Chapter 3

Sequences and finance

Chapter questions

- ▶ What is a sequence?
- ▶ How do we generate a sequence of numbers from a starting value and a rule?
- ▶ How do we identify particular terms in a sequence?
- ▶ What is recursion?
- ▶ What is an arithmetic sequence?
- ▶ What is a geometric sequence?
- ▶ How can I tabulate and graph an arithmetic or geometric sequence?
- ▶ How can I find a particular term of an arithmetic or geometric sequence using recursion?
- ▶ How can I generate an arithmetic or geometric sequence using recursion?
- ▶ How can recurrence relations be used to model simple interest, flat rate depreciation and unit-cost depreciation?
- ▶ How can recurrence relations be used to model compound interest and reducing-balance depreciation?
- ▶ How can a rule be used to find particular terms for linear growth and decay models?

In this chapter, we will be investigating sequences that can be generated by a rule. Some sequences make each new term by adding a constant amount. Others multiply each term by a fixed number to make the next term. These ideas have many applications to financial mathematics. In particular, the ideas of sequences are applied to calculating interest and loan repayments and the depreciation of assets.

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3A Number patterns

Learning intentions

- To be able to determine a simple rule for a sequence of numbers.
- To be able to generate a sequence from a starting number and a simple rule.

Sequences

A **sequence** is a list of numbers or symbols in a particular order. The numbers or items in a sequence are called the **terms** of the sequence. Each term is separated by a comma. If the sequence continues indefinitely, or if there are too many terms in the sequence to write them all, we use an *ellipsis* '. . . ' at the end of a few terms of the sequence, like this:

$$7, 3, 4, 11, 15, 24, \ldots$$

Sequences may be either generated randomly or by **recursion**, using a rule. For example, recording the numbers obtained while tossing a die would give a randomly generated sequence, such as:

$$3, 1, 2, 2, 6, 4, 3, \ldots$$

Because there is no pattern in the sequence, there is no way of predicting the next term in the sequence. Consequently, random sequences are of no relevance to this chapter.

When there is a pattern to the sequence, sequences can exhibit different behaviours such as terms increasing, decreasing or being constant (e.g. $3, 3, 3, 3, \ldots$). Sequences can also oscillate (meaning change or alternate) between two or more values (e.g. $1, -1, 1, -1, \ldots$) or have a limiting value where the sequence approaches a particular value. For example,

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

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Example 1 **Identifying behaviour of sequences**

Consider the following sequences and identify their behaviour as increasing, decreasing, constant or oscillating. Also state whether the sequence has a limiting value.

a $-3, -3, -3, -3, \ldots$ **b** $100, 90, 80, 70, \ldots$ **c** $1000, 100, 10, 1, 0.1, \ldots$

d $1, 4, 9, 25, 36, \ldots$ **e** $10, -8, 10, -8, \ldots$

Explanation	Solution
a Each term has the same value.	The sequence exhibits constant behaviour.
b The value of each term is decreasing by 10.	The sequence exhibits decreasing behaviour.
c The value of each term is divided by 10 and so the sequence is getting closer to zero.	The sequence exhibits decreasing behaviour and has a limiting value of zero.
d The value of each term is increasing.	The sequence exhibits increasing behaviour.
e The value of each term is alternating between 10 and -8.	The sequence exhibits oscillating behaviour.

Now try this 1 **Identifying behaviour of sequences (Example 1)**

Consider the following sequences and identify their behaviour as increasing, decreasing, constant or oscillating. Also state whether the sequence has a limiting value.

a $8, 4, 8, 4, 8, \ldots$ **b** $65, 70, 75, 80, \ldots$ **c** $1.1, 1.01, 1.001, 1.0001, \ldots$

Hint 1 Determine what the relationship is between each term.

Hint 2 Identify the general pattern of the sequence.

An introduction to recursion

Some sequences of numbers do display a pattern. For example, if our sequence starts with the number one, and we have the rule:

'add 2 to the current number,'

then we get the sequence:

$$1 \xrightarrow{+2} 3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9 \quad . \; . \; .$$

For example, to find the term after 9, just add 2 to 9, to get $9 + 2 = 11$.

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Recursion is a process of generating a sequence of terms from a given starting point and a rule. Different rules can be used, such as adding or subtracting a term, multiplying or dividing by a particular number, squaring numbers or even combining previous terms to find new terms. Knowing the starting term and the rule means that the next term can be found easily.

In this chapter we will look at sequences that can be generated by a rule.

Example 2 Looking for a recursive rule for a sequence of numbers

Look for a pattern or rule in each sequence and find the next number.

a 2, 8, 14, 20, . . .

b 5, 15, 45, 135, . . .

c 7, 4, 1, −2, . . .

Explanation

a 2, 8, 14, 20, . . .

1. Add 6 to make each new term.
2. Add 6 to 20 to make the next term, 26.

b 5, 15, 45, 135, . . .

1. Multiply by 3 to make each new term.
2. Multiply 135 by 3 to make the next term, 405.

c 7, 4, 1, −2, . . .

1. Subtract 3 each time to make the next term.
2. Subtract 3 from −2 to get the next term, −5.

Solution

$$2 \xrightarrow{+6} 8 \xrightarrow{+6} 14 \xrightarrow{+6} 20 \xrightarrow{+6} 26 \quad \ldots$$

The next number is 26.

$$5 \xrightarrow{\times 3} 15 \xrightarrow{\times 3} 45 \xrightarrow{\times 3} 135 \xrightarrow{\times 3} 405 \quad \ldots$$

The next number is 405.

$$7 \xrightarrow{-3} 4 \xrightarrow{-3} 1 \xrightarrow{-3} -2 \xrightarrow{-3} -5 \quad \ldots$$

The next number is −5.

Now try this 2 Looking for a recursive rule for a sequence of numbers (Example 2)

Look for a pattern or rule in each sequence and find the next number.

a 25, 22, 18, 15, . . .

b 1000, 200, 40, . . .

c 3, 6, 9, 12, 15, . . .

Hint 1 Calculate the difference between each consecutive term.

Hint 2 Determine the relationship between two consecutive terms.

Hint 3 Apply the rule that you find, so that you can determine the next term.

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Generating sequences from a starting value and a rule

Sequences can be generated from a starting value and a rule that tells us how to find the next value in the sequence.

Example 3 **Finding a sequence from a starting value and rule**

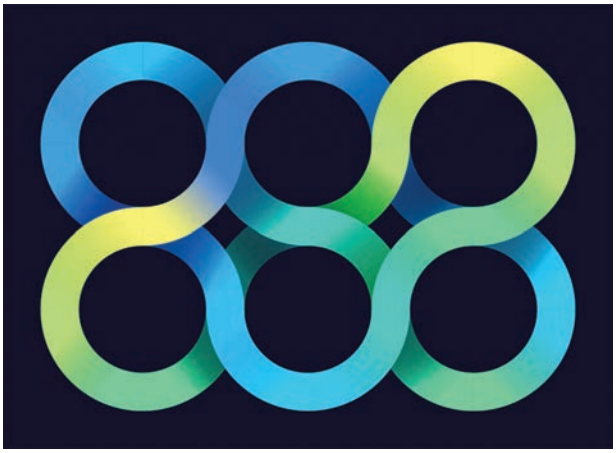
Write down the first five terms of the sequence with a starting value of 5 and the rule 'add 3 to each term'.

Explanation	Solution
1 Write down the starting value.	5
2 Apply the rule (add 3) to generate the next term.	$5+3=8$
3 Calculate three more terms.	$8+3 = 11$ $11+3 = 14$ $14+3 = 17$
4 Write your answer.	The sequence is 5, 8, 11, 14, 17.

Now try this 3 **Finding a sequence from a starting value and rule (Example 3)**

Write down the first five terms of the sequence with a starting value of 7 and the rule 'add 5 to each term'.

Hint 1 Write down the starting value.
Hint 2 Apply the rule to generate the next term.



Using repeated addition on a CAS calculator to generate a sequence

As we have seen, a recursive rule based on repeated addition, such as 'to find the next term, add 6', is a quick and easy way of generating the next few terms of a sequence. However, it becomes tedious to do by hand if we want to find, say, the next 20 terms.

Fortunately, your CAS calculator can semi-automate the process.

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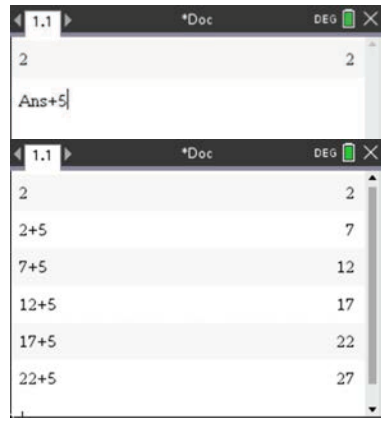
Using TI-Nspire CAS to generate the terms of an arithmetic sequence recursively

Generate the first six terms of the arithmetic sequence: 2, 7, 12, . . .

Steps



1 Press ⌂ on>**New > Add Calculator**.

2 Enter the value of the first term, 2. Press enter. The calculator stores the value, 2, as Answer.

3 The common difference for the sequence is 5. So, type in +**5**.

4 Press enter. The second term in the sequence, 7, is generated.

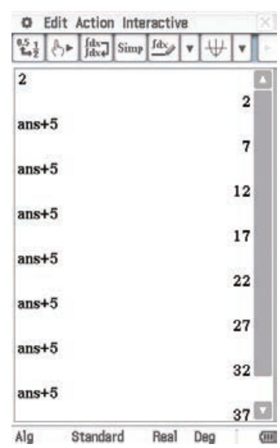
5 Pressing enter again generates the next term, 12. Keep pressing enter until the desired number of terms is generated.

6 Write down the terms. The first 6 terms of the sequence are: 2, 7, 12, 17, 22, 27.

Using Class Pad to generate the terms of an arithmetic sequence recursively

Generate the first six terms of the arithmetic sequence: 2, 7, 12, . . .

Steps



1 Tap Main $\sqrt{\alpha}$ to open the Main application.

2 Starting with a clean screen, enter the value of the first term, **2**. Press EXE. The calculator stores the value, 2, as **ans**.

3 The common difference for this sequence is 5. So, type **+ 5**. Then press EXE. The second term in the sequence, **7**, is displayed.

4 Pressing EXE again generates the next term, **12**. Keep pressing EXE until the required number of terms is generated.

5 Write down the terms. The first 6 terms of the sequence are: 2, 7, 12, 17, 22, 27.

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Section Summary

- A **sequence** is a list of numbers or symbols in a particular order.
- Each number or symbol that makes up a sequence is called a **term**.
- **Recursion** involves repeating the same calculation over and over, using the previous result to calculate the next result.
- Sequences can exhibit different behaviours such as increasing, decreasing, constant, oscillating or limiting values.

Exercise 3A

Building understanding

1 Fill in the boxes for the following sequences.

a $2, 5, \square, 11, \square, \ldots$

b $1, 3, 9, \square, 81, \ldots$

2 State the starting value, and determine the rule in the following sequences.

a $10, 8, 6, 4, 2, 0, -2, \ldots$

b $a, d, g, j, m, \ldots$

3 Consider the following sequence:

$$5, 11, 17, 23, 29, \ldots$$

a State the starting value of the sequence.

b Determine the rule for the sequence.

c If the sequence were to continue according to the rule, determine the value of the next three terms.

Developing understanding

Example 1 **4** Consider the following sequences, and identify their behaviour as increasing, decreasing, constant or oscillating. Also state whether the sequence has a limiting value.

a $4, 4, 4, 4, 4, \ldots$

b $1, 1, 2, 3, 5, \ldots$

c $100, 10, 100, 10, 100, \ldots$

d $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

e $-4, -6, -8, -10, -12, \ldots$

f $-20, -16, -12, -8, -4, \ldots$

Example 2 **5** Find the next term in each sequence.

a $3, 7, 11, 15, \ldots$

b $10, 9, 8, 7, \ldots$

c $1, 2, 1, 2, \ldots$

d $31, 24, 17, 10, \ldots$

e $16, 24, 32, 40, \ldots$

f $-2, 0, 2, -2, 0, \ldots$

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6 Find the next term in each sequence.

a 4, 8, 16, 32, . . . **b** 48, 24, 12, 6, . . . **c** 1, 3, 9, 27, . . .

d 243, 81, 27, 9, . . . **e** 1000, 500, 250, 125, . . . **f** 2, 6, 18, 54, . . .

7 Find the next term in each sequence.

a January, February, . . . **b** a, c, e, g, . . . **c** ♣, ♦, ♥, ♠, ♣, . . .

d ⇒, ⇐, ⇒, ⇐, . . . **e** Monday, Tuesday, . . . **f** ↑, →, ↓, ←, . . .

8 Describe how terms are generated in each number sequence, and give the next two terms.

a 5, 8, 11, 14, . . . **b** 19, 28, 37, 46, . . . **c** 38, 34, 30, 26, . . .

d 66, 58, 50, 42, . . . **e** 3, 6, 12, 24, . . . **f** 4, 12, 36, 108, . . .

g 128, 64, 32, 16, . . . **h** 3, –6, 12, –24, . . . **i** 1, 2, 3, 5, . . .

Example 3

9 Write down the first five terms of the sequence with a starting value of 3 and the rule 'add 2 to each term'.

10 Write down the first five terms of the sequence with a starting value of 90 and the rule 'subtract 6 from each term'.

11 Write down the first five terms of the sequence with a starting value of 5 and the rule 'multiply each term by 2, and then add 1'.

Testing understanding

12 Find the next term in each sequence.

a 1, 8, 27, 64, . . .

b 1, 2, 6, 24, . . .

c $-1, -2, -6, -24, -120, \ldots$

13 Describe how the terms are generated in each number sequence, and give the next two terms.

a 2, 4, 12, 48, . . .

b 1, 4, 9, 16, 25, . . .

c 1, 9, 25, 49, . . .

14 Consider the following sequence, and determine the next five terms.

$$1, -2, 4, -8, 16, \ldots$$

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3B Writing recurrence relations in symbolic form

Learning intentions

- To be able to number and name terms in a sequence.
- To be able to generate a sequence from a recurrence relation.

In this section, we will learn how to name and label each term to make them easier to refer to. We will also formalise how to express the starting value and rule for a sequence by writing down the **recurrence relation** for the sequence.

Numbering and naming the terms in a sequence

The symbols $t_0, t_1, t_2, \ldots$ are used as labels or names for the terms in the sequence. The numbers 0, 1, 2 are called *subscripts* which tell us how many times the rule has been applied. Because it signals the start of the sequence, t_0 is called the **starting** or **initial term** of the sequence. Note that this is different to taking the power of a term, such as $t^3 = t \times t \times t$.

For the sequence: 7, 3, 4, 11, 15, 24, . . . , we have:

Term number (n)	0	1	2	3	. . .
Term symbol	$t_0 = 7$	$t_1 = 3$	$t_2 = 4$	$t_3 = 11$	. . .
Term name	term 0	term 1	term 2	term 3	. . .

Example 4 Naming terms in a sequence

For the sequence: 2, 8, 14, 20, 26, 32, . . ., state the values of:

a t_1 **b** t_4 **c** t_5

Explanation

1 Write the name for each term under its value in the sequence.

2 Read the value of each required term.

Solution

2,	8,	14,	20,	26,	32
t_0	t_1	t_2	t_3	t_4	t_5

a $t_1 = 8$

b $t_4 = 26$

c $t_5 = 32$

Now try this 4 Naming terms in a sequence (Example 4)

For the sequence: 54, 50, 46, 42, 38, 34, 30, 26, . . ., state the values of:

a t_2 **b** t_5 **c** t_7

Hint 1 Remember that the initial term is t_0.
Hint 2 Write the name for each term under its value in the sequence.
Hint 3 Read off the value required for each term.

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Recurrence relations

A **recurrence relation** is a mathematical rule that is used to generate a sequence. It has two parts:

- a *starting point:* the value of the term at the start of the sequence
- a *rule*, that can be used to generate successive terms in the sequence.

For example, a recursion rule for the sequence: 2, 8, 14, 20, . . . , can be written as follows:

- Start with 2.
- To obtain the next term, add 6 to the current term, and repeat the process.

The recursion rule can be written in a more compact way using variables with subscripts.

Let t_n be the term in the sequence after n applications of the rule. Using this definition, the recurrence relation can be written as:

Starting value ($n = 0$)	Rule for generating the next term	Recurrence relation (Starting value and rule)
$t_0 = 2$	$t_{n+1} = t_n + 6$	$t_0 = 2, \quad t_{n+1} = t_n + 6$

Note that t can be replaced by any letter of the alphabet, and that each application of the rule is called an **iteration**.

Example 5 Generating a sequence from a recurrence relation

Write down the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 30, \qquad t_{n+1} = t_n - 5$$

Explanation	Solution
1 Write down the starting value.	$t_0 = 30$
2 Use the rule to find the next term, t_1.	$t_1 = 30 - 5 = 25$
3 Use the rule to determine three more terms.	$t_2 = t_1 - 5 = 25 - 5 = 20$ $t_3 = t_2 - 5 = 20 - 5 = 15$ $t_4 = t_3 - 5 = 15 - 5 = 10$
4 Write your answer.	The first five terms are: 30, 25, 20, 15, 10

Now try this 5 Generating a sequence from a recurrence relation (Example 5)

Write down the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 12, \qquad t_{n+1} = t_n + 5$$

Hint 1 Write down the starting value.
Hint 2 Use the rule to find the next term, t_1.
Hint 3 Use the rule to determine three more terms.

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Section Summary

- Terms in a sequence can be named and numbered, starting from zero and using subscripts, $t_0, t_1, t_2, \ldots$.
- A **recurrence relation** consists of both the starting value and a rule to generate successive terms in the sequence. For example,

$$t_0 = 2, \qquad t_{n+1} = t_n + 6$$

is the sequence with a starting value of 2 where each successive term is made by adding 6.

Exercise 3B

Building understanding

1 State the starting value, t_0, of the sequence 20, 19, 18, 17, ...

2 Consider the following sequence: $8, 4, 3, 11, 14, \ldots$.

a Rewrite the sequence.

b Write the name of each term below each term of the sequence.

3 Consider the following sequence: $9, 11, 13, 15, \ldots$.
Complete the following statements.

a The starting value of the sequence is ...

b To obtain the next term, add ... to the current term, and repeat the process.

Developing understanding

Example 4

4 Find the required terms from the sequence: 6, 11, 16, 21, 26, 31, 36 ...

a t_0 **b** t_3 **c** t_2 **d** t_4 **e** t_5 **f** t_6

5 For each sequence, state the value of the named terms:

i t_0 **ii** t_3 **iii** t_1

a 6, 10, 14, 18, ...

b 2, 8, 32, 128, ...

c 29, 22, 15, 8, ...

d 96, 48, 24, 12, ...

