - 1)(2) + (2x)(2x + 3)$

$\frac{dy}{dx} = 2x^2 - 2 + 4x^2 + 6x$

$\frac{dy}{dx} = 6x^2 + 6x - 2$

QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

- Substitute each of the functions being divided as "u" and "v"

- **Shortcut: Think "bottom times derivative of top, minus top times derivative of bottom, all divided by bottom squared."**
  i.e. the bottom function times the derivative of the top function, minus the top function times the derivative of the bottom function, all divided by the bottom function squared.

Eg. Differentiate $y = \frac{x^2 - 2}{x + 3}$

$u = x^2 - 2$ and $v = x + 3$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{(x+3)(2x) - (x^2 - 2)(1)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{2x^2 + 6x - x^2 + 2}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{x^2 + 6x + 2}{(x+3)^2}$$

Substitute into

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

Eg. Differentiate $y = \frac{x^2 - 2}{x + 3}$ **(shortcut)**

$$\frac{dy}{dx} = \frac{(x+3)\left(\frac{d}{dx}(x^2 - 2)\right) - (x^2 - 2)\left(\frac{d}{dx}(x+3)\right)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{(x+3)(2x) - (x^2 - 2)(1)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{2x^2 + 6x - x^2 + 2}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{x^2 + 6x + 2}{(x+3)^2}$$

TRIGONOMETRIC FUNCTIONS

$$y = \sin(x) \rightarrow \frac{dy}{dx} = \cos(x)$$

$$y = \cos(x) \rightarrow \frac{dy}{dx} = -\sin(x)$$

$$y = \tan(x) \rightarrow \frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

- NOTE: If the argument (what's inside the brackets) is not "x," the chain rule must be used.

Eg. Differentiate $y = 3\sin(x)$

$\frac{dy}{dx} = 3\cos(x)$ as the derivative of $\sin(x)$ is $\cos(x)$

Eg. Differentiate $y = 4\cos(2x)$

$$\frac{dy}{dx} = 4 \times -\sin(2x) \times 2$$

$$\frac{dy}{dx} = -8\sin(2x)$$

The argument is not just "x," must multiply by the derivative of what is in the brackets, i.e 2

Eg. Differentiate $y = 3\tan(x^2) + x$

$$\frac{dy}{dx} = 3 \times \frac{1}{\cos^2(x^2)} \times 2x$$

$$\frac{dy}{dx} = \frac{6x}{\cos^2(x^2)}$$

The argument is not just "x," must multiply by the derivative of what is in the brackets, i.e 2x

Eg. If $f:[0,2\pi]\to R, f(x)=2\sin(2x)$

a) State the coordinates of the stationary turning points

$$f'(x)=4\cos(2x)$$
$$0=4\cos(2x)$$
$$0=\cos(2x)$$

Stationary points when gradient = 0, therefore $f'(x)=0$

$$2x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$$
$$x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$$

$0\le x\le 2\pi$ → $0\le 2x\le 4\pi$, so the solutions of $0=\cos(2x)$ are between 0 and 4π

Therefore the coordinates are $(\frac{\pi}{4},2),(\frac{3\pi}{4},-2),(\frac{5\pi}{4},2),(\frac{7\pi}{4},-2)$

b) Find the gradient of $f(x)$ when $x=\frac{\pi}{3}$

$$f'(x)=4\cos(2x)$$
$$f'(\frac{\pi}{3})=4\cos(\frac{2\pi}{3})$$
$$f'(\frac{\pi}{3})=4\times-\frac{1}{2}$$
$$f'(\frac{\pi}{3})=-2$$

Eg. When $x=30^o$, there is a stationary turning point. When $x=90^o$ there is another turning point. If the graph follows the function $f:[0,2\pi]\to R, f(x)=\sin(ax)$, find the value of a.

NOTE: Trigonometric calculus only works if “x” is in radians NOT DEGREES

$$\frac{dy}{dx}=a\cos(ax)$$

$0=a\cos(\frac{\pi}{6}a)$ and $0=a\cos(\frac{\pi}{2}a)$

$0=\cos(\frac{\pi}{6}a)$ and $0=\cos(\frac{\pi}{2}a)$

$\frac{\pi}{6}a=\frac{\pi}{2},\frac{3\pi}{2}$ and $\frac{\pi}{2}a=\frac{\pi}{2},\frac{3\pi}{2}$

$a=3,9$ and $a=1,3$

Therefore $a=3$

$\frac{dy}{dx}=0$ when $x=\frac{\pi}{6}$ and $x=\frac{\pi}{2}$

When, $a=3$, there will be a stationary point when $x=\frac{\pi}{6}$ **and** $x=\frac{\pi}{2}$

EXPONENTIALS AND LOGS

$$y = e^x \quad \rightarrow \frac{dy}{dx} = e^x$$

$$y = e^{kx} \quad \rightarrow \frac{dy}{dx} = ke^{kx}$$

$$y = \log_e(x) \quad \rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$y = \log_e(f(x)) \quad \rightarrow \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Eg. Differentiate $y = 2e^{-3x}$

$$\frac{dy}{dx} = 2e^{-3x} \times -3$$

$$\frac{dy}{dx} = -6e^{-3x}$$

Eg. Differentiate $y = \log_e(2x)$

$$\frac{dy}{dx} = \frac{2}{2x}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Note: The derivative of $\log_e(ax) = \frac{1}{x}$

Eg. Differentiate $y = \log_e(x^2 - 2x + 1)$

$$\frac{dy}{dx} = \frac{2x-2}{x^2-2x+1}$$

Derivative of the argument divided by the argument

$$\frac{dy}{dx} = \frac{2(x-1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{2}{x-1}$$

Eg. Differentiate $y = \log_2 x$

Base is not "e" so must change the base using the change base rule $\log_e(x) = \frac{\log_a(x)}{\log_a(e)}$

$$y = \frac{\log_e x}{\log_e 2}$$

$$y = \frac{1}{\log_e 2} \times \log_e x$$

$$\frac{dy}{dx} = \frac{1}{x \log_e 2}$$

Eg. Differentiate $y = 2^x$

Base is not "e" so must change the base using the change base rule $a^x = e^{\ln(a)x}$

$$y = e^{\ln(2)x}$$

$$\frac{dy}{dx} = \ln(2)e^{\ln(2)x}$$

COMBINING FUNCTIONS

- Using chain, product and quotient rules on a variety of functions simultaneously

Eg. Differentiate $f(x) = \sin(e^{3x})$

$$f'(x) = \cos(e^{3x}) \times e^{3x} \times 3$$
$$f'(x) = 3e^{3x} \cos(e^{3x})$$

Differentiate the outside, then multiply by the derivative of the inside

Eg. Differentiate $f(x) = \cos(x^2)e^{3x-1}$

$$f'(x) = \cos(x^2) \times 3 \times e^{3x-1} + e^{3x-1} \times -\sin(x^2) \times 2x$$
$$f'(x) = 3e^{3x-1} \cos(x^2) - 2xe^{3x-1} \sin(x^2)$$

Product and chain rule. First function times derivative of the second plus second function times derivative of the first.

