Chapter 4

Matrices

Chapter questions

- What is a matrix?
- How is the order of a matrix defined?
- How are the positions of the elements of a matrix specified?
- What are the rules for adding and subtracting matrices?
- How do we multiply a matrix by a scalar?
- What is the method for multiplying a matrix by another matrix?
- What are transition matrices and how do we apply them?
- What are inverse matrices and how can they be used to solve equations?
- How can your CAS calculator be used to do matrix operations?

A **matrix** is a rectangular group of numbers, set out in rows and columns. A matrix (plural matrices) can be used to store information, solve sets of simultaneous equations or be used in various applications. We will explore some of these applications while learning the basic theory of matrices.

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4A The basics of a matrix

Learning intentions

- To be able to state the order of a given matrix.
- To be able to describe the location of an element in a matrix.

Matrices can be used to store and display information, for example, the sales of a market stall that operates on Friday and Saturday are recorded in matrix *A*.

$$A = \begin{array}{r} \\ \text{Friday} \\ \text{Saturday} \\ \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Shirts} & \textit{Jeans} & \textit{Belts} \end{array} \\ \left[\begin{array}{ccc} 6 & 8 & 4 \\ 3 & 7 & 1 \end{array}\right] \\ \begin{array}{ccc} \textit{column 1} & \textit{column 2} & \textit{column 3} \end{array} \end{array} \begin{array}{l} \\ \textit{row 1} \\ \textit{row 2} \\ \end{array}$$

Rows	Columns
Friday's sales are listed in row 1. Saturday's sales are listed in row 2.	The number of shirts sold is listed in column 1. The number of pairs of jeans sold is listed in column 2. The number of belts sold is listed in column 3.

We can read the following information from the matrix:

- on Friday, 8 pairs of jeans were sold
- on Saturday, 1 belt was sold
- the total number of items sold on Friday was $6 + 8 + 4 = 18$
- the total number of belts sold was $4 + 1 = 5$.

Order of a matrix

The **order** (or size) **of a matrix** is written as: number of rows × number of columns.

The number of rows is always given first, then the number of columns. For example, the order of matrix *A* in the market stall example above is 2×3; that is, 2 rows × 3 columns. It is called a 'two by three' matrix.

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Elements of a matrix

The numbers within a matrix are called its **elements**.

Matrices are usually named using capital letters, such as A, B, C. The corresponding lowercase letter with subscripts is used to denote an element of a matrix.

An element of a matrix

a_{ij} is the element in row i, column j of matrix A.

For example, in the matrix:

$$A = \begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix}$$

- element a_{13} is in row 1, column 3, and its value is 4
- element a_{22} is in row 2, column 2, and its value is 7.

Example 1 Interpreting the elements of a matrix

Matrix B shows the number of boys and girls in Years 10 to 12 at a particular school.

$$B = \begin{array}{c} \\ \textit{Year 10} \\ \textit{Year 11} \\ \textit{Year 12} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Boys} & \textit{Girls} \end{array} \\ \begin{bmatrix} 57 & 63 \\ 48 & 54 \\ 39 & 45 \end{bmatrix} \end{array}$$

a Give the order of matrix B.

b What information is given by the element b_{12}?

c Which element gives the number of girls in Year 12?

d How many boys are there in total?

e How many students are in Year 11?

Explanation	Solution
a Count the rows, count the columns.	The order of matrix B is 3×2.
b The element b_{12} is in row 1 and column 2. This is where the Year 10 row meets the Girls column.	There are 63 girls in Year 10.
c Year 12 is row 3. Girls are column 2.	b_{32} gives the number of Year 12 girls.
d The sum of the Boys column gives the total number of boys.	The total number of boys is 144.
e The sum of the Year 11 row gives the total number of students in Year 11.	There are 102 students in Year 11.

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Now try this 1 **Interpreting the elements of a matrix (Example 1)**

Matrix C shows the number of men and women working in three different departments of an organisation.

$$C = \begin{array}{r} \\ \textit{Finance} \\ \textit{Legal} \\ \textit{Sales} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Men} & \textit{Women} \end{array} \\ \begin{bmatrix} 21 & 32 \\ 11 & 18 \\ 62 & 41 \end{bmatrix} \end{array}$$

a Give the order of matrix C.

b What information is given by the element c_{21}?

c Which element gives the number of men working in Sales?

d How many women work in the three departments in total?

Hint 1 Determine the number of rows and columns to find the order.

Hint 2 Remember that rows come before columns.

Hint 3 To find the totals, add up all the relevant numbers from a row or a column.

Row matrices

A **row matrix** has a *single row* of elements.