6 Find the required terms from the sequence: 8, 12, 16, 20, ...

a t_3 **b** t_2 **c** t_0 **d** t_4 **e** t_5 **f** t_8

7 For each sequence, state the value of the named terms:

i t_0 **ii** t_3 **iii** t_6

a 14, 20, 26, 32, ...

b 2, 6, 18, 54, ...

c 40, 32, 24, 16, ...

d 8000, 4000, 2000, 1000, ...

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Example 5

8 Write down the first five terms of the sequence defined by the following recurrence relations, showing the values of the first four iterations.

a $t_0 = 1, \quad t_{n+1} = t_n + 2$

b $V_0 = 100, \quad V_{n+1} = V_n - 10$

c $P_0 = 52, \quad P_{n+1} = P_n + 12$

9 Rewrite the following recursion relations in symbolic form, where V_n represents the value after n applications of the rule.

a The starting value is 3, and the rule is 'add 7 to the current term and repeat the process'.

b The starting value is 9, and the rule is 'add 4 to the current term and repeat the process'.

c The starting value is 16, and the rule is 'subtract 3 from the current term and repeat the process'.

10 Generate the first 5 terms for each of the sequences listed in the question above.

11 State the recurrence relation in symbolic form for the following sequences.

a $11, 15, 19, 23, \ldots$

b $43, 39, 35, 31, \ldots$

c $3, -1, -5, -9, \ldots$

Testing understanding

12 The following recurrence relation can generate a sequence of numbers.

$$T_0 = 20, \qquad T_{n+1} = T_n + 3$$

a State the term name for the term that has a value of 23.

b State the term name for the term that has a value of 32.

c State the term name for the term that has a value of 44.

13 The first five terms of a sequence are: 4, 9, 19, 34, 54, ...

a State the starting value of the sequence.

b Explain why the next term cannot be generated by adding or subtracting a value to or from the current term.

c Explain why the next term cannot be generated by multiplying or dividing the current term by a value.

d Determine the rule, in words, to generate the sequence.

e State the recurrence relation in symbolic form to generate the sequence, where V_n represents the value after n iterations.

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3C An introduction to arithmetic sequences

Learning intentions

- To be able to find the common difference in an arithmetic sequence.
- To be able to identify an arithmetic sequence.
- To be able to tabulate and graph an arithmetic sequence.

A sequence that can be generated by adding or subtracting a fixed amount to or from the previous term is called an **arithmetic sequence**. For example, the sequence 2, 6, 10, 14, 18, . . . is arithmetic because each successive term can be found by adding 4.

$$2 \xrightarrow{+4} 6 \xrightarrow{+4} 10 \xrightarrow{+4} 14 \xrightarrow{+4} 18$$

Other examples include:

Sequence	Rule
2, 7, 12, . . .	'to find the next term in the sequence, add 5 to the current term'
200, 300, 400, . . .	'to find the next term in the sequence, add 100 to the current term'
50, 45, 40, . . .	'to find the next term in the sequence, subtract 5 from the current term'
100, 99, 98, . . .	'to find the next term in the sequence, subtract 1 from the current term'

The common difference

The fixed amount we add or subtract to form an arithmetic sequence recursively is called the **common difference**. The symbol D is often used to represent the common difference.

If the sequence is *known* to be arithmetic, the common difference can be calculated by simply computing the difference between any pair of successive terms.

Common difference, D

In an **arithmetic sequence**, the fixed number added to (or subtracted from) each term to make the next term is called the **common difference**, D, where:

$$D = \text{any term} - \text{previous term}$$

For example, the common difference for the arithmetic sequence: 30, 25, 20, . . . is:

$$D = t_1 - t_0 = 25 - 30 = -5.$$

Often it is not necessary to calculate the common difference in this formal way. It may be easy to see what amount has been repeatedly added (or subtracted) to make each new term.

Example 6 **Finding the common difference in an arithmetic sequence**

Find the common difference in the following arithmetic sequences and use it to find the next term in each of the sequences below:

a 2, 5, 8, . . . **b** 25, 23, 21, . . .

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Explanation	Solution
1 Because we know the sequence is arithmetic, all we need to do is find the difference in value between term t_0 and t_1.	**a** $D = t_1 - t_0 = 5 - 2 = 3$
2 To find t_3, add the common difference to t_2.	$t_3 = t_2 + D = 8 + 3 = 11$
	b $D = t_1 - t_0 = 23 - 25 = -2$
	$t_3 = t_2 + D = 21 + (-2) = 19$

Now try this 6 **Finding the common difference in an arithmetic sequence (Example 6)**

Find the common difference in the following arithmetic sequence and use it to find the next term in the sequence.

31, 27, 23, 19, 15, 11, . . .

Hint 1 Find the common difference by finding the difference between two consecutive terms.

Hint 2 Apply the common difference to the last term to find the next term.

Identifying arithmetic sequences

If a sequence is arithmetic, the difference between successive terms will be constant. We can use this idea to see whether or not a sequence is arithmetic.

Example 7 **Identifying an arithmetic sequence**

Which of the following sequences is arithmetic?

a 21, 28, 35, 42, . . . **b** 2, 6, 18, 54, . . .

Explanation	Solution
a 1 Determine whether the difference between successive terms is constant.	21, 28, 35, 42, . . . Differences: 28 − 21 = 7 35 − 28 = 7 42 − 35 = 7
2 Write your conclusion.	As the differences between successive terms are constant, the sequence is arithmetic.
b 2, 6, 18, 54, . . .	
1 Determine whether the difference between successive terms is constant.	Differences: 6 − 2 = 4 18 − 6 = 12 54 − 18 = 36
2 Write your conclusion.	As the differences between successive terms are not constant, the sequence is not arithmetic.

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Now try this 7 **Identifying an arithmetic sequence (Example 7)**

Which of the following sequences is arithmetic?

a 2, 4, 8, 16, . . .

b 58, 53, 48, 43, . . .

Hint 1 Determine the difference between successive terms.

Hint 2 If the difference is constant (the same), then the sequence is arithmetic.

Tables and graphs of arithmetic sequences

The terms of a sequence can be tabulated, highlighting that each input (n value) has a corresponding value (t_n). This tells us that a sequence can be thought of as a **function**. For example, the sequence 3, 6, 9, 12, 15, . . . can be tabulated as shown below:

Term number, n	0	1	2	3	4	5	6	7	8	9
Term value, t_n	3	6	9	12	15	18	21	24	27	30

If we plot the values of the terms of an arithmetic sequence (t_n) against their number (n) or the number of applications of the rule, we will find that the points lie on a straight line. We could anticipate this because, as we progress through the sequence, the value of successive terms increases by the same amount (the common difference, D).

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The advantage of graphing a sequence is that the straight line required for an arithmetic sequence is immediately obvious, and any exceptions would stand out very clearly. An upward slope indicates **linear growth** and a downward slope reveals **linear decay**.

Example 8 Graphing the terms of an increasing arithmetic sequence ($D > 0$)

The sequence 4, 7, 10, . . . is arithmetic with common difference $D = 3$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Explanation

a Show the term numbers and values of the first four terms in a table.

1 Write the term numbers in the top row of the table.

2 Write the values of the terms in the bottom row.

b Use the table to plot the graph.

1 Use the horizontal axis, n, for the term numbers.
Use the vertical axis for the value of each term, t_n.

2 Plot each point from the table.

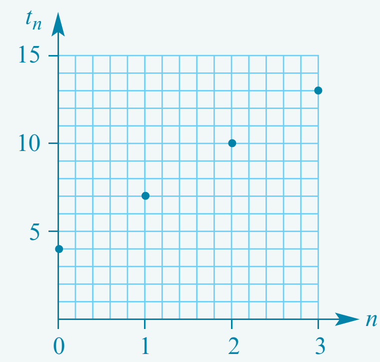
c Describe the graph.

1 Are the points along a straight line or a curve?

2 Is the line of the points rising (positive slope) or falling (negative slope)?

Solution

Term number, n	0	1	2	3
Term value, t_n	4	7	10	13



The points of an arithmetic sequence with $D = 3$ lie along a rising straight line.
The line has a positive slope.

Now try this 8 Graphing the terms of an increasing arithmetic sequence ($D > 0$) (Example 8)

The sequence 2, 8, 14, . . . is arithmetic with common difference $D = 6$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Hint 1 Plot each of the points on the graph carefully.

Hint 2 If a point does not lie on the straight line, check your work carefully.

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Example 9 Graphing the terms of a decreasing arithmetic sequence ($D < 0$)

The sequence 9, 7, 5, . . . is arithmetic with common difference $D = -2$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Explanation

a Show the term numbers and values of the first four terms in a table.

1 Write the term numbers in the top row of the table.

2 Write the values of the terms in the bottom row.

b Use the table to plot the graph.

1 Use the horizontal axis, n, for the term numbers.
Use the vertical axis for the value of each term, t_n.

2 Plot each point from the table.

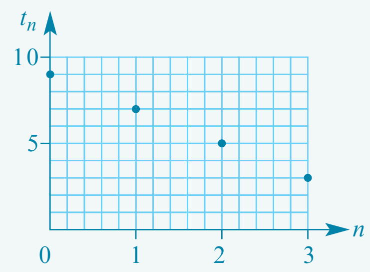
c Describe the graph.

1 Are the points along a straight line or a curve?

2 Is the line of the points rising (positive slope) or falling (negative slope)?

Solution

Term number, n	0	1	2	3
Term value, t_n	9	7	5	3



The points of an arithmetic sequence with $D = -2$ lie along a falling straight line.
The line has a negative slope.

Now try this 9 Graphing the terms of a decreasing arithmetic sequence ($D < 0$) (Example 9)

The sequence 10, 7, 4, . . . is arithmetic with common difference $D = -3$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Hint 1 Plot each of the points on the graph carefully.

Hint 2 If a point does not lie on the straight line, check your work carefully.

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Graphs of arithmetic sequences are:

- points along a line with positive slope, when a constant amount is added ($D > 0$)
- points along a line with negative slope, when a constant amount is subtracted ($D < 0$).

A line with **positive** slope rises from left to right. A **negative** slope falls from left to right.

Section Summary

- A sequence is **arithmetic** if it is generated by adding or subtracting a fixed amount to or from the current term.
- In an arithmetic sequence, the **common difference**, D, is the difference between any two successive terms.
- Graphs of arithmetic sequences are:
  - points along a line with positive slope (rising from left to right), when a constant amount is added ($D > 0$),
  - points along a line with negative slope (falling from left to right), when a constant amount is subtracted ($D < 0$).

Skill-sheet

Exercise 3C

Building understanding

1 What is the value of term t_3 in each of the following sequences:

a $2, 4, 6, 8, \ldots$

b $1, 4, 9, 16, \ldots$

c $51, 48, 45, 42, \ldots$

2 For each of the following sequences, find the difference between t_0 and t_1 and the difference between t_1 and t_2.

a $11, 15, 19, 23, 27, 31, \ldots$

b $3, 1, -1, -3, -5, \ldots$

3 Consider the following sequence:

$$4, 11, 18, 25, 32, \ldots$$

a Copy down the sequence and label each term.

b State the value of t_3.

c Determine the difference between each pair of consecutive terms, and decide if the sequence is arithmetic.

d If the sequence were to continue, determine the value of t_5 and t_6.

Developing understanding

Example 6

4 For each of these arithmetic sequences, find the common difference and the value of t_3.

a $5, 11, 17, 23, \ldots$ **b** $17, 13, 9, 5, \ldots$ **c** $11, 15, 19, 23, \ldots$

d $8, 4, 0, -4, \ldots$ **e** $35, 30, 25, 20, \ldots$ **f** $1.5, 2, 2.5, 3, \ldots$

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5 Give the next two terms in each of these arithmetic sequences.

a 17, 23, 29, 35, ... **b** 14, 11, 8, 5, ... **c** 2, 1.5, 1.0, 0.5, ...
d 27, 35, 43, 51, ... **e** 33, 21, 9, −3, ... **f** 0.8, 1.1, 1.4, 1.7, ...

Example 7

6 Find out which of the sequences below is arithmetic. Give the common difference for each sequence that is arithmetic.

a 8, 11, 14, 17, ... **b** 7, 15, 22, 30, ... **c** 11, 7, 3, −1, ...
d 12, 9, 6, 3, ... **e** 16, 8, 4, 2, ... **f** 1, 1, 1, 1, ...

7 Use your CAS calculator to generate the first five terms of the following arithmetic sequences.

a 3, 8, ... **b** 16, 9, ... **c** 1.6, 3.9, ...
d 8.7, 5.6, ... **e** 293, 226, ...

8 Using your CAS calculator:

a generate the first 7 terms of the arithmetic sequence 1, 6, ... and write down t_6.
b generate the first 13 terms of the arithmetic sequence 45, 43, ... and write down t_{12}.
c generate the first 11 terms of the arithmetic sequence 15, 14, ... and write down t_{10}.
d generate the first 16 terms of the arithmetic sequence 0, 3, ... and write down t_{15}.

9 Initially, Fumbles Restaurant had 320 wine glasses. After one week, they only had 305 wine glasses. On average, 15 glasses are broken each week.
Using your CAS calculator, determine how many weeks it takes at that breakage rate for there to be only 200 glasses left?

10 Elizabeth intially had 100 songs on her Favourites playlist. Each month she added seven more songs.
Using your CAS calculator:

a determine the number of songs she had in her Favourites playlist after each of the first 4 months.
b determine the number of songs she had in her Favourites playlist by the end of the year.

Example 8
Example 9

11 For the following arithmetic sequences:

a 3, 5, 7, ...
b 11, 8, 5, ...

i write down the next term.
ii write down the first four terms in a table, indicating the value n associated with each term.
iii use the table to plot a graph.
iv describe the graph.

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Testing understanding

12 Find the missing terms in the following arithmetic sequences.

a 8, 13, 18, 23, □, □, . . .
b 14, 8, 2, −4, □, □, . . .
c 6, 15, □, □, 42, . . .
d 23, 18, □, □, 3, −2, . . .
e 3, □, □, 27, 35, 43, . . .
f □, □, 29, 37, 45, 53, . . .
g □, □, 7, −4,−15, −26, . . .
h 36, □, 22, □, 8, 1, . . .
i 15, □, 31, □, 47, □, . . .

13 Consider the following sequence:

$$-2, -5, -8, -11, \ldots$$

a Calculate the common difference, D, for the arithmetic sequence.
b If you were to calculate t_{100}, how much would you need to add or subtract from the starting term?
c Determine t_{100}.
d Determine t_{200}.

3D Arithmetic sequences using recursion

Learning intentions

- To be able to generate an arithmetic sequence using a recurrence relation.
- To be able to use the rule for the nth term to solve problems involving arithmetic sequences.

Using a recurrence relation to generate and analyse an arithmetic sequence

Consider the arithmetic sequence below:

$$10, 15, 20, 25, 30, \ldots$$

In words, the recursion relation that can be used to generate this sequence is:

'start the sequence with 10'

'to find the next term, add 5 to the current term, and keep repeating the process'.

Labelling the terms $t_0, t_1, t_2, \ldots$ and following this process we have:

$t_0 = 10$	initial or starting value
$t_1 = 10 + 5 = 15$	after 1 application of the rule
$t_2 = 15 + 5 = 20$	after 2 applications of the rule
$t_3 = 20 + 5 = 25$	after 3 applications of the rule
$t_4 = 25 + 5 = 30$	after 4 applications of the rule

and so on until we have the rule $t_{n+1} = t_n + 5$ after n applications of the rule.

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The **recurrence relation** is a precise and compact way of expressing the starting value and rule that generates this sequence. For this example,

$$t_0 = 10, \qquad t_{n+1} = t_n + 5 \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

The recurrence relation for generating an arithmetic sequence

The recurrence relation:

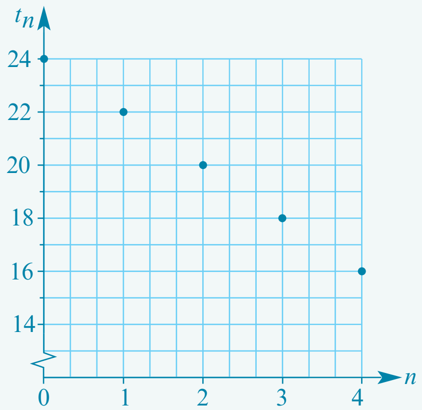
$$t_0 = a, \qquad t_{n+1} = t_n + D$$

can be used to generate an arithmetic sequence with first term $t_0 = a$ and common difference, D.

Example 10 Using a recurrence relation to generate an arithmetic sequence

Generate and graph the first five terms of the arithmetic sequence defined by the recurrence relation:

$$t_0 = 24, \qquad t_{n+1} = t_n - 2$$

Explanation	Solution
1 Write down the recurrence relation.	$t_0 = 24, \qquad t_{n+1} = t_n - 2$
2 Write down the starting term.	$t_0 = 24$
3 Use the rule, which translates into 'to find the next term, subtract two from the previous term', to generate the first five terms in the sequence.	$t_1 = t_0 - 2 = 24 - 2 = 22$ $t_2 = t_1 - 2 = 22 - 2 = 20$ $t_3 = t_2 - 2 = 20 - 2 = 18$ $t_4 = t_3 - 2 = 18 - 2 = 16$
4 To graph the terms, plot t_n against n.	

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Now try this 10 **Using a recurrence relation to generate an arithmetic sequence (Example 10)**

Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 7, \qquad t_{n+1} = t_n + 3$$

Hint 1 Write down the starting term and then use the rule to find the next four terms.

Hint 2 To graph the terms, plot t_n against n.

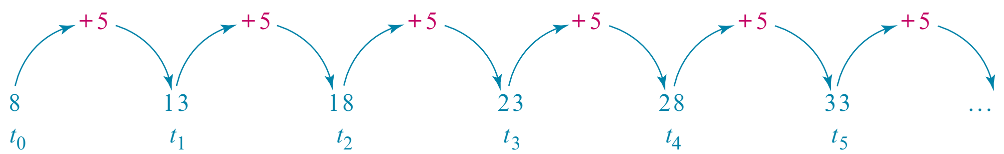
Finding the *n*th term in an arithmetic sequence

Repeated addition can be used to find each new term in an arithmetic sequence, but this process is very tedious for finding a term such as t_{50}. Instead, a general rule can be found to calculate any term, t_n, using n, the number of times the recursion rule is applied, the value of the first term, a, and the common difference, D.

Consider the arithmetic sequence: 8, 13, 18, 23, 28, 33, . . . which is defined by:

$t_0 = 8, \qquad t_{n+1} = t_n + 5 \qquad \text{for } n = 0, 1, 2, 3, \ldots.$

This is illustrated pictorially in the diagram below.



Using the information from the diagram, we can write recursively:

$t_0 = 8 + 0 \times 5 = 8$ after 0 applications of the rule

$t_1 = 8 + 1 \times 5 = 13$ after 1 application of the rule

$t_2 = 8 + 2 \times 5 = 18$ after 2 applications of the rule

$t_3 = 8 + 3 \times 5 = 23$ after 3 applications of the rule.

A pattern emerges which suggests that after n applications of the recursion rule,

$t_n = 8 + n \times 5$ after n applications of the rule.

Using this rule, t_n can be found without having to find all previous values in the sequence.

Thus, using the rule:

$$t_{50} = 8 + 50 \times 5 = 258$$

This rule can be generalised to apply to any situation involving the recursive generation of an arithmetic sequence.

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Rule for finding the *n*th term of an arithmetic sequence

The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = t_n + D$$

can be used to generate an arithmetic sequence with a starting value, $t_0 = a$, and a common difference, D.

The rule for directly calculating the term t_n in this sequence is generated from the recurrence relation:

$$t_n = a + nD$$

where n is the term number, $n = 0, 1, 2, 3, \ldots$.

Example 11 Finding term *n* of an arithmetic sequence

Consider the recurrence relation:

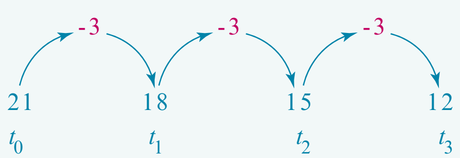
$$t_0 = 21, \qquad t_{n+1} = t_n - 3$$

Find t_{20}.

Explanation

1 State the initial value, a, and the common difference, D.

2 The term t_{20} requires 20 applications of the rule.



3 Substitute the values of a, D and n into $t_n = a + nD$.

4 Evaluate.

5 Write your answer.

Solution

$a = 21, D = -3$

$n = 20$

$$\begin{aligned} t_n &= a + nD \\ t_{20} &= 21 + 20(-3) \\ &= -39 \end{aligned}$$

$t_{20} = -39$

Now try this 11 Finding term *n* of an arithmetic sequence (Example 11)

Consider the recurrence relation:

$t_0 = 54, \qquad t_{n+1} = t_n + 7.$

Find t_{50}.

Hint 1 State the initial term of the sequence, a.

Hint 2 State the common difference, D.

Hint 3 Use the rule to find t_{50} ($n = 50$).

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Section Summary

- The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = t_n + D$$

can be used to generate an arithmetic sequence with starting value $t_0 = a$ and common difference, D.

- The rule for directly calculating term t_n in this sequence is:

$$t_n = a + n \times D$$

where n is the term number $0, 1, 2, 3, \ldots$.

Skill-sheet

Exercise 3D

Building understanding

1 Give the value of a and D in each of the following arithmetic sequences.

a $7, 11, 15, 19, \ldots$ **b** $8, 5, 2, -1, \ldots$ **c** $14, 23, 32, 41, \ldots$
d $62, 35, 8, -19, \ldots$ **e** $-9, -4, 1, 6, \ldots$ **f** $-13, -19, -25, \ldots$

2 For each of the arithmetic sequences in Question 1, use the value of a and D that you found and the rule $t_n = a + n \times D$ to find the value of t_5.

3 For the following sequence, state the value of a, D and n required to find t_{20}.

$$12, 22, 32, 42, \ldots$$

4 For the recurrence relation: $t_0 = 7, t_{n+1} = t_n + 3$, complete the blanks using the rule: $t_n = t_0 + n \times D$ to find the value of t_2, t_3, t_4 and t_{20}.

$$t_1 = t_0 + 1 \times 3 = 7 + 1 \times 3 = 10$$
$$t_2 = t_0 + \ldots \times 3 = \ldots + \ldots \times 3 = \ldots$$
$$t_3 = t_0 + \ldots \times 3 = \ldots + \ldots \times 3 = \ldots$$
$$t_{20} = t_0 + \ldots \times 3 = \ldots + \ldots \times 3 = \ldots$$

Developing understanding

Example 10

5 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:
$t_0 = 15, \qquad t_{n+1} = t_n + 5 \qquad$ where $n \geq 0$.
b Calculate the value of t_{44} in the sequence.

6 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:
$t_0 = 60, \qquad t_{n+1} = t_n - 5 \qquad$ where $n \geq 0$.
b Calculate the value of t_{10} in the sequence.

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7 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:
$t_0 = 15, \quad t_{n+1} = t_n + 35 \quad$ where $n \geq 0$.

b Calculate the value of t_{15} in the sequence.

Example 11

8 Find the required term in each of these arithmetic sequences.

a 18, 21, 24, 27, ... Find t_{35}.
b −14, −6, 2, 10, ... Find t_{41}.
c 27, 14, 1, −12, ... Find t_{37}.
d 16, 31, 46, 61, ... Find t_{29}.
e −19, −23, −27, −31, ... Find t_{26}.
f 0.8, 1.5, 2.2, 2.9, ... Find t_{36}.
g 82, 68, 54, 40, ... Find t_{21}.
h 9.4, 8.8, 8.2, 7.6, ... Find t_{29}.

9 Find t_{40} in an arithmetic sequence that starts at 11 and has a common difference of 8.

10 The first term in an arithmetic sequence is 27 and the common difference is 19. Find t_{100}.

11 A sequence started at 100 and had 7 subtracted each time to make new terms. Find t_{20}.

Testing understanding

12 When planted, a tree was initially 1.50 m high. It grew 0.75 m in each subsequent year. How high was the tree 18 years after it was planted?

13 In an arithmetic sequence, $t_4 = 10$ and $t_8 = 18$.

a Using the equation $t_n = a + nD$, substitute in t_4 and $n = 4$ to form an equation in terms of a and D.

b Using the equation $t_n = a + nD$, substitute in t_8 and $n = 8$ to form an equation in terms of a and D.

c Use simultaneous equations to solve the two equations you found in part **a** and **b** to find a and D.

d Check your answer by finding t_5 and t_9 using the rule and the values of a and D.

e Hence, write down the first three terms of the arithmetic sequence.

14 Consider the following arithmetic sequence:

$$5, 7, 9, 11, \ldots$$

How many terms in this sequence are less than 25?

3E Finance applications using arithmetic sequences and recurrence relations

Learning intentions

- ► To be able to use a recurrence relation to model simple interest.
- ► To be able to use a recurrence relation to model flat rate depreciation.
- ► To be able to use a recurrence relation to model unit cost depreciation.
- ► To be able to use a rule to determine the nth term for linear growth or decay.

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This section is concerned with the use of arithmetic sequences and recurrence relations to model simple interest, flat rates and unit cost depreciating value of assets. The skill sheet available for this section through the Interactive Textbook also contains non-financial applications.

Using recurrence relations to model simple interest

Linear growth in a sequence occurs when a quantity increases by the same amount at regular intervals, for example, the payment of simple interest on an investment or the amount that is owed on a simple-interest loan.

Simple interest

Simple interest is usually determined by multiplying the annual **interest rate**, $r\%$, by the original amount borrowed (invested) called the **principal**, P, for each year of the loan.

The value after $n + 1$ years is the value after n years plus the interest in the previous year. This gives the recurrence relation:

$$V_0 = P, \qquad V_{n+1} = V_n + D$$

where V_n is the value after n years and $D = \frac{r}{100} \times V_0$.

Note: Remember that the interest rate is a percentage, so we write this as $r\% = \frac{r}{100}$.

Simple interest is usually applied to smaller loans, such as car loans, or investments that are short term. Compound interest, discussed later in this chapter, is used for larger loans over a long period, such as for mortgages.

Example 12 Finding the amount of simple interest each year

An investment of \$3500 pays interest at the rate of 4.2% per annum in the form of simple interest. Find the amount of interest paid each year.

Explanation	Solution
1 Define the symbol V_n in this model.	V_n is the value of the investment after n years.
2 Write down the value of V_0, the principal of the investment.	$V_0 = 3500$
3 Write down the interest rate and use it to calculate the value of $D = \frac{r}{100} \times V_0$.	$r = 4.2\%$ $D = \frac{4.2}{100} \times 3500 = \$147$
4 Determine the interest paid each year.	Thus, each year, the investment earns \$147 in simple interest.

Now try this 12 Finding the amount of simple interest each year (Example 12)

An investment of \$4600 pays interest at the rate of 5.1% per annum in the form of simple interest. Find the amount of interest paid each year.

Hint 1 State the value of the annual interest rate, r, and the principal, P.

Hint 2 Calculate $D = \frac{r}{100} \times V_0$.

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If the investment was for part of the year, for example, 6 months, then we would multiply $\frac{r}{100} \times Vo$ by the fraction of the year required, for example, $\frac{1}{2}$. In Example 12, interest over 6 months would then be \$73.50.

Example 13 Using a recurrence relation to model linear growth: simple interest

The following recurrence relation can be used to model a simple interest investment of \$2000, paying interest at the rate of 7.5% per annum:

$$V_0 = 2000, \qquad V_{n+1} = V_n + 150$$

where V_n is the value of the investment after n years.

Note: The amount of interest that is paid each year is found by multiplying the annual interest rate by the principal:

$$\frac{r}{100} \times V_0 = \frac{7.5}{100} \times 2000 = 150$$

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$2750 in value?

Explanation

a 1 Write down the recurrence relation. The recurrence relation tells you that: 'to find the next value, add 150 to the current value'.

2 With $V_0 = 2000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

This can also be done on your CAS calculator as shown opposite.

b This question is best answered using your CAS calculator to generate the values of successive terms and counting the number of steps (years) until the value of the investment is \$2750.

Write your conclusion.

Note: Count the number of times 150 has been added, *not* the number of terms on the calculator screen.

Solution

a 1 $V_0 = 2000, \qquad V_{n+1} = V_n + 150$

2
$V_0 = 2000$
$V_1 = V_0 + 150 = 2000 + 150 = \$2150$
$V_2 = V_1 + 150 = 2150 + 150 = \$2300$
$V_3 = V_2 + 150 = 2300 + 150 = \$2450$

2000	2000
2000 + 150	2150
2150 + 150	2300
2300 + 150	2450

b

2000	2000
2000 + 150	2150
2150 + 150	2300
2300 + 150	2450
2450 + 150	2600
2600 + 150	2750

The investment will have a value of \$2750 after 5 years.

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Now try this 13 **Using a recurrence relation to model linear growth: simple interest (Example 13)**

The following recurrence relation can be used to model a simple interest investment of \$3000, paying interest at the rate of 5.2% per annum.

$$V_0 = 3000, \qquad V_{n+1} = V_n + 156$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$4092 in value?

Hint 1 Make sure you understand the recurrence relation and how 156 was calculated.

Hint 2 Use the starting value and the recurrence relation to generate the terms V_1, V_2 and V_3.

Hint 3 To find when the investment will reach the required value, continue to add \$156, and then count the number of times that it was added to the initial starting value.

Using recurrence relations to model flat rate depreciation

Like linear growth, linear decay can be represented as a recurrence relation. For example, a car that is purchased for \$80 000 will be worth far less in a few years' time. One way that depreciation is calculated is based on the age of the asset. This is called **flat rate** or **fixed rate** depreciation. The value of an asset, V_n, in period n, reduces in value by a fixed amount, D, each period. The fixed amount may be given or it may be calculated based on a percentage rate, r, of the original value, V_0. Thus, $D = \frac{r}{100} \times V_0$.

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Example 14 Using a recurrence relation to model linear decay: flat rate depreciation

The following recurrence relation can be used to model the flat rate depreciation of a car purchased for \$18 500, depreciating at a flat rate of 10% per year.

$$V_0 = 18\,500, \qquad V_{n+1} = V_n - 1850$$

In the recurrence relation, V_n is the value of the car after n years.

Note: The car depreciates \$1850 per year since the annual interest rate ($r = 10\%$) multiplied by the principal ($P = \$18\,500$) is $\frac{10}{100} \times \$18\,500 = \$1850$.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b How long will it take for the car's value to depreciate to zero?

Explanation

a 1 Write down the recurrence relation. The recurrence relation tells you that: 'to find the next value, subtract 1850 from the current value'.

Solution

$V_0 = 18\,500, \quad V_{n+1} = V_n - 1850$

2 With $V_0 = 18\,500$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

$V_0 = 18\,500$

$V_1 = V_0 - 1850 = 18\,500 - 1850 = \$16\,650$

$V_2 = V_1 - 1850 = 16\,650 - 1850 = \$14\,800$

$V_3 = V_2 - 1850 = 14\,800 - 1850 = \$12\,950$

This can also be done on your CAS calculator.

18500	18500
18500 – 1850	16650
16650 – 1850	14800
14800 – 1850	12950

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b This question is best answered using your calculator and counting the number of steps (years) until the depreciated value of the car is \$0.

18500	18500
18500 – 1850	16650
16650 – 1850	14800
14800 – 1850	12950
12950 – 1850	11100
11100 – 1850	9250
9250 – 1850	7400
7400 – 1850	5550
5550 – 1850	3700
3700 – 1850	1850
1850 – 1850	0

Write your conclusion.

Note: Count the number of times 1850 has been subtracted, *not* the number of terms on the calculator screen.

The car will have a value of zero after 10 years.

Now try this 14 **Using a recurrence relation to model linear decay: flat rate depreciation (Example 14)**

The following recurrence relation can be used to model the flat rate depreciation of office furniture, purchased for \$3700, depreciating at a flat rate of 7% per year.

$$V_0 = 3700, \quad V_{n+1} = V_n - 259$$

In the recurrence relation, V_n is the value of the office furniture after n years.

a Use the recurrence relation to find the value of the office furniture after 1, 2 and 3 years.

b How long will it take for the office furniture's value to depreciate to less than \$1000?

Hint 1 Use the starting value and the recurrence relation to generate the terms V_1, V_2 and V_3.

Hint 2 To find when the investment will reach the required value, continue to subtract \$259, and then count the number of times that it was subtracted from the initial starting value.

Using recurrence relations to model unit cost depreciation

A second way that depreciation can be calculated is based on how often the asset is used, rather than based on its age. In the instance of a car, the value of the car may depreciate based on the number of kilometres that the car has travelled. This is called **unit cost depreciation**. In this method, V_n represents the value of the asset after n uses, and D is the cost per unit of each use.

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Example 15 Using a recurrence relation to model linear decay: unit cost depreciation

A car with a purchase price of \$32 000 depreciates at a unit cost of \$150 per 1000 kilometres.

a State the recurrence relation for unit cost depreciation of the car, where V_n represents the value of the car after n thousand kilometres.

b Use the recurrence relation to find the value of the car after 1000, 2000 and 3000 kilometres.

c How many kilometres is the car expected to travel before its value depreciates by at least \$1000?

Explanation

a 1 State the starting value and the common difference for each unit of 1000 kilometres.

Solution: $V_0 = 32\,000, \quad D = 150$

2 Use the common difference, $D = 150$, and the starting value, $V_0 = \$32\,000$, to write down the recurrence relation.

Solution: $V_0 = 32\,000, \quad V_{n+1} = V_n - 150$

b With $V_0 = 32\,000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

Solution:

$V_0 = \$32\,000$

$V_1 = V_0 - 150 = 32\,000 - 150 = \$31\,850$

$V_2 = V_1 - 150 = 31\,850 - 150 = \$31\,700$

$V_3 = V_2 - 150 = 31\,700 - 150 = \$31\,550$

This can also be done on your CAS calculator.

32000	32000
32000 – 150	31850
31850 – 150	31700
31700 – 150	31550

c This question is best answered using your calculator and counting the number of steps until the depreciated value of the car is less than \$31 000.

32000	32000
32000 – 150	31850
31850 – 150	31700
31700 – 150	31550
31550 – 150	31400
31400 – 150	31250
31250 – 150	31100
31100 – 150	30950

Write your conclusion.

Solution: The car will have depreciated by at least \$1000 after 7000 kilometres.

Note: Count the number of times 150 has been subtracted, *not* the number of terms on the calculator screen.

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Now try this 15 **Using a recurrence relation to model linear decay: unit cost depreciation (Example 15)**

An office printer purchased for \$9200 depreciates at a unit cost of \$15 per 100 000 sheets printed.

a State the recurrence relation for unit cost depreciation for the office printer, where V_n represents the value of the office printer after n hundred thousand sheets have been printed.

b Use the recurrence relation to find the value of the office printer after 100 000, 200 000 and 300 000 sheets.

c How many sheets can be printed before the office printer's value depreciates to less than \$9000?

Hint 1 Find the starting value and common difference for the office printer to state the recurrence relation.

Hint 2 Use the starting value and the recurrence relation to generate the terms V_1, V_2 and V_3.

Hint 3 To find when the printer will reach the required value, continue to subtract the common difference and then count the number of times that it was subtracted from the initial starting value.

A rule for term *n* in a sequence modelling linear growth or decay recursively

While we can generate as many terms as we like in a sequence using a recurrence relation for linear growth and decay, it is possible to derive a rule for calculating any term in the sequence directly, using the rule established earlier.

For example, if \$1000 is invested in a simple-interest investment paying 5% interest per annum, the value of the investment increases by \$50 per year. This means that the value of the investment, V_n, after n years is $V_n = 1000 + n \times 50$.

This rule can be readily generalised to apply to any situation involving linear growth or decay, as follows:

Rule for term *n* in a sequence used to model linear growth or decay

Let V_n be the value of the nth term of the sequence used to model linear growth or decay.

The value of the nth term in this sequence generated by the recurrence relation:

$$V_0 = \text{starting or initial value}, \quad V_{n+1} = V_n + D$$

is given by:

$$V_n = V_0 + n \times D$$

For linear growth, $D > 0$.

For linear decay, $D < 0$.

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Example 16 Using a rule for determining the *n*th term for linear growth or decay

The following recurrence relation can be used to model a simple-interest investment of $4000, paying interest at the rate of 6.5% per year.

$$V_0 = 4000, \qquad V_{n+1} = V_n + 260$$

a How much interest is added to the investment each year?

b Use a rule to find the value of the investment after 15 years.

c Use a rule to find when the value of the investment first exceeds $10 000.

Explanation

a This value can be read directly from the recurrence relation or calculated by finding 6.5% of $4000 = 0.065 × 4000 = $260.

b Because it is linear growth, use the rule: $V_n = V_0 + n \times D$
Here $V_0 = 4000$, $n = 15$ and $D = 260$.

c Substitute $V_n = 10\,000$, $V_0 = 4000$ and $D = 260$ into the rule: $V_n = V_0 + n \times D$, and solve for n.

Write your conclusion.

Note: Because the interest is only paid into the account after a whole number of years, any decimal answer will need to be *rounded up* to the next whole number.

Solution

a $260

b
$$V_n = V_0 + n \times D$$
$$V_{15} = 4000 + 15 \times 260$$
$$= \$7900$$

c
$$10\,000 = 4000 + n \times 260$$
$$6000 = n \times 260$$
$$n = \frac{6000}{260}$$
$$= 23.07\ldots \text{ years}$$

The value of the investment will first exceed $10 000 after 24 years.

Now try this 16 Using a rule for determining the *n*th term for linear growth or decay (Example 16)

The following recurrence relation can be used to model a simple-interest investment of $12 000, paying interest at the rate of 4.8% per year.

$$V_0 = 12\,000, \qquad V_{n+1} = V_n + 576$$

a How much interest is added to the investment each year?

b Use a rule to find the value of the investment after 10 years.

c Use a rule to find when the value of the investment first exceeds $20 000.