In matrix A, the Friday sales from the market stall can be represented as a 1 × 3 *row* matrix.

$$\begin{array}{c|ccc|} & \textit{Shirts} & \textit{Jeans} & \textit{Belts} \\ \textit{Friday} & 6 & 8 & 4 \end{array}$$

Column matrices

A **column matrix** has a *single column* of elements.

In matrix A, the sales of jeans from the market stall can be represented as a 2 × 1 *column* matrix.

$$\begin{array}{c|c|} & \textit{Jeans} \\ \textit{Friday} & 8 \\ \textit{Saturday} & 7 \end{array}$$

Square matrices

In **square matrices**, the number of *rows* equals the number of *columns*.

Here are three examples.

$$\underset{1 \times 1}{[9]} \qquad \underset{2 \times 2}{\begin{bmatrix} 5 & 4 \\ 6 & 2 \end{bmatrix}} \qquad \underset{3 \times 3}{\begin{bmatrix} 0 & 4 & 3 \\ 4 & 1 & 6 \\ 3 & 6 & 7 \end{bmatrix}}$$

A square matrix is **symmetric** if all elements in the matrix can be swapped, as follows: $a_{ij} = a_{ji}$. The third square matrix above is an example of a symmetric matrix.

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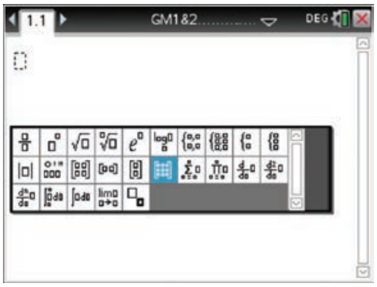
How to enter a matrix using the TI-Nspire CAS

Enter the matrix $B = \begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$ into the TI-Nspire CAS. Display the element b_{21}.

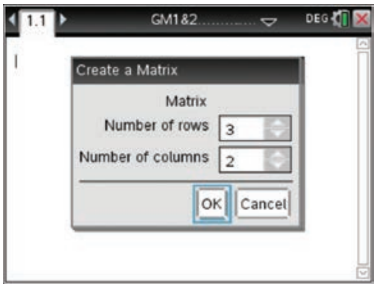
Steps

1 Press on **>New Document>Add Calculator**.

2 Press ⊞ and use the cursor ▼ ▶ arrows to highlight the matrix template shown. Press enter.
Note: Math Templates can also be accessed by pressing ctrl + menu **>Math Templates**.
Note: You can also press menu> 7> 1> 1

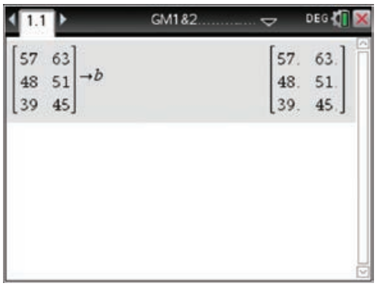


3 Press ◀ then ▲ or ▼ to select the **Number of rows** required (the number of rows in this example is 3).
Press tab to move to the next entry and repeat for the **Number of columns** (the number of columns in this example is 2).
Press tab to highlight **OK** and press enter.



Create a Matrix
Matrix
Number of rows 3
Number of columns 2

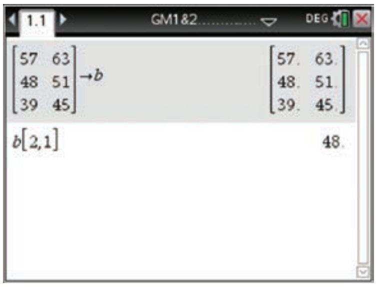
4 Type the values into the matrix template. Use tab or the arrow keys to move to the required position in the matrix to enter each value.
When the matrix has been completed, press tab or ▶ to move outside the matrix, and press ctrl+var followed by B. This will store the matrix as the variable **b**. Press enter.



$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix} \to b$ $\begin{bmatrix} 57. & 63. \\ 48. & 51. \\ 39. & 45. \end{bmatrix}$

5 When you type B (or b) in the CAS calculator,
it will paste in the matrix $\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$.

6 To display element b_{21} (the element in position Row 2, Column 1), type in **b[2,1]** and press enter.



$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix} \to b$ $\begin{bmatrix} 57. & 63. \\ 48. & 51. \\ 39. & 45. \end{bmatrix}$

b[2,1] 48.

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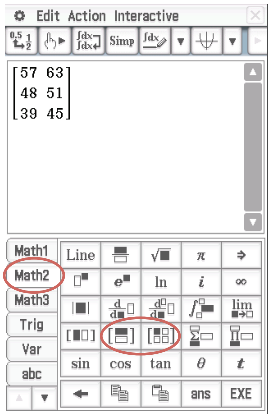
How to enter a matrix using the ClassPad

Enter the matrix $B = \begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$ into the ClassPad calculator.