Hint 1 To find the amount of interest each year, you should find 4.8% of the initial investment, $12 000.

Hint 2 Construct the rule using the initial value, the common difference and the value of $n = 10$, to find the value of the investment after 10 years.

Hint 3 To find the value of n, substitute in the value $V_n = 20\,000$.

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Section Summary

- **Linear growth and decay** can be modelled by V_0 = initial or starting value, $V_{n+1} = V_n + D$, where V_n is the value after n years and D is the amount that is added each time period. Thus, $V_n = V_0 + n \times D$. For linear growth, $D > 0$. For linear decay, $D < 0$.
- **Simple interest** is an example of linear growth, where a fixed amount of interest is earned each period and found by multiplying the interest rate, r, by the initial amount, V_0. That is, $D = \frac{r}{100} \times V_0$.
- **Linear depreciation** is an example of linear decay, where the value of the asset declines by a fixed amount. It can be calculated based on the age of the asset (**flat rate** or **fixed rate**) or the use of the asset (**unit cost**).

Skill-sheet

Exercise 3E

Building understanding

1 An investment of \$5000 is made that pays interest of 4% per annum. How much interest does the investment pay each year?

2 A computer initially cost \$3000 but depreciates at a flat rate of 12% per year. How much does the computer depreciate by each year?

3 Let V_n be the value of term n in the sequence used to model linear growth or decay, and D be the common difference.
A recurrence relation for simple interest is given by V_0 = starting value and $V_{n+1} = V_n + D$.
Using this recurrence relation, term n in the sequence can be given by:

$$V_n = \ldots + \ldots \times D$$

Developing understanding

Example 12 4 Find the amount of simple interest that is paid each year for the following investments.

a \$10 000 investment at 7.3% per annum.
b \$16 500 investment at 3.8% per annum.
c \$214 600 investment at 5.4% per annum.

Example 13 5 The following recurrence relation can be used to model a simple-interest investment of \$10 000, paying interest at the rate of 4.5% per year.

$$V_0 = 10\,000, \qquad V_{n+1} = V_n + 450$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.
b When will the investment reach \$14 500 in value?

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6 The following recurrence relation can be used to model a simple-interest investment.

$$V_0 = 8000, \qquad V_{n+1} = V_n + 400$$

In the recurrence relation, V_n is the value in dollars of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$12 000 in value?

c How much was invested at the start?

d What was the interest rate?

Example 14 **7** A tractor costs \$90 000 when new. Its value depreciates at a flat rate of 10% or \$9000 per year.

Let V_n be the value (in dollars) of the tractor after n years.

A recurrence relation that models the depreciating value of this tractor over time is:

$$V_0 = 90\,000, \qquad V_{n+1} = V_n - 9000$$

a Use the recurrence relation to find the value of the tractor after three years.

b The tractor will be sold after 5 years. How much will it be worth then?

c If the tractor continues to depreciate at the same rate for the rest of its life, how many years will it take to have zero value?

d A different model of tractor costs \$95 000 and depreciates at a flat rate of 12% of its original value per year. Write down a recurrence relation to model this situation.

8 Let V_n be the value (in dollars) of a computer after n years.

A recurrence relation that models the depreciating value of this computer over time is:

$$V_0 = 2400, \qquad V_{n+1} = V_n - 300$$

a What was the value of the computer when it was new?

b By how much (in dollars) did the computer depreciate each year?

c What was the percentage flat rate of depreciation?

d After how many years will the value of the computer be \$600?

e When will the computer devalue to half of its original price?

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Example 15 **9** A commercial sewing machine costs \$18 000 when new. Its value depreciates by \$200 for every 1000 pairs of jeans that are sewn using it.
Let V_n be the value (in dollars) of the sewing machine after n thousand pairs of jeans have been sewn. A recurrence relation that models the depreciating value of the sewing machine over time is:

$$V_0 = 18\,000, \qquad V_{n+1} = V_n - 200$$

a Use the recurrence relation to find the value of the sewing machine after it has sown 5000 pairs of jeans.
b The sewing machine will be sold after it has produced 20 000 pairs of jeans. How much will it be worth then?
c If the sewing machine continues to depreciate at the same rate for the rest of its life, how many pairs of jeans will it take to have zero value?
d A different model of sewing machine costs \$20 000 and depreciates by \$250 for every 1000 pairs of jeans that are sewn using it. Write down a recurrence relation to model this situation.

10 Let V_n be the value (in dollars) of a scissor-lift after n thousand uses.
A recurrence relation that models the depreciating value of the scissor-lift over time is:

$$V_0 = 26\,500, \qquad V_{n+1} = V_n - 70$$

a What was the value of the scissor-lift when it was new?
b By how much (in dollars) did the scissor-lift depreciate after every 1000 uses?
c After how many uses will the scissor-lift devalue to less than half of its original price?

Example 16 **11** The following recurrence relation can be used to model a simple-interest investment of \$32 000, paying interest at the rate of 2.5% per year. Let V_n be the value of the investment after n years.

$$V_0 = 32\,000, \qquad V_{n+1} = V_n + 800$$

a How much interest is added to the investment each year?
b Use a rule to find the value of the investment after 15 years.
c Use a rule to find when the value of the investment first reaches \$40 000.

12 The following recurrence relation can be used to model the flat rate depreciation of a motorbike purchased for \$4000, depreciating at a flat rate of 12.5% per year. Let V_n be the value of the motorbike after n years.

$$V_0 = 4000, \qquad V_{n+1} = V_n - 500$$

a How much does the motorbike depreciate by each year?
b Use a rule to find the value of the motorbike after 4 years.
c Use a rule to find when the depreciated value of the bike is zero.

Testing understanding

13 Bruce invests a certain amount of money and receives \$459 each year in simple interest. After 5 years, Bruce has \$10 795 including the initial amount and 5 years of interest. Determine how much Bruce initially invested.

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3F An introduction to geometric sequences

Learning intentions

- To be able to find the common ratio in a geometric sequence.
- To be able to identify a geometric sequence.
- To be able to use a CAS calculator to generate a geometric sequence.
- To be able to graph an increasing or decreasing geometric sequence.

The common ratio, *R*

In a **geometric sequence**, each new term is made by multiplying the previous term by a fixed number, called the **common ratio**, ***R***. This repeating or recurring process is another example of a sequence generated by recursion.

In the sequence:

$$\underset{t_0}{2,} \xrightarrow{\times 3} \underset{t_1}{6,} \xrightarrow{\times 3} \underset{t_2}{18,} \xrightarrow{\times 3} \underset{t_3}{54,} \xrightarrow{\times 3} \ldots$$

each new term is made by multiplying the previous term by 3. The common ratio is 3.

In the sequence:

$$\underset{t_0}{120,} \xrightarrow{\times \frac{1}{2}} \underset{t_1}{60,} \xrightarrow{\times \frac{1}{2}} \underset{t_2}{30,} \xrightarrow{\times \frac{1}{2}} \underset{t_3}{15,} \xrightarrow{\times \frac{1}{2}} \ldots$$

each new term is made by halving the previous term. In this sequence we are multiplying each term by $\frac{1}{2}$, which is equivalent to dividing by 2. The common ratio is $\frac{1}{2}$.

New terms in a geometric sequence $t_0, t_1, t_2, t_3, \ldots$ are made by multiplying the previous term by the common ratio, R.

Common ratio, *R*

In a **geometric sequence**, the **common ratio**, R, is found by dividing the next term by the current term.

$$\text{Common ratio } R = \frac{\text{any term}}{\text{the previous term}} = \frac{t_1}{t_0} = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \ldots$$

As preparation for our study of growth and decay, we will be using common ratios which are greater than zero: $R > 0$.

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Example 17 Finding the common ratio, *R*

Find the common ratio in each of the following geometric sequences.

a 3, 12, 48, 192, . . . **b** 16, 8, 4, 2, . . .

Explanation	Solution
a 3, 12, 48, 192, . . .	
1 The common ratio is equal to any term divided by the previous term.	Common ratio, $R = \frac{t_1}{t_0} = \frac{12}{3} = 4$
2 Check that multiplying by 4 makes each new term.	$3 \xrightarrow{\times 4} 12 \xrightarrow{\times 4} 48 \xrightarrow{\times 4} 192 \xrightarrow{\times 4} \ldots$
3 Write your answer.	The common ratio is 4.
b 16, 8, 4, 2, . . .	
1 Find the common ratio, *R*.	Common ratio, $R = \frac{t_1}{t_0} = \frac{8}{16} = \frac{1}{2}$
2 Check that multiplying by $\frac{1}{2}$ makes each new term.	$16 \xrightarrow{\times \frac{1}{2}} 8 \xrightarrow{\times \frac{1}{2}} 4 \xrightarrow{\times \frac{1}{2}} 2 \xrightarrow{\times \frac{1}{2}} \ldots$
3 Write your answer.	The common ratio is $\frac{1}{2}$.

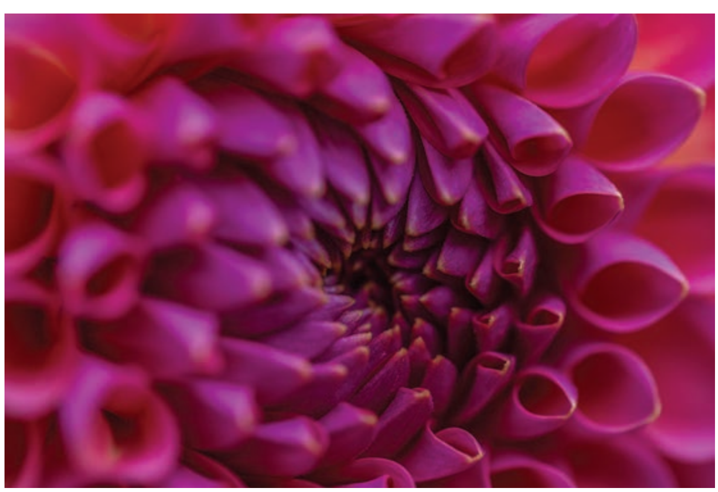
Now try this 17 Finding the common ratio, *R* (Example 17)

Find the common ratio in each of the following geometric sequences.

a 4, 20, 100, 500, . . . **b** 243, 81, 27, 9, . . .

Hint 1 To find the common ratio, calculate any term divided by the previous term.

Hint 2 Check your answer by multiplying each term by the common ratio.



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Identifying geometric sequences

To identify a sequence as geometric, it is necessary to find a common ratio between successive terms.

Example 18 Identifying a geometric sequence

Which of the following is a geometric sequence?

a 2, 10, 50, 250, . . . **b** 3, 6, 18, 36, . . .

Explanation	Solution
a 2, 10, 50, 250, . . .	
1 Find the ratio (multiplier) between successive terms.	$R = \frac{t_1}{t_0}$ $R = \frac{t_2}{t_1}$ $R = \frac{t_3}{t_2}$ $R = \frac{10}{2}$ $R = \frac{50}{10}$ $R = \frac{250}{50}$ $R = 5$ $R = 5$ $R = 5$
2 Check that the ratios are the same.	The common ratio is 5.
3 Write your conclusion.	The sequence is geometric.
b 3, 6, 18, 36, . . .	
1 Find the ratio between successive terms.	$R = \frac{t_1}{t_0}$ $R = \frac{t_2}{t_1}$ $R = \frac{t_3}{t_2}$ $R = \frac{6}{3}$ $R = \frac{18}{6}$ $R = \frac{36}{18}$ $R = 2$ $R = 3$ $R = 2$
2 Are the ratios the same?	The ratios are not the same.
3 Write your conclusion.	The sequence is not geometric.

Now try this 18 Identifying a geometric sequence (Example 18)

Which of the following is a geometric sequence?

a 3, 9, 12, 15, . . . **b** 2, 4, 8, 16, . . .

Hint 1 Find the ratio between successive terms.
Hint 2 Check if the ratios are the same.
Hint 3 If the ratios are the same then the sequence is a geometric sequence.

The common ratio between two terms can also be based on a percentage. That is, terms in a sequence can increase or decrease by a percentage.

For example, successive terms can increase by 20%, meaning that to find the next term, we multiply by $1 + \frac{20}{100} = 1.2$, so the common ratio is 1.2.

Thus, the following sequence has an increase by 20%: 10, 12, 14.4, 17.28, 20.736,. . .

Using repeated multiplication on a CAS calculator to generate a geometric sequence

Using a recursive rule based on repeated multiplication, such as 'to find the next term, multiply by 2', is a quick and easy way of generating the next few terms of a geometric sequence. It would be tedious to find the next 50 terms.

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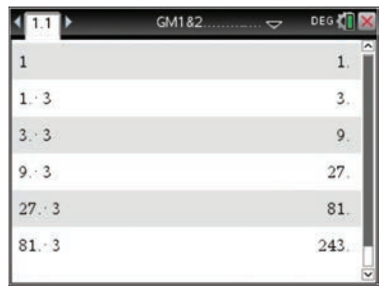
Fortunately, your CAS calculator can semi-automate the process of performing multiple repeated multiplications and do this very quickly.

How to use recursion to generate the terms of a geometric sequence with the TI-Nspire CAS

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, . . .

Steps

1 Press [on] >**New Document> Add Calculator**.
2 Enter the value of the first term **1**. Press [enter]. The calculator stores the value, 1, as Answer (you cannot see this yet).
3 The common ratio for the sequence is 3. So, type in **×3**.



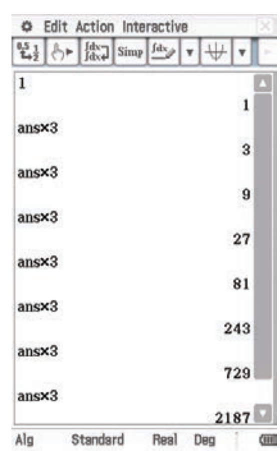
4 Press [enter]. The second term in the sequence, 3, is generated.
5 Pressing [enter] again generates the next term, 9. Keep pressing [enter] until the desired number of terms is generated.
6 Write down the first six terms of the sequence.

The first six terms of the sequence are: 1, 3, 9, 27, 81, 243.

How to use recursion to generate the terms of a geometric sequence with the ClassPad

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, . . .

Steps



1 Tap Main $\sqrt{\alpha}$ to open the **Main** application.
2 Starting with a clean screen, enter the value of the first term, **1**. Press [EXE]. The calculator stores the value, 1, as **answer**. (You can't see this yet.)
3 The common ratio for this sequence is 3. So, type **× 3**. Then press [EXE]. The second term in the sequence (i.e. **3**) is displayed.
4 Pressing [EXE] again generates the next term, **9**. Keep pressing [EXE] until the required number of terms is generated.
5 Write down the first six terms of the sequence. The first six terms of the sequence are: 1, 3, 9, 27, 81, 243.

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Graphs of geometric sequences

In contrast with the straight-line graph of an arithmetic sequence, the values of a geometric sequence lie along a curve. Graphing the values of a sequence is a valuable tool for understanding applications involving growth and decay.

The graph of a geometric sequence clearly displays a curve of increasing values associated with growth or decreasing values indicating decay. As we will see, this depends on the value of the common ratio, R.

Example 19 Graphing an increasing geometric sequence ($R > 1$)

Consider the geometric sequence: 2, 6, 18, . . .

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Explanation

a **1** Find the common ratio using $R = \frac{t_1}{t_0}$.

2 Check that this ratio makes the given terms.

3 Multiply 18 by 3 to make the next term, 54.

4 Write your answer.

b **1** Number the positions along the top row of the table.

2 Write the terms in the bottom row.

c **1** Use the horizontal axis, n, for the position of each term.
Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.

d Describe the pattern revealed by the graph.

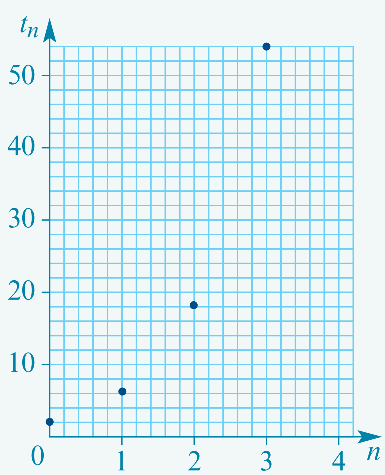
Solution

Common ratio, $R = \frac{t_1}{t_0} = \frac{6}{2} = 3$

$$2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18$$

The next term is 54.

Position, n	0	1	2	3
Term, t_n	2	6	18	54



The values lie along a curve, and they are increasing.

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Now try this 19 Graphing an increasing geometric sequence ($R > 1$) (Example 19)

Consider the geometric sequence: 1, 3, 9, . . .

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Hint 1 Find the common ratio so you can find the next term.

Hint 2 When graphing the points, remember to use the horizontal axis for the position of each term, n, and the vertical axis for t_n.

Example 20 Graphing a decreasing geometric sequence ($0 < R < 1$)

Consider the geometric sequence: 32, 16, 8, . . .

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Explanation

a **1** Find the common ratio, using $R = \frac{t_1}{t_0}$.

2 Check that this ratio makes the given terms.

3 Multiply 8 by $\frac{1}{2}$ to make the next term, 4.

4 Write your answer.

Solution

Common ratio, $R = \frac{t_1}{t_0} = \frac{16}{32} = \frac{1}{2}$

$$32 \xrightarrow{\times\frac{1}{2}} 16 \xrightarrow{\times\frac{1}{2}} 8$$

The next term is 4.

Explanation

b **1** Number the positions along the top row of the table.

2 Write the terms in the bottom row.

Solution

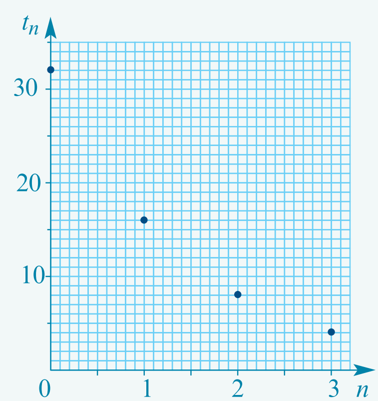
Position, n	0	1	2	3
Term, t_n	32	16	8	4

Explanation

c **1** Use the horizontal axis, n, for the position of each term.
Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.

Solution



Explanation

d Describe the pattern revealed by the graph.

Solution

The graph is a curve with values decreasing and approaching zero.

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Now try this 20 **Graphing a decreasing geometric sequence ($0 < R < 1$) (Example 20)**

Consider the geometric sequence: 64, 16, 4, . . .

a Find the next term.

b Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

c Use the table to plot the graph.

d Describe the graph.

Hint 1 Find the common ratio so you can find the next term.

Hint 2 When graphing the points, remember to use the horizontal axis for the position of each term, n, and the vertical axis for t_n.

Graphs of geometric sequences (for R positive)

Graphs of geometric sequences for $R > 0$ are:

- *increasing* when R is greater than 1 ($R > 1$),
- *decreasing* towards zero when R is less than 1 and greater than zero ($0 < R < 1$).

Section Summary

- In a geometric sequence, the common ratio is found by dividing the next term by the current term.

$$\text{Common ratio, } R = \frac{\text{any term}}{\text{the previous term}} = \frac{t_1}{t_0} = \frac{t_2}{t_1} = \ldots$$

- A geometric sequence is increasing if $R > 1$ or is decreasing towards zero if $0 < R < 1$.

Exercise 3F

Building understanding

1 Simplify the following fractions.

a $\frac{3}{6}$ **b** $\frac{81}{3}$ **c** $\frac{27}{9}$ **d** $\frac{4}{64}$

2 Calculate the ratio between the first two numbers in the following geometric sequences.

a 4, 8, 16, 32, . . .

b 1, 3, 9, 27, . . .

c 16, 8, 4, 2, . . .

d 48, 24, 12, 6, . . .

3 Decide if the following sequences are arithmetic or geometric.

a 2, 4, 6, 8, . . .

b 1, 4, 16, 64, . . .

c 729, 243, 81, 27, . . .

d 100, 95, 90, 85, . . .

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Developing understanding

Example 17 **4** Find the common ratio for each of the following geometric sequences.

a 3, 6, 12, 24, ...
b 64, 16, 4, 1, ...
c 6, 30, 150, 750, ...
d 2, 8, 32, 128, ...
e 32, 16, 8, 4, ...
f 2, 12, 72, 432, ...
g 10, 100, 1000, 10 000, ...
h 3, 21, 147, 1029, ...

Example 18 **5** Identify which of the following sequences are geometric. Give the common ratio for each sequence that is geometric.

a 4, 8, 16, 32, ...
b 1, 3, 9, 27, ...
c 5, 10, 15, 20, ...
d 5, 15, 45, 135, ...
e 24, 12, 6, 3, ...
f 3, 6, 12, 18, ...
g 4, 8, 12, 16, ...
h 27, 9, 3, 1, ...
i 2, 4, 8, 16, ...

6 Find the missing terms in each of these geometric sequences.

a 7, 14, 28, ☐, ☐, ...
b 3, 15, 75, ☐, ☐, ...
c 4, 12, ☐, ☐, 324, ...
d ☐, ☐, 20, 40, 80, ...
e 2, ☐, 32, 128, ☐, ...
f 3, ☐, 27, ☐, 243, 729, ...

7 Use your CAS calculator to generate each sequence, and find t_6, the sixth term.

a 7, 35, 175, ...
b 3, 18, 108, ...
c 96, 48, 24, ...
d 4, 28, 196, ...
e 160, 80, 40, ...
f 11, 99, 891, ...

Example 19 **8** Consider each of the geometric sequences below. For each one:

i Find the next term.
ii Show the first four terms in a table.
iii Use the table to plot a graph.
iv Describe the graph.

a 3, 6, 12, ...
b 2, 10, 50, ...

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Example 20

9 Consider each of the geometric sequences below. For each one:

i Find the next term.
ii Show the first four terms in a table.
iii Use the table to plot a graph.
iv Describe the graph.

a 8, 4, 2, . . .
b 81, 27, 9, . . .

10 Consider a geometric sequence that starts with the value of 20 where each subsequent term increases by 10%.

a State what each term must be multiplied by to find the next term.
b Calculate the next three terms.

11 Consider a geometric sequence that starts with the value of 100 where each subsequent term decreases by 10%.

a State what each term must be multiplied by to find the next term.
b Calculate the next three terms.

12 Consider the geometric sequence 10 000, 12 000, 14 400, 17 280, ...

a Calculate the ratio between the first two numbers of the sequence.
b Calculate the ratio between t_2 and t_1, and between t_3 and t_2.
c State the percentage increase between t_0 and t_1.
d Use the percentage increase (found in **c**) or the common ratio (found in part **a** or **b**) to calculate the next term.

Testing understanding

13 A sequence is generated from the recurrence relation $V_0 = 500$, $V_{n+1} = 0.4V_n - 5$.

a Use your CAS calculator to generate the first five terms of the sequence.
b Identify whether the sequence is arithmetic or geometric or neither.
c Explain why a term greater than 800 will never occur in the sequence.
d How many iterations are required to generate the first negative term?

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3G Recursion with geometric sequences

Learning intentions

- To be able to generate a geometric sequence using a recurrence relation.
- To be able to find the nth term in a geometric sequence using a recurrence relation.

Using a recurrence relation to generate and analyse a geometric sequence

Consider the geometric sequence below:

$$2, 6, 18, \ldots$$

We can continue to generate the terms of this sequence by recognising that it uses the rule:

'start the sequence with 2'
'to find the next term, multiply the current term by 3, and keep repeating the process.'

Label the terms $t_0, t_1, t_2, \ldots$ and following this process we have:

$t_0 = 2$	initial or starting value
$t_1 = 3 \times t_0 = 6$	after 1 application of the rule
$t_2 = 3 \times t_1 = 18$	after 2 applications of the rule
$t_3 = 3 \times t_2 = 54$	after 3 applications of the rule
$t_4 = 3 \times t_3 = 162$	after 4 applications of the rule

and so on, until we have the rule $t_{n+1} = 3 \times t_n$ after n applications of the rule.

A **recurrence relation** is a way of expressing the starting value and the rule that generates this sequence in precise mathematical language.

The recurrence relation that generates the sequence $2, 6, 18, \ldots$ is:

$$t_0 = 2, \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots.$$

The rule tells us that:
'the first term is 2, and each subsequent term is equal to the current term multiplied by 3.'

The recurrence relation for generating a geometric sequence

The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = Rt_n$$

can be used to generate a geometric sequence with the first term, $t_0 = a$, and the common ratio, R.

Example 21 Using a recurrence relation to generate a geometric sequence

Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 5, \qquad t_{n+1} = 2t_n$$

Explanation

1 Write down the recurrence relation.

2 Write down the first term.

3 Use the rule, which translates into: 'to find the next term, multiply the previous term by 2' to generate the first five terms in the sequence.

4 To graph the terms, plot t_n against n for $0 \le n \le 4$.

Solution

$t_0 = 5, \qquad t_{n+1} = 2t_n$

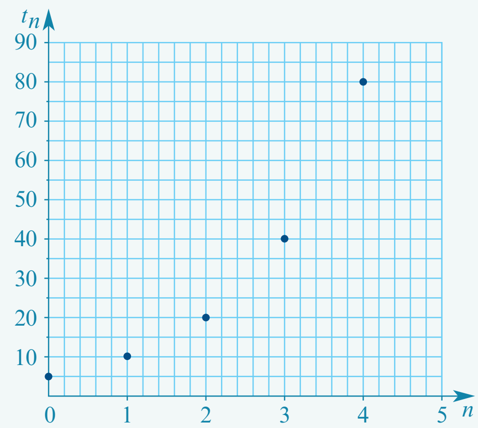
$t_0 = 5$

$t_1 = 2t_0 = 2 \times 5 = 10$

$t_2 = 2t_1 = 2 \times 10 = 20$

$t_3 = 2t_2 = 2 \times 20 = 40$

$t_4 = 2t_3 = 2 \times 40 = 80$



Now try this 21 Using a recurrence relation to generate a geometric sequence (Example 21)

Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 1000, \qquad t_{n+1} = 0.1t_n$$

Hint 1 Write down the starting term and then use the rule to find the next four terms.

Hint 2 To graph the terms, plot t_n against n.

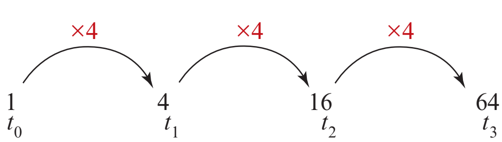
Finding the *n*th term in a geometric sequence

Repeated multiplication can be used to find each new term in a geometric sequence, but this process is very tedious for finding a term such as t_{50}. Instead, a general rule can be found to calculate any term, t_n, using: n, the number of times the recursion rule is applied, the value of the first term, a, and the common ratio, R.

Consider the geometric sequence: $1, 4, 16, 64, \ldots$ which is defined by

$$t_0 = 1, \qquad t_{n+1} = 4t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots.$$

This is illustrated pictorially in the diagram below:



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Using the information from the diagram, we can write recursively:

$t_0 = 1$	after 0 applications of the rule
$t_1 = 4 \times t_0 = 4^{1} \times t_0 = 4$	after 1 application of the rule
$t_2 = 4 \times 4 \times t_0 = 4^{2} \times t_0 = 16$	after 2 applications of the rule
$t_3 = 4 \times 4 \times 4 \times t_0 = 4^{3} \times t_0 = 64$	after 3 applications of the rule

A pattern emerges which suggests that, after n applications of the recursion rule:

$t_n = 4^n \times t_0$ after n applications of the rule.

Using this rule, t_n can be found without having to find all previous values in the sequence. Thus, using the rule:

$$t_{10} = 4^{10} \times 1 = 1\,048\,576$$

This rule can be generalised to apply to any situation involving the recursive generation of a geometric sequence.

Rule for finding the *n*th term of a geometric sequence

The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = R \times t_n$$

can be used to generate a geometric sequence with a starting value, $t_0 = a$, and a common ratio, R.

The rule for directly calculating term t_n in this sequence is generated by a recurrence relation:

$$t_n = R^n \times a$$

where n is the term number, $n = 0, 1, 2, 3, \ldots$.

Example 22 Finding the *n*th term of a geometric sequence

Consider the following recurrence relation:

$$t_0 = 2, \qquad t_{n+1} = 3t_n$$

Find t_{12}.

Explanation	Solution
1 State the initial value, a, and the common ratio, R.	$a = 2 \qquad R = 3$
2 The term t_{12} requires 12 applications of the rule.	$n = 12$
3 Substitute the values of a, R and n into $t_n = R^n \times a$.	$t_n = R^n \times a$ $t_{12} = 3^{12} \times 2$
4 Evaluate.	$= 1\,062\,882.$
5 Write your answer.	$t_{12} = 1\,062\,882.$

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Now try this 22 **Finding the *n*th term of a geometric sequence (Example 22)**

Consider the following recurrence relation:

$$t_0 = 5, \qquad t_{n+1} = 2t_n$$

Find t_{20}.

Hint 1 Write down the starting term and the common ratio.

Hint 2 Substitute the values of a, R and n into $t_n = R^n \times a$.

Section Summary

- The recurrence relation:

$$t_0 = a, \qquad t_{n+1} = R \times t_n$$

can be used to generate a geometric sequence with a starting value, $t_0 = a$, and a common ratio of R.

- The rule for directly calculating term t_n in this sequence is:

$$t_n = R^n \times t_0$$

where n is the term number, $n = 0, 1, 2, 3, \ldots$

Skill-sheet

Exercise 3G

Building understanding

1 Give the value of a and R in each of the following geometric sequences.

a $2, 6, 18, 54, \ldots$

b $5, 20, 80, 320, \ldots$

c $5, 10, 20, 40, \ldots$

d $3, 12, 48, 192, \ldots$

2 For each of the geometric sequences in Question 1, use the value of a and R that you found and the rule: $t_n = R^n \times t_0$, to find the value of t_5.

3 For the recurrence relation: $t_0 = 6, t_{n+1} = 2 \times t_n$, complete the blanks using the rule: $t_n = R^n \times t_0$, to find the value of t_1, t_2, t_3 and t_{20}.

$$t_1 = R^1 \times t_0 = 2^1 \times 6 = 2 \times 6 = 12$$
$$t_2 = R^2 \times t_0 = 2^{\cdots} \times 6 = \ldots \times \ldots = \ldots$$
$$t_3 = R^3 \times t_0 = 2^{\cdots} \times 6 = \ldots \times \ldots = \ldots$$
$$t_{20} = R^{\cdots} \times t_0 = \ldots^{\cdots} \times 6 = \ldots \times \ldots = \ldots$$

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Developing understanding

Example 21

4 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 2 \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

b Calculate the value of t_{10}.

5 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 3 \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

b Calculate the value of t_{12}.

6 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 4 \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

b Calculate the value of t_{10}.

7 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 100 \qquad t_{n+1} = \frac{1}{10}t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

b Calculate the value of t_{10}.

8 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_0 = 100 \qquad t_{n+1} = \frac{1}{2}t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

b Calculate the value of t_{15}.

Example 22

9 Consider the following recurrence relation:

$$t_0 = 10, \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Use your CAS calculator to find the value of the following terms:

a t_3 **b** t_5 **c** t_{12} **d** t_{15} **e** t_{20} **f** t_{25}

10 Use your CAS calculator to find t_{30} for each of the following recurrence relations:

a $t_0 = 6, \qquad t_{n+1} = 2t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

b $t_0 = 4, \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

c $t_0 = 2000, \qquad t_{n+1} = 0.5t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

d $t_0 = 10\,000, \qquad t_{n+1} = 0.5t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$

11 Find t_{20} in a geometric sequence that starts at 4 and has a common ratio of 2.

12 The first term in a geometric sequence is 5 and has a common ratio of 2. Find t_{10}.

13 A sequence starts at 5000 and is divided by 2 each time to make a new term. Find t_{20}.

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Testing understanding

14 A piece of paper had an area of 1m^2. It was cut in half each day with one half thrown away and the other half retained for the next day. What was the area of the paper that was retained at the end of the 7th day, in m^2?

15 In a geometric sequence, $t_2 = 12$ and $t_5 = 96$.

a Using the equation $t_n = R^n \times t_0$, substitute $t_2 = 12$ and $n = 2$ to form an equation in terms of R and t_0.

b Using the equation $t_n = R^n \times t_0$, substitute $t_5 = 96$ and $n = 5$ to form an equation in terms of R and t_0.

c Use simultaneous equations to solve the two equations you found in part **a** and **b** to find R and t_0.

d Hence, write down the recurrence relation.

e Check your answer by finding t_3 and t_4 using the rule and the values of R and t_0.

f Hence, write down the first six terms (from t_0) of the sequence.

3H Finance applications using geometric sequences and recurrence relations

Learning intentions

- To be able to use a recurrence relation to model compound interest for investments or loans.
- To be able to write a recurrence relation to model a loan that compounds with a different compounding period.
- To be able to use a recurrence relation to model reducing-balance depreciation.

The skill sheet available for this section through the Interactive Textbook also contains non-financial applications.

Geometric growth and decay

Geometric growth and **decay** are also commonly found in the world around us. Geometric growth or decay in a sequence occurs when the quantity being modelled increases or decreases by the same percentage at regular intervals. Everyday examples include the payment of compound interest or the depreciation of the value of a new car by a constant percentage of its depreciated value each year. This method of depreciation is commonly called **reducing-balance depreciation**.

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A recurrence model for geometric growth and decay

Geometric growth or decay in a sequence occurs when quantities increase or decrease by the same percentage at regular intervals.

Recall that the recurrence relation for a geometric sequence consists of the starting value and a rule to generate the next term. Consider the following two recurrence relations:

$$V_0 = 10, \qquad V_{n+1} = 5V_n$$
$$V_0 = 10, \qquad V_{n+1} = 0.5V_n$$

Both of these rules generate a geometric sequence, but the first relation generates a sequence that *grows* geometrically while the second relation generates a sequence that *decays* geometrically.

As a general rule, if R is a constant, the recurrence relation rule:

- $V_{n+1} = RV_n$ for $R > 1$, can be used to generate *geometric growth*.
- $V_{n+1} = RV_n$ for $0 < R < 1$, can be used to generate *geometric decay*.

Compound interest investments and loans

While interest is sometimes calculated using simple interest models, it is more commonly calculated using **compound interest**, where any interest earned after one period is added to the principal and then contributes to the earnings of interest in the next time period.

This means that the amount of interest earned in each period increases over time. The value of the investment grows geometrically.

For example, the investment of \$8000 that pays 5% interest per annum, compounding yearly, increases in value by 5% each year. This can be modelled using a recurrence relation as follows:

$$V_0 = 8000, \qquad V_{n+1} = V_n + 0.05V_n$$

or more compactly,

$$V_0 = 8000, \qquad V_{n+1} = 1.05V_n$$

where V_n is the value of the investment after n years. Note that in this example, $R = 1.05 > 1$, telling us that we have a model of geometric growth.

The recurrence relation for compounding interest investments and loans that compound yearly

Let V_n be the value of the investment after n years.

Let r be the percentage interest per compound period.

The recurrence model for the value of the investment after n compounding periods is:

$$V_0 = \text{principal}, \qquad V_{n+1} = R \times V_n$$

where:

$$R = 1 + \frac{r}{100}$$

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Example 23 Using a recurrence relation to model geometric growth: a compound interest investment (1)

The following recurrence relation can be used to model a compound interest investment of \$1000, paying interest at the rate of 8% per annum.

$$V_0 = 1000, \qquad V_{n+1} = 1.08V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years, to the nearest cent.

b Determine when the value of the investment will first exceed \$1500.

Explanation

a **1** Write down the recurrence relation.

2 With $V_0 = 1000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

This can also be done with your CAS calculator.

b This is best done using your CAS calculator and counting the number of steps (years) until the value of the investment first exceeds \$1500.

Write your conclusion.

Note: Count the number of times the value of the investment is increased by 8%.

Solution

a $V_0 = 1000 \quad V_{n+1} = 1.08V_n$

$V_0 = 1000$
$V_1 = 1.08V_0 = 1.08 \times 1000 = \$1080$
$V_2 = 1.08V_1 = 1.08 \times 1080 = \$1166.40$
$V_3 = 1.08V_2 = 1.08 \times 1166.40 = \$1259.71$

1000	1000
1000 · 1.08	1080
1080 · 1.08	1166.4
1166.4 · 1.08	1259.71

b

1000	1000
1000 · 1.08	1080
1080 · 1.08	1166.4
1166.4 · 1.08	1259.71
1259.712 · 1.08	1360.49
1360.48896 · 1.08	1469.33
1469.3280768 · 1.08	1586.87

The investment will first exceed \$1500 after 6 years.

Now try this 23 Using a recurrence relation to model geometric growth: a compound interest investment (Example 23)

The following recurrence relation can be used to model a compound interest investment of \$3000, paying interest at the rate of 4% per annum.

$$V_0 = 3000, \qquad V_{n+1} = 1.04V_n$$

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In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years, to the nearest dollar.

b Determine when the value of the investment will first exceed \$3600.

Hint 1 Write down the recurrence relation and use this to calculate V_1, V_2 and V_3.

Hint 2 Use your CAS calculator to find how many times the rule needs to be applied to get to at least \$3600.

Often, compound interest investments and loans accrue over periods of less than a year, and so this must be taken into account when we formulate a general recurrence relation.

Example 24 Using a recurrence relation to model geometric growth: a compound interest investment (2)

Sally borrows \$6000 from a bank. Interest will accrue at the rate of 4.2% per annum.

Let V_n be the value of the loan after n compounding periods.