Steps

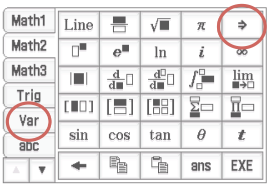
1. Open the soft Keyboard in the **Main** application $\sqrt{\alpha}$.
2. Select the Math2 keyboard.
3. Tap the 2 × 2 matrix followed by the 2 × 1 matrix icon. This will add a third row and create a 3 × 2 matrix.
4. Enter the values of $\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$.

   Note: Tap in each new position to enter the new value or use the cursor key on the hard keyboard to navigate to a new position.

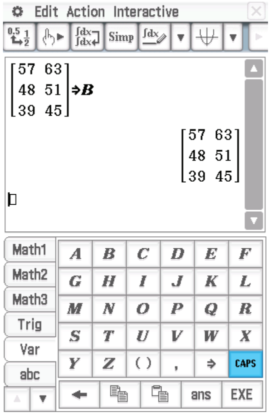


$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$

5. To assign the matrix the variable name B:
   - **a** Move the cursor to the very right-hand side of the matrix.
   - **b** From the keyboard, tap the variable assignment key ⇒, followed by the var, then CAPS (for uppercase letters) and ***B***. Press EXE to confirm your choice.



Note: Until it is reassigned, ***B*** will represent the matrix as defined above.



$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix} \Rightarrow B$

$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$

Section Summary

- A **matrix** is a rectangular array of numbers, set out in rows and columns within square brackets. The rows are horizontal; the columns are vertical.
- The **order** (size) of a matrix is the number of rows × the number of columns.
- The element a_{ij} is the number in row i and column j of the matrix.
- A **row** matrix has a single row of elements.
- A **column** matrix has a single column of elements.
- A **square** matrix has an equal number of rows and columns.

Exercise 4A

Building understanding

1 Consider the following matrix:

$$A = \begin{bmatrix} 3 & 4 & 12 & 1 \\ 0 & 7 & 9 & 10 \end{bmatrix}$$

a State the number of rows in matrix A.

b State the number of columns in matrix A.

c State the order of matrix A.

d State the number of elements in matrix A.

2 Students in Year 11 were surveyed about whether they played a musical instrument or sport. The results were recorded in matrix S.

$$S = \begin{array}{c} \\ Girls \\ Boys \end{array} \begin{array}{c} \begin{array}{cc} Music & Sport \end{array} \\ \begin{bmatrix} 38 & 53 \\ 45 & 36 \end{bmatrix} \end{array}$$

a How many girls play sport?

b How many boys play sport?

c How many students from Year 11 play a musical instrument?

Developing understanding

Example 1

3 Matrix C is shown on the right.

$$C = \begin{bmatrix} 2 & 4 & 16 & 7 \\ 6 & 8 & 9 & 3 \\ 5 & 6 & 10 & 1 \end{bmatrix}$$

a Write down the order of matrix C.

b State the value of:

i c_{13} **ii** c_{24} **iii** c_{31}

c Write down the sum of the elements in row 3.

d Write down the sum of the elements in column 2.

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4 For each of the following matrices:

i state the order **ii** write down the values of the required elements.

a $A = \begin{bmatrix} 5 & 6 & 8 \\ 4 & 7 & 9 \end{bmatrix}$ Find a_{12} and a_{22}

b $B = \begin{bmatrix} 6 & 8 & 2 \end{bmatrix}$ Find b_{13} and b_{11}

c $C = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 8 & -4 \end{bmatrix}$ Find c_{32} and c_{12}

d $D = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}$ Find d_{31} and d_{11}

e $E = \begin{bmatrix} 10 & 12 \\ 15 & 13 \end{bmatrix}$ Find e_{21} and e_{12}

f $F = \begin{bmatrix} 8 & 11 & 2 & 6 \\ 4 & 1 & 5 & 7 \\ 6 & 14 & 17 & 20 \end{bmatrix}$ Find f_{34} and f_{23}

5 Name which of the matrices in Question **4** are:

a row matrices **b** column matrices **c** square matrices.

6 Which three of these matrices is symmetric?

A $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

7 For matrix D, state the values of the following elements.

$$D = \begin{bmatrix} 3 & 4 & 6 & 11 & 2 \\ 5 & 1 & 9 & 10 & 4 \\ 8 & 7 & 2 & 0 & 1 \\ 6 & 8 & 5 & 8 & 2 \end{bmatrix}$$

a d_{23} **b** d_{45}

c d_{11} **d** d_{24}

e d_{42}

8 Consider the following matrices.

$$A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 5 & 3 \\ -3 & 4 & 8 \\ 7 & 6 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ -5 \end{bmatrix} \quad C = \begin{bmatrix} 8 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 4 & -3 & 0 & 1 & 9 \\ 6 & 11 & 2 & 7 & 5 \end{bmatrix}$$

a Write down the order of each matrix A, B, C and D.

b Identify the elements: a_{32}, b_{21}, c_{11} and d_{24} of matrices A, B, C and D respectively.