Write down a recurrence relation to model the value of Sally's loan if interest is compounded:

a yearly **b** quarterly **c** monthly

Explanation	Solution
a 1 Define the variable V_n. The compounding period is **yearly**.	Let V_n be the value of Sally's loan after n years.
2 Determine the value of R.	The interest rate is 4.2% per annum. $R = 1 + \frac{4.2}{100} = 1.042$
3 Write the recurrence relation.	$V_0 = 6000, \quad V_{n+1} = 1.042 \times V_n$
b 1 Define the variable V_n. The compounding period is **quarterly**.	Let V_n be the value of Sally's loan after n quarters.
2 Determine the value of R.	The interest rate is 4.2% per annum. The quarterly interest rate is $\frac{4.2}{4} = 1.05$ $R = 1 + \frac{1.05}{100} = 1.0105$
3 Write the recurrence relation.	$V_0 = 6000, \quad V_{n+1} = 1.0105 \times V_n$
c 1 Define the variable V_n. The compounding period is **monthly**.	Let V_n be the value of Sally's loan after n months.
2 Determine the value of R.	The interest rate is 4.2% per annum. The monthly interest rate is $\frac{4.2}{12} = 0.35$ $R = 1 + \frac{0.35}{100} = 1.0035$
3 Write the recurrence relation.	$V_0 = 6000, \quad V_{n+1} = 1.0035 \times V_n$

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Now try this 24 **Using a recurrence relation to model geometric growth: a compound interest investment (2) (Example 24)**

Valentina borrows $25 000 from a bank. Interest accrues at the rate of 4.8% per annum.

Let V_n be the value of the loan after n compounding periods.

Write down a recurrence relation to model the value of Valentina's loan if interest is compounded:

a yearly **b** quarterly **c** monthly

Hint 1 Define the variable V_n given the compounding period.
Hint 2 Determine the value of R.
Hint 3 Write the recurrence relation.

Reducing-balance depreciation

We have already considered two different methods of depreciation - flat rate depreciation and unit cost depreciation - both examples of linear decay. **Reducing-balance depreciation** is another method of depreciation where the value of an asset decays geometrically. Each year, the value will be reduced by a percentage, $r\%$, of the previous year's value. The calculations are very similar to compounding interest, but with decay in value rather than growth.

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The recurrence relation for reducing-balance depreciation

Let V_n be the value of the asset after n years.

Let r be the annual percentage depreciation.

The recurrence model for the value of the asset after n years is:

$$V_0 = \text{ initial value}, \qquad V_{n+1} = R \times V_n$$

where

$$R = 1 - \frac{r}{100}$$

Example 25 Using a recurrence relation to model geometric decay: reducing-balance depreciation

A car is purchased for \$18 500. The following recurrence relation can be used to model the car's value as it depreciates by 10% of its value each year.

$$V_0 = 18\,500, \qquad V_{n+1} = 0.9 \times V_n$$

In the recurrence relation, V_n is the value of the car after n years.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b When will the value of the car first be worth less than \$10 000?

Explanation

a **1** Write down the recurrence relation.

2 The recurrence relation tells you that: 'to find the next value, multiply the current value by 0.90'.
With $V_0 = 18\,500$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

This can also be done with your calculator.

b This is best done using your calculator and counting the number of times the car's value has been decreased by 10% until the value of the car is first less than \$10 000.

Write your conclusion.

Solution

a $V_0 = 18\,500, \quad V_{n+1} = 0.90V_n$

$V_0 = 18\,500$
$V_1 = 0.9V_0 = 0.9 \times 18\,500 = \$16\,650$
$V_2 = 0.9V_1 = 0.9 \times 16\,650 = \$14\,985$
$V_3 = 0.9V_2 = 0.9 \times 14\,985$
$= \$13\,486.50$

18500	18500
18500 · 0.9	16650
16650 · 0.9	14985
14985 · 0.9	13486.5

b

18500	18500
18500 · 0.9	16650
16650 · 0.9	14985
14985 · 0.9	13486.5
13486.5 · 0.9	12137.85
12137.85 · 0.9	10924.07
10924.065 · 0.9	9831.66

The value of the car is first less than \$10 000 after 6 years.

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Now try this 25 **Using a recurrence relation to model geometric decay: reducing-balance depreciation (Example 25)**

A ute is purchased for \$53 800. The following recurrence relation can be used to model the ute's value as it depreciates by 8% of its value each year.

$$V_0 = 53\,800, \qquad V_{n+1} = 0.92 \times V_n$$

In the recurrence relation, V_n is the value of the ute after n years.

a Use the recurrence relation to find the value of the ute after 1, 2 and 3 years to the nearest cent.

b When will the value of the ute first be worth less than \$30 000?

Hint 1 Write down the recurrence relation.

Hint 2 Note that depreciation of 8% leaves you with 92% of the value.

Hint 3 Apply the rule to find the value of V_1, V_2 and V_3.

Hint 4 Use your CAS calculator to apply the rule to find the first time that the value of the asset falls below \$30 000.

Section Summary

- If R is a constant, the recurrence relation rule:
  - $V_{n+1} = RV_n$ for $R > 1$, can be used to generate *geometric growth*.
  - $V_{n+1} = RV_n$ for $0 < R < 1$, can be used to generate *geometric decay*.
- Let V_n be the value of the investment after n years and r be the percentage interest per compound period. The recurrence model for the value of the investment after n compounding periods is:

$$V_0 = \text{ principal} \qquad V_{n+1} = R \times V_n, \text{ where } R = 1 + \frac{r}{100}$$

- Let V_n be the value of the asset after n years and r be the annual percentage depreciation. The recurrence model for the value of the asset after n periods is:

$$V_0 = \text{ initial value} \qquad V_{n+1} = R \times V_n, \text{ where } R = 1 - \frac{r}{100}$$

Skill-sheet

Exercise 3H

Building understanding

1 Determine which of the following recurrence relations model geometric growth, geometric decay or neither.

a $V_0 = 5, \quad V_{n+1} = 4V_n \quad \text{for } n = 0, 1, 2, 3, \ldots$

b $V_0 = 30, \quad V_{n+1} = \frac{1}{5}V_n \quad \text{for } n = 0, 1, 2, 3, \ldots$

c $V_0 = 2, \quad V_{n+1} = V_n - 2 \quad \text{for } n = 0, 1, 2, 3, \ldots$

d $V_0 = 100, \quad V_{n+1} = V_n + 10 \quad \text{for } n = 0, 1, 2, 3, \ldots$

e $V_0 = 10, \quad V_{n+1} = 0.2V_n \quad \text{for } n = 0, 1, 2, 3, \ldots$

f $V_0 = 3, \quad V_{n+1} = 1.1V_n \quad \text{for } n = 0, 1, 2, 3, \ldots$

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2 The following recurrence relation is used to model a compound interest investment.

$$V_0 = 5000, \qquad V_{n+1} = 1.05V_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

where V_n is the value of the investment after n years.

a State the amount of money that is initially invested.

b State the annual interest rate that is applied to the investment.

c State the value of the investment after 1 year.

3 The following recurrence relation is used to model balance-reducing depreciation on an asset.

$$V_0 = 40\,000, \qquad V_{n+1} = 0.9V_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

where V_n is the value of the asset after n years.

a State the initial value of the asset.

b State the annual percentage depreciation of the asset.

c State the value of the asset after 1 year.

4 Consider an investment with an interest rate of 4.5% per annum. Calculate the interest rate for the compounding period if interest is compounded:

a quarterly **b** monthly

Developing understanding

Example 23

5 The following recurrence relation can be used to model a compound interest investment of \$10 000, paying interest at the rate of 4.5% per year.

$$V_0 = 10\,000, \qquad V_{n+1} = 1.045V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years. Round your answers to the nearest cent.

b When will the value of the investment first exceed \$12 000 in value?

6 The following recurrence relation can be used to model a compound interest investment of \$200 000, paying interest at the rate of 5.2% per year.

$$V_0 = 200\,000, \qquad V_{n+1} = 1.052V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years. Round your answers to the nearest cent.

b When will the value of the investment first exceed \$265 000 in value?

7 The following recurrence relation can be used to model a compound interest investment.

$$V_0 = 8000, \qquad V_{n+1} = 1.075V_n$$

In the recurrence relation, V_n is the value, in dollars, of the investment after n years.

a How much was invested at the start?

b Use the recurrence relation to determine when the value of the investment first exceeds \$10 000.

c What was the annual interest rate?

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8 The following recurrence relation can be used to model a compound interest investment of \$100 000, paying interest at the rate of 6.3% per year.

$$V_0 = 100\,000, \quad V_{n+1} = 1.063V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years. Round your answer to the nearest cent.

b When will the value of the investment be more than double the initial investment?

Example 24 **9** Simon invests \$80 000 with a bank. He will be paid interest at the rate of 6% per year, compounding monthly.
In the recurrence relation, V_n is the value of the investment after n months.

a Determine the monthly interest rate.

b Write a recurrence relation to model Simon's investment.

c Use the recurrence relation to find the value of the investment after 1, 2 and 3 months.

Example 25 **10** A tractor costs \$90 000 when new. Its value depreciates at a reducing-balance rate of 15% per year.
Let V_n be the value (in dollars) of the tractor after n years.
A recurrence relation that models the depreciating value of this tractor over time is:

$$V_0 = 90\,000, \quad V_{n+1} = 0.85V_n$$

a Use the recurrence relation to find the value of the tractor at the end of each year for the first five years. Round your answers to the nearest cent.

b The tractor will be sold after 8 years. How much will it be worth then? Round your answer to the nearest cent.

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11 Let V_n be the value (in dollars) of a refrigerator after n years.
A recurrence relation that models the depreciating value of this refrigerator over time is:

$$V_0 = 1200, \quad V_{n+1} = 0.56V_n$$

a What was the value of the refrigerator when it was new?
b After how many years will the value of the refrigerator first be less than \$200?
c When will the refrigerator devalue to less than half of its new price?
d What was the percentage rate of depreciation?

12 A car, purchased new for \$84 000, will be depreciated using a reducing-balance depreciation method with an annual depreciation rate of 3.5%. Let V_n be the value (in dollars) of the car after n years.

a What was the value of the car when it was new?
b How much is the car worth after one year?
c Write a recurrence relation to model the value of the car from year to year.
d Confirm that your recurrence relation is correct by calculating the value of the car after one year, using the recurrence relation.
e Generate a sequence of numbers that represents the value of the car from year to year for 5 years in total, starting with the initial value. Write the values of the terms of the sequence to the nearest cent.
f How much has the value of the car declined by after five years?

Testing understanding

13 Jackson has \$20 000 to invest at the bank for 5 years. The bank offers a number of investment options.

A Jackson can invest his money and receive 7% simple interest per annum.
B Jackson can invest his money and receive 6.5% interest per annum, compounded each year.
C Jackson can invest his money and receive 6% interest per annum, compounded each month.

Determine which of these options gives Jackson the largest value at the end of five years, and how much interest he will earn from this option.

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3I Finding term *n* in a sequence modelling geometric growth and decay

Learning intentions

- To be able to find the value of an investment with compounding interest or an asset with reducing-balance depreciation after a certain length of time.
- To be able to find the amount of interest earned in a time period on an investment with compounding interest.

Earlier in this chapter, we found that the nth term of a geometric sequence is generated by the rule:

$$V_n = R^n \times V_0$$

This rule works for both geometric growth or decay because growth or decay depends on the value of R rather than the specification of the rule. This general rule can be applied to compound interest loans and investment as well as reducing-balance depreciation.

Compound interest loans and investment

Let V_0 be the amount borrowed or invested (principal).

Let r be the interest rate per compounding period.

The value of a compound-interest loan or investment after n compounding periods, V_n, is given by the rule:

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

Reducing-balance depreciation

Let V_0 be the purchase price of the asset.

Let r be the annual percentage rate of depreciation.

The value of an asset after n years, V_n, is given by the rule:

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

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Example 26 **Using a rule for determining the value of an investment with reducing-balance depreciation over time**

a The following recurrence relation can be used to model a compound interest investment of \$1000, paying interest at the rate of 10% per year.

$$V_0 = 1000, \qquad V_{n+1} = 1.1V_n$$

Use a rule to find the value of the investment after 15 years, to the nearest dollar.

b The following recurrence relation can be used to model the reducing-balance depreciation of a car purchased for \$18 500, where the value of the car depreciates at the rate of 10% per year.

$$V_0 = 18\,500, \qquad V_{n+1} = 0.9V_n$$

Use a rule to find the value of the car after 12 years, to the nearest dollar.

Explanation	Solution
a Use the rule: $V_n = R^n V_0$, where $V_0 = 1000$, $n = 15$ and $R = 1.1$	$V_n = R^n \times V_0$ $= 1.1^{15} \times 1000$ $= \$4177.24\ldots$ $= \$4177$ (nearest dollar)
b Use the rule: $V_n = R^n V_0$, where $V_0 = 18\,500$, $n = 12$ and $R = 0.90$	$V_n = R^n \times V_0$ $= 0.9^{12} \times 18\,500$ $= \$5224.94\ldots$ $= \$5225$ (nearest dollar)

Now try this 26 **Using a rule for determining the value of an investment with reducing-balance depreciation over time (Example 26)**

a The following recurrence relation can be used to model a compound interest investment of \$5000, paying interest at the rate of 6% per year.

$$V_0 = 5000, \qquad V_{n+1} = 1.06V_n$$

Use a rule to find the value of the investment after 20 years, to the nearest dollar.

b The following recurrence relation can be used to model the reducing-balance depreciation of a car purchased for \$37 500, where the value of the car depreciates at the rate of 15% per year.

$$V_0 = 37\,500, \qquad V_{n+1} = 0.85V_n$$

Use a rule to find the value of the investment after 10 years, to the nearest dollar.

Hint 1 Write down the rule $V_n = R^n V_0$.
Hint 2 Substitute in the value for n, R and V_0.
Hint 3 Remember to round your answer to the nearest dollar.

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Example 27 **Using a rule to analyse an investment**

A principal value of \$20 000 is invested in an account, earning compound interest at the rate of 6% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.06^n \times 20\,000$$

a Find the value of the investment after 5 years, to the nearest cent.

b Find the amount of interest earned after 5 years, to the nearest cent.

c Find the amount of interest earned in the fifth year, to the nearest cent.

d If the interest compounds monthly instead of yearly, write down a rule for the value of the investment after n months.

e Use this rule to find the value of the investment after 5 years (60 months).

Explanation	Solution
a **1** Substitute $n = 5$ into the rule for the value of the investment.	$V_5 = 1.06^5 \times 20\,000$ $= 26\,764.511552$
2 Write your answer, rounded to the nearest cent.	After 5 years, the value of the investment is \$26 764.51, to the nearest cent.
b To find the total interest earned in 5 years, subtract the principal from the value of the investment after 5 years.	Amount of interest $= 26\,764.51 - 20\,000$ $= 6764.51$ After 5 years, the amount of interest earned is \$6764.51.
c **1** The amount of interest earned in the fifth year is equal to the difference between the value of the investment at the end of the fourth and fifth year.	
2 Calculate V_4 to the nearest cent.	$V_4 = 1.06^4 \times 20\,000$ $V_4 = 25\,249.54$ to the nearest cent.
3 Calculate the difference between V_4 and V_5.	$V_5 - V_4 = 26\,764.51 - 25\,249.54$ $= 1514.97$
4 Write your answer.	Interest earned in the fifth year was \$1514.97.

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d **1** Define the symbol V_n.	Let V_n be the value of the investment after n months.
2 Determine the value of r. To convert to a monthly interest rate, divide the annual rate by 12.	$r = \frac{6}{12} = 0.5\%$
3 The general rule for the nth term is: $V_n = R^n V_0$, where $R = 1 + r/100$. In this investment, r is the monthly interest rate. Hence, determine R.	$R = 1 + \frac{0.5}{100}$ $= 1.005$
4 Substitute $R = 1.005$ and $V_0 = 20\,000$ into the rule to find the value for V_n.	$V_n = 1.005^n \times 20\,000$
e Substitute $n = 60$ (5 years = 60 months), $R = 1.005$ and $V_0 = 20\,000$ into the rule to find the value for V_{60}.	$V_{60} = 1.005^{60} \times 20\,000 = \$26\,977.00$

Now try this 27 Using a rule to analyse an investment (Example 27)

A principal value of $40 000 is invested in an account earning compound interest at the rate of 3% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.03^n \times 40\,000$$

a Find the value of the investment after 10 years, to the nearest cent.

b Find the amount of interest earned after 10 years, to the nearest cent.

c Find the amount of interest earned in the tenth year, to the nearest cent.

d If the interest compounds monthly instead of yearly, write down a rule for the value of the investment after n months.

e Use this rule to find the value of the investment after 10 years (120 months).

Hint 1 Substitute $n = 10$ into the rule: $V_n = R^n \times V_0$.

Hint 2 Remember that the interest earned is the change in the value of the investment.

Hint 3 When interest is earned at different time periods, the effective interest rate, r, must be calculated.

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Credit cards

Calculating credit card debt is an application of **compound interest**, where interest is calculated **daily**.

Calculating credit card debt

If a credit card debt of $\$P$ accumulates at the rate of $r\%$ per annum, compounding daily, then the amount of debt accumulated, A, after n days is given by:

$$A = P\left(1 + \frac{r/365}{100}\right)^n = P\left(1 + \frac{r}{36\,500}\right)^n$$

and the amount of interest payable after n days is given by:

$$I = A - P$$

Note: To determine the daily interest rate, the annual interest rate, r, is divided by 365. (1 year = 365 days)

Example 28 Calculating credit card interest

Determine how much interest is payable on a credit card debt of \$5630 at an interest rate of 17.8% per annum, compounding daily, for 27 days. Give your answer to the nearest cent.

Explanation	Solution
1 Calculate the value of debt after 27 days using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$	$P = \$5630$, $r = 17.8\%$ and $n = 27$ $A = P\left(1 + \frac{r}{36\,500}\right)^n$ $= 5630\left(1 + \frac{17.8}{36\,500}\right)^{27}$ $= \$5704.60$ to the nearest cent
2 The amount of interest payable is obtained by subtracting the original debt, P, from the value of the debt after 27 days.	$I = A - P$ $= \$5704.60 - \$5630.00$ $= \$74.60$

Now try this 28 Calculating credit card interest (Example 28)

Determine how much interest is payable on a credit card debt of \$3120 at an interest rate of 15.2% per annum, compounding daily, for 35 days.

Hint 1 Calculate the value of the debt for 35 days using the rule $A = P\left(1 + \frac{r}{36\,500}\right)^n$.

Hint 2 Calculate the amount of interest by finding the difference between A and P.

Most credit cards offer an interest-free period, which means that if you pay for your purchase within that time you won't pay any interest. This includes the statement period and some additional days to pay the full balance before an interest rate applies. The actual number of interest-free days varies, depending on when you make your purchase and the number of days remaining in your statement period.

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Example 29 Calculating credit card interest with an interest-free period

Janelle pays for her holiday, costing \$1500, to Bali using her credit card. Her bank offers a 30-day statement period plus a further 25 days interest free. After that time, the bank charges interest at a rate of 20% per annum, compounding daily.

Janelle makes the purchase on 17 August, which is day 10 of her statement period. She intends to pay off the credit card on 1 November. At this date, how much will she need to pay back? (Assume no interest is payable on the last day).

Explanation	Solution
1 Determine the number of interest-free days.	Janelle has 20 + 25 days = 45 interest-free days.
2 Determine the number of days for which interest is payable. Since the purchase was made on 17 August, start counting from 18 August.	August: 18th–31st = 14 days September: 1st–30th = 30 days October: 1st–31st = 31 days Total days = 14 + 30 + 31 = 75 Interest payable days = 75 – 45 = 30
3 Calculate the amount payable using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$.	$P = \$1500, r = 20\%, n = 30$ $A = 1500\left(1 + \frac{20}{36\,500}\right)^{30}$ $= \$1524.85$ to the nearest cent Janelle pays back \$1524.85.

Now try this 29 Calculating credit card interest with an interest-free period (Example 29)

Lars buys a new camera that costs \$5300 using his credit card. His bank offers a 30-day statement period plus a further 15 days interest free. After that time, the bank charges interest at a rate of 17% per annum, compounding daily.

Lars purchases the camera on 18 July, which is day 20 of his statement period. He intends to pay off the credit card on 14 September. At this date, how much will he need to pay back?

Hint 1 Determine the number of interest-free and interest-payable days.

Hint 2 Calculate the total amount payable using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$.

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Purchase and investment options

There are often several options available to consumers to pay for a product. This includes paying cash, taking out a loan, using a credit card or using a 'buy-now, pay-later' option.

To determine the best option, the overall cost of each should be calculated. For example, paying cash means not bearing any interest rates or fees but requires the individual to have the cash available. Taking out a loan will require the individual to pay interest, and a 'buy-now, pay-later' option requires regular payments with fees.

Example 30 Purchase options

A purchase of \$2000 is made. Calculate the cost of the purchase under each of the following options.

a Cash: Pay the amount of the purchase in cash at the time of the purchase.

b Simple-interest loan: A simple-interest loan with annual interest rate of 8%, with the interest and principal paid back after two years.

c Compound-interest loan: A compound-interest loan with annual interest rate of 6%, with interest accruing annually and interest and principal payable after two years.

d Credit card: Use a credit card with a statement length of 30 days plus an additional 15 days, after which an interest rate of 20% per annum, compounding daily, is applied. The purchase is made on day 15 of the statement, and the full amount plus interest is repaid after two years.

e Buy-now, pay-later: An initial fee of \$30 and a monthly fee of \$10, paid each month for two years when the principal will be paid back.

f Establish which option is cheapest when cash is not available.

Explanation	Solution
a Only the purchase price is required.	Pay \$2000 at the time of purchase.
b **1** Calculate the amount of interest paid annually, using $\frac{r}{100} \times V_0$.	$\frac{8}{100} \times \$2000 = \$160$
2 Calculate the total amount payable at the end of two years ($n = 2$), using the recursion relation for $V_n = 2000 + 160n$.	$V_2 = 2000 + 160 \times 2$ $= \$2320$ The total cost of a simple-interest loan is \$2320.
c **1** Calculate the common ratio using $1 + \frac{r}{100}$.	With an interest rate of 6%, the common ratio is 1.06.
2 Calculate the total amount payable at the end of two years, using the rule $V_n = 1.06^n \times 2000$.	$V_2 = 1.06^2 \times 2000$ $= \$2247.20$ The total cost of a compound-interest loan is \$2247.20.

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d **1** Calculate the number of days that interest is payable (2 years = 730 days).	730 days - (15 days left on statement + 15 days of interest free) = 700 days
2 Calculate the amount payable using the rule: $A = P\left(1 + \frac{r}{36\,500}\right)^n$.	$A = 2000\left(1 + \frac{20}{36\,500}\right)^{700}$ $= \$2934.70$ The total cost of using the credit card is \$2934.70.
e **1** Calculate the total fees payable.	$30 + 24 \times 10 = \$270$
2 Calculate the total amount payable at the end of two years by adding the total fees and the purchase price.	\$270 + \$2000 = \$2270 The total cost of buy-now, pay-later is \$2270.
f Compare the cost of each option.	The best option is a compound-interest loan, costing \$2247.20.

Now try this 30 Purchase options (Example 30)

A purchase of \$3000 is made. Calculate the cost of the purchase under each of the following options.

a Cash: Pay the amount of the purchase in cash at the time of the purchase.

b Simple-interest loan: A simple-interest loan with annual interest rate of 7%, with the interest and principal paid back after two years.

c Compound-interest loan: A compound-interest loan with annual interest rate of 6%, with interest accruing annually and the interest and principal payable after two years.

d Credit card: Use a credit card with a statement length of 30 days plus an additional 12 days, after which an interest rate of 19% per annum, compounding daily, is applied. The purchase is made on day 20 of the statement, and the full amount plus interest is repaid 100 days later.

e Buy-now, pay-later: An initial fee of \$20 and a monthly fee of \$12, paid each month for two years when the principal will be paid back.

f Establish which option is cheapest when cash is not available.

Hint 1 Calculate the total cost of each of the five options.

Hint 2 Determine which is the cheapest, other than cash.

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Inflation: Effect on prices and purchasing power

Inflation is a term that describes the continuous upward movement in the general level of prices. This has the effect of steadily *reducing* the **purchasing power** of your money; that is, what you can actually buy with your money.

In the early 1970s, inflation rates were very high, up to around 16% and 17%. Between 2009 and 2020, inflation in Australia has been low but continued to fluctuate, ranging from 0.9% and 3.3%.

Example 31 Determining the effect of inflation on prices over a short period of time

Suppose that inflation is recorded as 2.7% in 2022 and 3.5% in 2023, and that a loaf of bread costs \$2.20 at the end of 2021. If the price of bread increases with inflation, what will be the price of the loaf at the end of 2023?

Explanation	Solution
1 Determine the increase in the price of the loaf of bread at the end of 2022 after a 2.7% increase.	Increase in price (2022): $= 2.20 \times \frac{2.7}{100} = 0.06$
2 Calculate the price at the end of 2022.	$\text{Price}_{2022} = 2.20 + 0.06 = \$2.26$
3 Determine the increase in the price of the loaf of bread at the end of 2023 after a further 3.5% increase.	Increase in price (2023) $= 2.26 \times \frac{3.5}{100}$ $= 0.08$
4 Calculate the price at the end of 2023.	$\text{Price}_{2023} = 2.26 + 0.08 = \$2.34$

Now try this 31 Determining the effect of inflation on prices over a short period of time (Example 31)

Suppose that inflation is recorded as 1.3% in 2021 and 2.0% in 2022, and that a meat pie costs \$4.10 at the end of 2020. If the price of a meat pie increases with inflation, what will be the price of the pie at the end of 2022?

Hint 1 Determine the price of the pie at the end of 2021 after the increase of 1.3%.
Hint 2 Determine the price of the pie at the end of 2022 after a further increase of 2.0%.

While the difference in price in one or two years does not seem like a lot, over the long term, the impact of inflation on prices can be significant.

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Example 32 Determining the effect of inflation on prices over a long period

Suppose that a one-litre carton of milk costs \$1.70 today.

a What will be the price of the one-litre carton of milk in 20 years' time if the average annual inflation rate is 2.1%?

b What will be the price of the one-litre carton of milk in 20 years' time if the average annual inflation rate is 6.8%?

Explanation	Solution
a 1 This is the equivalent of investing \$1.70 at 2.1% interest, compounding annually, so we can use the compound interest formula.	$A = P \times \left(1 + \frac{r}{100}\right)^t$
2 Substitute $P = 1.70$, $t = 20$ and $r = 2.1$ in the formula to find the price in 20 years.	$\text{Price} = 1.70 \times \left(1 + \frac{2.1}{100}\right)^{20}$ $= \$2.58$ to the nearest cent
b Substitute $P = 1.70$, $t = 20$ and $r = 6.8$ in the formula and evaluate.	$\text{Price} = 1.70 \times \left(1 + \frac{6.8}{100}\right)^{20}$ $= \$6.34$ to the nearest cent

Now try this 32 Determining the effect of inflation on prices over a long period (Example 32)

Suppose that a movie ticket costs \$12 today.

a What will be the price of the movie ticket in 20 years' time if the average annual inflation rate is 1.8%?

b What will be the price of the movie ticket in 20 years' time if the average annual inflation rate is 6.3%?

Hint 1 Use the rule: $A = P \times \left(1 + \frac{r}{100}\right)^t$

Another way of looking at the effect of inflation on our money is to consider what a sum of money today would buy in the future.

If you put \$100 in a box under the bed and leave it there for 10 years, what could you buy with the \$100 in 10 years' time? To find out, we need to 'deflate' this amount back to current-day purchasing power dollars, which can be done using the compound-interest formula.

Suppose there has been an average inflation rate of 4% over the 10-year period. Substituting $A = 100$, $r = 4$ and $t = 10$ gives:

$$100 = P \times \left(1 + \frac{4}{100}\right)^{10} = P \times (1 + 0.04)^{10}$$

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Rearranging this equation or using your CAS calculator to solve it, gives:

$$P = \frac{100}{(1+0.04)^{10}} = \$67.56 \text{ to the nearest cent}$$

That is, the money that was worth \$100 when it was put away has a purchasing power of only \$67.58 after 10 years if the inflation rate has averaged 4% per annum.

Example 33 Investigating purchasing power

If savings of \$100 000 are hidden in a mattress in 2021, what is the purchasing power of this amount in 8 years' time if the average inflation rate over this period is 3.7%? Give your answer to the nearest dollar.

Explanation	Solution
1 Write the compound-interest formula with P (the purchasing power, which is unknown), $A = 100\,000$ (current value), $r = 3.7$ and $t = 8$.	$A = P \times \left(1 + \frac{r}{100}\right)^t$ $100\,000 = P \times \left(1 + \frac{3.7}{100}\right)^8$
2 Use your CAS calculator to solve this equation for P, and write your answer.	The purchasing power of \$100 000 in 8 years is \$74 777, to the nearest dollar.

Now try this 33 Investigating purchasing power (Example 33)

If savings of \$60 000 are hidden in a mattress in 2022, what is the purchasing power of this amount in 10 years' time if the average inflation rate over this period is 2.2%? Give your answer to the nearest dollar.

Hint 1 Use the rule: $P = \dfrac{A}{\left(1 + \frac{r}{100}\right)^n}$

Section Summary

- Let V_0 be the amount borrowed or invested (principal) and r be the interest rate per compounding period. The value of a compound-interest loan or investment after n compounding periods, V_n, is given by:

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

- Let V_0 be the purchase price of the asset and r be the annual percentage rate of depreciation. The value of an asset after n years, V_n, is given by the rule:

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

- If a credit card debt of \$$P$ accumulates at the rate of r% per annum, compounding daily, then the amount of debt accumulated after n days is given by:

$$A = P\left(1 + \frac{r/365}{100}\right)^n = P\left(1 + \frac{r}{36\,500}\right)^n$$

  and the amount of interest payable after n days is given by: $I = A - P$.
- Inflation is a term that describes the continuous upward movement in the general level of prices.

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Exercise 3I

Building understanding

1 Consider the following recurrence relation.

$$t_0 = 5, \qquad t_{n+1} = 3t_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Find the value of:

a t_1 **b** t_3 **c** t_5 **d** t_7

2 Use the rule to find the value of $V_4 = R^4 V_0$.

a $V_0 = 5, V_{n+1} = 3V_n$
b $V_0 = 10, V_{n+1} = 2V_n$
c $V_0 = 1, V_{n+1} = 0.5V_n$
d $V_0 = 200, V_{n+1} = 0.25V_n$

3 For the following recurrence relation:

$$V_0 = 5000, \qquad V_{n+1} = 1.05V_n \qquad \text{for } n = 0, 1, 2, 3, \ldots,$$

use the rule: $V_n = R^n \times V_0$, to find the following to two decimal places.

a V_6 **b** V_{10} **c** V_{100}

Developing understanding

Example 26

4 The following recurrence relation can be used to model a compound-interest investment: $V_0 = 10\,000, V_{n+1} = 1.1V_n$, where V_n is the value of the investment after n years.

a How much money was initially invested?
b What was the annual interest rate for this investment?
c Write down a rule for the value of the investment after n years.
d Use the rule to find V_5, giving your answer to the nearest dollar.
e What does your answer to part **d** tell you?

5 The following recurrence relation can be used to model a compound-interest investment: $V_0 = 12\,000, V_{n+1} = 1.08V_n$, where V_n is the value of the investment after n years.

a How much money was initially invested?
b What was the annual interest rate for this investment?
c Write down a rule for the value of the investment after n years.
d Use the rule to find the value of the investment after 4 years, giving your answer to the nearest dollar.

6 The following recurrence relation can be used to model reducing-balance depreciation of a car: $V_0 = 18\,500, V_{n+1} = 0.9V_n$, where V_n is the value of the car after n years.

a How much was the car initially worth?
b What was the annual interest percentage depreciation?
c Write down a rule for the value of the car after n years.
d Use the rule to find V_5, giving your answer to the nearest dollar.
e What does your answer to part **d** tell you?

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7 The following recurrence relation can be used to model reducing-balance depreciation of a boat: $V_0 = 9500, V_{n+1} = 0.95V_n$, where V_n is the value of the boat after n years.

a How much was the boat initially worth?

b What was the annual interest percentage depreciation?

c Write down a rule for the value of the boat after n years.

d Use the rule to find V_{10}, giving your answer to the nearest dollar.

e What does your answer to part **d** tell you?

8 The following recurrence relation can be used to model a loan with compound interest: $V_0 = 520\,000, V_{n+1} = 0.9965V_n$, where V_n is the value of the investment after n months.

a How much money was initially borrowed?

b What was the monthly interest rate for this loan?

c Write down a rule for the value of the loan after n months.

d Use the rule to find V_6, giving your answer to the nearest dollar.

e What does your answer to part **d** tell you?

Example 27

9 A principal value of \$10 000 is invested in an account earning compound interest at the rate of 4.5% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.045^n \times 10\,000$$

a Find the value of the investment after 5 years to the nearest cent.

b Find the amount of interest earned after 5 years to the nearest cent.

c Find the amount of interest earned in the fifth year to the nearest cent.

d Let I_n be the value of the investment after n months, when the investment compounds monthly instead of yearly. Write down a rule for the value of the investment after n months.

e Use this rule to find the value of the investment after 5 years (60 months) to the nearest cent.

10 A principal value of \$300 000 is invested in an account earning compound interest at the rate of 9% per annum. The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.09^n \times 300\,000$$

a Find the value of the investment after 10 years to the nearest cent.

b Find the amount of interest earned after 10 years to the nearest cent.

c Find the amount of interest earned in the tenth year to the nearest cent.

d Let I_n be the value of the investment after n months, when the investment compounds monthly instead of yearly. Write down a rule for the value of the investment after n months.

e Use this rule to find the value of the investment after 10 years (120 months) to the nearest cent.

11 A commercial printing machine was purchased (new) for \$24 000. It depreciates in value at a rate of 9.5% per year, using a reducing-balance depreciation method. Let V_n be the value of the printer after n years.

a Write down a rule for the value of the printer after n years.

b Use the rule to find the value of the printer after 5 years to the nearest cent.

c What is the total depreciation of the printer over 5 years?

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Example 30 **12** An item that costs \$5000 is to be purchased, and the customer can choose between four different payment options including a simple-interest loan, a compound-interest loan, a credit card or a buy-now, pay-later scheme as listed below.

a Calculate the total cost of a simple-interest loan with interest of 6% per annum, paid at the end of three years. Give your answer to two decimal places.

b Calculate the total cost of a compound-interest loan with interest of 5.5% per annum, paid at the end of three years. Give your answer to two decimal places.

c Credit card: Use a credit card with a statement length of 30 days plus an additional 12 days, after which an interest rate of 20% per annum, compounding daily, is applied. The purchase is made on day 18 of the statement, and the full amount plus interest is repaid 800 days later.

d Calculate the total cost of a buy-now, pay-later option with an initial fee of \$50 and a monthly fee of \$8 over three years. Give your answer to two decimal places.

e Determine which option is cheapest for the customer.

Example 28 **13** Determine the amount of interest payable on the following credit card debts.

a \$2000 at an interest rate of 18.9% per annum for 52 days

b \$785 at an interest rate of 24% per annum for 200 days

c \$12 000 at an interest rate of 22.5% per annum for 60 days

d \$837 at an interest rate of 21.7% per annum for 90 days

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Example 29 **14** Matt has two credit cards, each with different borrowing terms.

- Credit card A charges 22% p.a. interest and offers up to 60 days interest free.
- Credit card B charges 19% p.a. but only offers 40 days interest free.

He wishes to buy an item costing \$2000 on his credit card which he will purchase at the beginning of the statement period, whichever card he uses, so as to have the maximum interest-free days. Which credit card should he use:

a if he is going to pay off the card 30 days after purchase?

b if he is going to pay off the card 60 days after purchase?

c if he is going to pay off the card 90 days after purchase?

d if he is going to pay off the card 240 days after purchase?

Example 31 **15** Suppose that inflation is recorded as 2.7% in 2017 and 3.5% in 2018, and that a magazine costs \$3.50 at the end of 2016. Assume that the price increases with inflation.

a What will be the price of the magazine at the end of 2017?

b What will be the price of the magazine at the end of 2018?

Example 32 **16** Suppose that the cost of petrol per litre is \$1.80 today.

a What will be the price of petrol per litre in 20 years' time if the average annual inflation rate is 1.9%?

b What will be the price of petrol per litre in 20 years' time if the average annual inflation rate is 7.1%?

Example 33 **17** If savings of \$200 000 are hidden in a mattress today, what is the purchasing power of that money in 10 years' time:

a if the average inflation rate over the 10 year period is 3%?

b if the average inflation rate over the 10 year period is 13%?

Testing understanding

18 Carina invested \$7500 at 6.25% per annum, compounding each year. If the investment now amounts to \$10 155.61, to the nearest cent, for how many years was it invested?

19 Tyson's grandparents invested \$5000 on his behalf when he was born. If it was invested at 4.75% per annum, compounding yearly, how old would Tyson be when the investment first exceeds \$20 000?

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Key ideas and chapter summary

Assign-ment

Sequence A **sequence** is a list of numbers or symbols in a particular order.

Term Each number or symbol that makes up a sequence is called a **term**.

Arithmetic sequence In an **arithmetic sequence**, each new term is made by adding or subtracting a fixed number, called the **common difference**, D, to or from the previous term.

Recurrence relation for an arithmetic sequence A **recurrence relation for an arithmetic sequence** has the form

$$t_0 = a, \quad t_{n+1} = t_n + D$$

where D = common difference, and a = first term.
The rule for finding t_n, the nth term in an arithmetic sequence, is:

$$t_n = a + nD$$

Graph of arithmetic sequence For an arithmetic sequence graph:

- the values lie along a straight line
- there are increasing values when $D > 0$ (positive slope)
- there are decreasing values when $D < 0$ (negative slope).

Linear growth An arithmetic sequence can be used to model **linear growth** where the value is increasing. That is, $V_0 =$ starting value, $V_{n+1} = V_n + D$, where V_n is the value after n years, and D is the amount that is added each time period. Thus, $V_n = V_0 + n \times D$.

Linear decay An arithmetic sequence can be used to model **linear decay** where the value is decreasing. That is, $V_0 =$ starting value, $V_{n+1} = V_n - D$, where V_n is the value after n years, and D is the amount that is subtracted each time period. Thus, $V_n = V_0 - n \times D$.

Principal The **principal** is the initial amount that is invested or borrowed.

Balance The **balance** is the value of a loan or investment at any time during the loan or investment period.

Interest The fee that is added to a loan or investment for borrowing or investing is called the **interest**.

Simple interest **Simple interest** is an example of linear growth where a fixed amount of interest is earned each period and found by calculating $\frac{r}{100} \times V_0$.

Depreciation **Depreciation** occurs when the value of an asset decreases over a period of time.

Flat rate depreciation **Flat rate depreciation** is an example of linear decay where a constant amount is subtracted from the value of an item at regular intervals of time.

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Unit-cost depreciation

Unit-cost depreciation is an example of linear decay that is calculated based on units of use rather than by time. The value of the asset declines by a constant amount for each unit of use (e.g. per 100 kilometres).

Geometric sequence

In a **geometric sequence**, each term is made by multiplying the previous term by a fixed number, called the common ratio, R.
For example: 5, 20, 80, 320, . . . is made by multiplying each term by 4.
The common ratio, R, is found by dividing any term by its previous term, e.g. $\frac{t_2}{t_1}$.
In our example: $R = \frac{20}{5} = 4$

Recurrence relation for a geometric sequence

A **recurrence relation for a geometric sequence** has the form

$$t_0 = a, \qquad t_{n+1} = R \times t_n$$

where R = common ratio, and a = first term.
The rule for finding t_n, the nth term, in a geometric sequence is:

$$t_n = R^n \times t_0$$

Graph of geometric sequence

For a graph of a geometric sequence:

- the values are increasing values when $R > 1$
- the values decrease towards zero when $0 < R < 1$.

Geometric growth and decay

A geometric sequence can be used to model **geometric growth** (when $R > 1$) or **geometric decay** (when $0 < R < 1$).

Reducing-balance depreciation

When the value of an item decreases as a percentage of its value after each time period, it is said to be depreciating using a reducing-balance method. **Reducing-balance depreciation** is an example of geometric decay.

Compounding period

Interest rates are usually quoted as annual rates (per annum). Interest is sometimes calculated more regularly than once a year, for example, each quarter, month, fortnight, week or day. The time period for the calculation is called the **compounding period**.

Credit

Credit is an advance of money from a financial institution, such as a bank, that does not have to be paid back immediately but which attracts interest after an interest-free period.

Inflation

Inflation is the continuous upward movement of the economy that increases prices over time or, conversely, decreases the spending power of money over time.

Purchasing power

Purchasing power describes what you can actually buy with your money.

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Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

3A **1 I can identify the behaviour of a sequence.**

e.g. Consider the following sequence and identify its behaviour as increasing, decreasing, constant, oscillating or having a limiting value: 100, 90, 80, 70, ...

3A **2 I can determine a simple rule for a sequence of numbers.**

e.g. Look for a pattern or rule in the following sequence, and hence find the next number: 3, 7, 11, 15, 19, ...

3A **3 I can generate a sequence from a starting number and a simple rule.**

e.g. Write down the first five terms of the sequence with a starting value of 3 and the rule 'add 2 to each term'.

3B **4 I can number and name terms in a sequence.**

e.g. For the sequence: 4, 8, 12, 16, . . ., state the values of t_0 and t_3.

3B **5 I can generate a sequence from a recurrence relation.**

e.g. Write down the first five terms of the sequence defined by the recurrence relation: $t_0 = 12$ and $t_{n+1} = t_n + 7$, showing the values of the first four iterations.

3C **6 I can find the common difference in an arithmetic sequence.**

e.g. Find the common difference for the arithmetic sequence: 22, 25, 28, ..., and use it to find the next term in the sequence.

3C **7 I can identify an arithmetic sequence.**

e.g. Determine whether the following sequence is an arithmetic sequence: 1, 3, 9, 27, ...

3C **8 I can tabulate and graph an arithmetic sequence.**

e.g. The sequence, 5, 8, 11, ... is an arithmetic sequence with a common difference of 3. Construct a table showing the term number, n, and its value, t_n, for the first four terms in the sequence. Use the table to plot the graph.

3D **9 I can generate an arithmetic sequence using a recurrence relation.**

e.g. Generate and graph the first four terms of the sequence defined by the recurrence relation: $t_0 = 7, t_{n+1} = t_n + 3$.

3D **10 I can find the *n*th term to solve problems involving arithmetic sequences.**

e.g. Consider the recurrence relation: $t_0 = 40$, $t_{n+1} = t_n - 3$. Find t_{20}.

3E **11 I can find the amount of simple interest each year.**

e.g. An investment of \$2200 pays interest at the rate of 5% per annum in the form of simple interest. Find the amount of interest paid each year.

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3E **12 I can use a recurrence relation to model simple interest.** ☐

e.g. The following recurrence relation can be used to model a simple-interest investment of \$1600, paying interest at a rate of 5% per annum:

$V_0 = 1600, \qquad V_{n+1} = V_n + 80.$

Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

3E **13 I can use a recurrence relation to model flat rate depreciation.** ☐

e.g. The following recurrence relation can be used to model a simple flat rate depreciation of a jet ski, purchased for \$18 000, depreciating at a flat rate of 15% per year.

$V_0 = 18\,000, \qquad V_{n+1} = V_n - 2700.$

Use the recurrence relation to find the value of the jet ski after 1, 2 and 3 years.

3E **14 I can use a recurrence relation to model unit cost depreciation.** ☐

e.g. The following recurrence relation can be used to model the unit cost depreciation of a ute, purchased for \$22 000, depreciating at a unit cost of \$200 per 10 000 km:

$V_0 = 22\,000, \qquad V_{n+1} = V_n - 200.$

Use the recurrence relation to find the value of the ute after 10 000, 20 000 and 30 000 kilometres.

3E **15 I can use a rule to determine the *n*th term for linear growth or decay.** ☐

e.g. The following recurrence relation can be used to model a simple-interest investment of \$300 000, paying interest at the rate of 3.5% per year:

$V_0 = 300\,000, \qquad V_{n+1} = V_n + 10\,500.$

Determine how much interest is added to the investment each year, and then use the rule to find the value of the investment after 12 years.

3F **16 I can find the common ratio in a geometric sequence.** ☐

e.g. For the sequence: $4, 8, 16, 32, \ldots$, state the common ratio.

3F **17 I can identify whether a sequence is a geometric sequence.** ☐

e.g. Determine whether the following sequence is a geometric sequence: $1, 3, 9, 27, \ldots$.

3F **18 I can use a CAS calculator to generate a geometric sequence.** ☐

e.g. Use your CAS calculator to generate the first 5 terms of a sequence that starts with 4 and has a common ratio of 2.

3F **19 I can graph an increasing or decreasing geometric sequence.** ☐

e.g. Consider the geometric sequence: $2, 10, 50, 250, \ldots$. Find the next term and construct a table for the first 5 terms. Use the table to plot the graph of the sequence.

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3G **20** I can generate a geometric sequence using a recurrence relation. ☐

e.g. Write down the first five terms in the geometric sequence generated by the following recurrence relation: $t_0 = 8, \quad t_{n+1} = 1.5 \times t_n$.

3G **21** I can find the *n*th term in a geometric sequence using a recurrence relation. ☐

e.g. Find the fifth term in the geometric sequence generated by the following recurrence relation: $t_0 = 10, \quad t_{n+1} = 2t_n$.

3H **22** I can use a recurrence relation to model compound interest for investments or loans. ☐

e.g. Sam invests $5000, paying interest at a rate of 6% per annum, compounding annually. Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

3H **23** I can write a recurrence relation to model a loan that compounds with a different compounding period. ☐

e.g. Jack borrows $2000. He will pay interest at the rate of 4.2% per annum. Write down a recurrence relation to model the value of Jack's loan if interest is compounded quarterly.

3H **24** I can use a recurrence relation to model reducing-balance depreciation. ☐

e.g. Tyson purchases a car for $27 500. The following recurrence relation can be used to model the car's value as it depreciates by 10% of its value each year:

$$V_0 = 27\,500, \quad V_{n+1} = 0.9V_n.$$

Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

3I **25** I can use the rule for determining the value of an investment with compounding interest or an asset with reducing-balance depreciation after a certain length of time. ☐

e.g. Adrian invests $250 000 in an account earning compound interest at the rate of 6% per annum. Use a rule to find the value of the investment after 15 years.

3I **26** I can use a rule to analyse an investment. ☐

e.g. Rohan invests $100 000 in an account earning compound interest at the rate of 3.5% per annum. Find the amount of interest earned in the fifth year of the investment.

3I **27** I can find the interest paid on a credit card. ☐

e.g. Determine how much interest is payable on a credit card debt of $4780 at an interest rate of 15.3% per annum for 24 days. Give your answer to the nearest cent.

3I **28** I can calculate credit card interest with an interest-free period. ☐

e.g. Jane pays $2700 for a holiday using her credit card. Her bank offers a 30-day statement period plus a further 15 days interest-free period. After that, the bank charges 18% per annum, compounding daily. If she pays for the holiday on day 12 of her statement period, how much does she pay back if she pays 84 days later?

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3I **29** I can determine the best purchase option. ☐

e.g. A purchase of \$2000 is made. Calculate the cost of a simple-interest loan with an annual interest rate of 8%, paid back after 2 years, and the cost of a compound-interest loan with annual interest rate of 6.5% per annum, compounding annually, and repaid after 2 years. Which is cheapest?

3I **30** I can determine the effect of inflation on prices over a short period of time. ☐

e.g. Suppose inflation is recorded as 2.2% in 2020 and 3.1% in 2021, and that a certain chocolate bar cost \$2.40 at the start of 2020. If the price of the chocolate bar increases with inflation, what will be the price at the end of 2021?

3I **31** I can determine the effects of inflation on prices over a long period of time. ☐

e.g. Suppose that one litre of milk costs \$1.90 today. What will be the price in 20 years' time if the average annual inflation is 3.8%.

3I **32** I can find the purchasing power of money. ☐

e.g. What is the purchasing power of \$80 000 in 10 years' time if the average inflation rate over this period is 3.2%?

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Multiple-choice questions

1 In the sequence: 1, 4, 7, 10, 13, . . . the next term is

A 1 **B** 4 **C** 5 **D** 13 **E** 16

2 Which of the following is an arithmetic sequence?

A 2, 4, 8, . . . **B** 2, 6, 18, . . . **C** 2, 4, 6, . . . **D** 2, 3, 5, . . . **E** 2, 4, 7, . . .

3 In the sequence: 27, 19, 11, 3, . . . the value of the common difference, D, is

A −24 **B** −8 **C** −5 **D** 3 **E** 8

4 In the sequence given by the recurrence relation: $t_0 = 63$, $t_{n+1} = t_n - 7$; what is the value of t_{15}?

A −42 **B** −35 **C** 15 **D** 161 **E** 168

5 A new boat is purchased for $110 000 with a loan. The loan is to be repaid with simple interest of 4.6% per annum. The total amount to be repaid after three years is

A $5060 **B** $15 180 **C** $110 000 **D** $120 120 **E** $125 180

6 Which of the following is a geometric sequence?

A 2, 4, 8, . . . **B** 2, 6, 10, . . . **C** 2, 4, 6, . . . **D** 2, 3, 5, . . . **E** 2, 4, 7, . . .

7 In the sequence: 27, 9, 3, 1, . . . the value of the common ratio, R, is

A −18 **B** −3 **C** 3 **D** $\frac{1}{3}$ **E** 18

8 In the sequence given by the recurrence relation: $t_0 = 10\,000$, $t_{n+1} = 0.2t_n$; what is the value of t_8?

A 0.0256 **B** 0.128 **C** 2000 **D** 20 000 **E** 2 560 000

9 The following recurrence relation can be used to model a compound-interest investment of $12 000 at the rate of 6% per annum. $V_0 = 12\,000$, $V_{n+1} = 1.06 \times V_n$. The value of the investment after 3 years, to the nearest cent, is

A $2.59 **B** $2160 **C** $2292.19 **D** $14 292.19 **E** $14 292.20

10 A new car is purchased for $135 000 with a loan. The loan is to be repaid with compound interest of 12.5% per annum. The total interest paid in the first three years, to the nearest cent, is

A $16 875.00 **B** $21 357.42 **C** $59 625.00 **D** $57 216.80 **E** $192 216.80

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Short-answer questions

1 Describe how terms are generated in each number sequence and give the next two terms.

a 2, 5, 8, 11, . . . **b** 47, 56, 65, 74, . . . **c** 16, 60, 16, 60, . . .
d 2, 6, 18, 54, . . . **e** 1000, 500, 250, 125, . . .

2 Given that the initial term in the sequence is t_0, for each sequence, state the value of the named terms

i t_0 **ii** t_3 **iii** t_5

a 12, 18, 24, 30, . . . **b** 20, 18, 16, 14, . . .
c 2, 10, 50, 250, . . .

3 Find t_{20} in the sequence: 7, 11, 15, 19, ...

4 The first term in an arithmetic sequence is 32 and the common difference is 11. Find t_{100}.

5 Generate and graph the first four terms in the sequence defined by

$$V_0 = 23, \qquad V_{n+1} = V_n + 7$$

6 The following recurrence relation can be used to model a simple-interest investment of \$20 000, paying interest at the rate of 4% per year.

$$V_0 = 20\,000, \qquad V_{n+1} = V_n + 800$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.
b When will the investment reach more than \$30 000?

7 Let V_n be the value (in dollars) of a printer after n years.
A recurrence relation that models the depreciating value of this printer over time is:

$$V_0 = 2000, \qquad V_{n+1} = V_n - 250$$

a What was the value of the printer when it was new?
b By how much (in dollars) did the printer depreciate each year?
c What was the percentage flat rate of depreciation?
d After how many years will the value of the printer be \$600?
e When will the printer devalue to half of its new price?

8 Generate and graph the first five terms of the geometric sequence defined by the recurrence relation

$$t_0 = 3, \qquad t_{n+1} = 2t_n.$$

9 Consider the recurrence relation given by

$$V_0 = 1000, \qquad V_{n+1} = 0.1V_n.$$

a Generate the first five terms of the geometric sequence defined by the recurrence relation.
b Graph the first five terms.
c Calculate the value of V_{10}.

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10 The first term in a geometric sequence is 8 and there is a common ratio of 2. Let t_n be the term after n applications of the rule. Find t_{10}.

11 The following recurrence relation can be used to model a compound-interest investment of \$20 000, paying interest at the rate of 4% per year.

$$V_0 = 20\,000, \qquad V_{n+1} = 1.04V_n$$

where V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach more than \$30 000?

12 Let V_n be the value (in dollars) of a work-station after n years.
A recurrence relation that models the depreciating value of this work-station over time is:

$$V_0 = 2000, \qquad V_{n+1} = 0.9V_n$$

a What was the value of the work-station when it was new?

b By how much (in dollars) did the work-station depreciate in the first year?

c What was the percentage flat rate depreciation?

d After how many years will the value of the work-station first be less than \$600?

e When will the work-station first devalue to less than half of its new price?

13 A given geometric sequence has a common ratio of 4. If $V_5 = 2048$, find the recurrence relation for the sequence.

Written-response questions

1 Chao wishes to buy a motor scooter which costs \$4100 when new. Its value depreciates at a flat rate of 8% per year.

a How much does the value of the motor scooter decline each year?

b How much is the motor scooter worth after one year?

c How much is the motor scooter worth after two years?

d Complete the following equations, where V_t is the value of the motor scooter at time t:
$V_1 = V_0 - \ldots$ and $V_2 = \ldots - \ldots$.

e Write the recurrence relation that models the depreciation of the motor scooter over time.

f After how many years will the value of the scooter first be less than \$1000?

2 Consider the recurrence relation:

$$t_0 = 5, \qquad t_{n+1} = t_n + 4.$$

a Construct a table showing the term number, n, and its value, t_n, for the first four terms.

b Use the table to plot the graph.

c Describe the graph and hence the sequence.

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3 Let V_n be the value (in dollars) of a delivery van after n years.
A recurrence relation that models the flat rate depreciating value of the van over time is:

$$V_0 = 64\,000, \qquad V_{n+1} = V_n - 3200$$

a What was the value of the van when it was new?

b By how much (in dollars) did the van depreciate each year?

c What was the percentage flat rate depreciation that was applied to the van?

d After how many years will the value of the van first be less than $40 000?

e When will the van devalue to half its original price?

4 A machine is purchased for $30 000 and depreciates by $0.15 for each unit that it produces.
Let V_n be the value (in dollars) of the machine after n units are produced.
A recurrence relation that models the unit cost depreciating value of the machine is:

$$V_0 = 30\,000, \qquad V_{n+1} = V_n - 0.15$$

a How much is the machine worth after it produces 1000, 2000 and 3000 units?

b If the machine produces 20 000 units each year, find the value of the machine after 3 years.

c How many units will the machine need to produce before it is devalued to half of its original price?

5 The following recurrence relation can be used to model a compound-interest investment.

$$V_0 = 6200, \qquad V_{n+1} = 1.08V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a How much was initially invested?

b What was the interest rate for the investment?

c What was the value of the investment after 1, 2 and 3 years, to the nearest cent?

d How much interest did the investment earn in each of the first three years? Give your answer to the nearest cent.

6 Hugh bought a motorbike for $9200 brand new. The bike depreciates at a reducing-balance rate of 15% per annum. Let V_n be the value of Hugh's bike after n years.

a How much does the value of the motorbike decline in the first year?

b Write down a recurrence relation to model the value of Hugh's motorbike after n years.

c Correct to the nearest dollar, how much is the motorbike worth after five years?

d After how many years will the value of the motorbike first be less than $5000?

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7 Renee has \$20 000 to invest for five years and must choose which investment offers the best deal. The bank offers the following investment options:

A Renee can invest her money and receive 5.5% simple interest per annum.

B Renee can invest her money and receive 5% interest per annum, compounding annually.

C Renee can invest her money and receive 4.5% interest per annum, compounding monthly.

Considering these options:

a How much would each option pay in interest for the first year, to the nearest dollar?

b Which option gives Renee the highest value at the end of five years?

c How much interest, in total, will Renee earn from this option? Give your answer to the nearest dollar.

8 The following recurrence relation can be used to model reducing-balance depreciation of a car:

$$V_0 = 22\,500, \qquad V_{n+1} = 0.85V_n$$

where V_n is the value of the car after n years.

a State the initial value of the car.

b What was the annual depreciation rate of the car?

c Write down a rule for the value of the car after n years.

d Use the rule to find V_5, giving your answer to the nearest dollar.

e After how many years will the value of the car be less than half of its starting value?

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