9 Some students were asked which of four sports they preferred to play, and the results were entered in the following matrix.

$$S = \begin{array}{r} \\ \textit{Year 10} \\ \textit{Year 11} \\ \textit{Year 12} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Tennis} & \textit{Basketball} & \textit{Football} & \textit{Hockey} \end{array} \\ \begin{bmatrix} 19 & 18 & 31 & 14 \\ 16 & 32 & 22 & 12 \\ 21 & 25 & 5 & 7 \end{bmatrix} \end{array}$$

a How many Year 11 students preferred basketball?

b Write down the order of matrix S.

c What information is given by s_{23}?

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10 Matrix F shows the number of hectares of land used for different purposes on two farms, X and Y. Row 1 represents Farm X and row 2 represents Farm Y. Columns 1, 2 and 3 show the amount of land used for wheat, cattle and sheep (W, C, S) respectively, in hectares.

$$F = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} W & C & S & \\ \end{array}$$

$$F = \begin{bmatrix} 150 & 300 & 75 \\ 200 & 0 & 350 \end{bmatrix}\begin{array}{c} X \\ Y \end{array}$$

a How many hectares are used on:

i Farm X for sheep? **ii** Farm X for cattle?

iii Farm Y for wheat?

b Calculate the total number of hectares used on both farms for wheat.

c Write down the information that is given by:

i f_{22} **ii** f_{13} **iii** f_{11}

d Which element of matrix F gives the number of hectares used:

i on Farm Y for sheep? **ii** on Farm X for cattle?

iii on Farm Y for wheat?

e State the order of matrix F.

Testing understanding

11 Given the information provided, construct a matrix, B, to show the number of pies and sausage rolls sold by two bakeries last Saturday.

- Row 1 represents Bakery 1 and row 2 represents Bakery 2.
- Column 1 shows the number of pies, and column 2 shows the number of sausage rolls.
- Bakery 2 sold 165 pies and 181 sausage rolls.
- Bakery 1 sold 30 more pies than Bakery 2.
- Bakery 1 sold 40 fewer sausage rolls than Bakery 2.

12 A group of Year 11 students were surveyed about their preferred activity on a school trip. The results were entered into the following matrix, but unfortunately, some of the data was missing.

$$S = \begin{array}{c} \\ \textit{Boys} \\ \textit{Girls} \end{array}\begin{array}{c} \begin{array}{cccc} \textit{Ski} & \textit{Mountain bike} & \textit{Hike} & \textit{Kayak} \end{array} \\ \begin{bmatrix} 28 & \dots & 29 & \dots \\ \dots & 83 & \dots & 31 \end{bmatrix} \end{array}$$

Use the following information to complete the matrix.

- A total of 137 students wanted to go mountain bike riding.
- s_{23} is half of s_{11}.
- The total number of boys surveyed was 146.
- The number of girls who wanted to go skiing was 5 more than the number of girls who wanted to go kayaking.

4B Adding and subtracting matrices

Learning intentions

- To be able to add and subtract matrices.
- To be able to identify a zero matrix.

Matrices of the same order can be added or subtracted by following a simple rule.

Rules for adding and subtracting matrices

1. Matrices are added by adding the elements that are in the same positions.
2. Matrices are subtracted by subtracting the elements that are in the same positions.
3. **Matrix addition and subtraction** can only be done if the two matrices have the *same order*.

Example 2 Addition and subtraction of matrices

Complete the following addition and subtraction of matrices.

a $\begin{bmatrix}2&4\\5&1\end{bmatrix}+\begin{bmatrix}9&8\\9&-1\end{bmatrix}$

b $\begin{bmatrix}7&3\\2&8\\1&0\end{bmatrix}-\begin{bmatrix}4&2\\-1&9\\3&7\end{bmatrix}$

Explanation	**Solution**
a **1** Write the addition.	$\begin{bmatrix}2&4\\5&1\end{bmatrix}+\begin{bmatrix}9&8\\9&-1\end{bmatrix}$
2 Add the elements that are in the same positions.	$=\begin{bmatrix}2+9&4+8\\5+9&1+(-1)\end{bmatrix}$
3 Evaluate each element.	$=\begin{bmatrix}11&12\\14&0\end{bmatrix}$
b **1** Write the subtraction.	$\begin{bmatrix}7&3\\2&8\\1&0\end{bmatrix}-\begin{bmatrix}4&2\\-1&9\\3&7\end{bmatrix}$
2 Subtract the elements that are in the same positions, and evaluate each element.	$=\begin{bmatrix}7-4&3-2\\2-(-1)&8-9\\1-3&0-7\end{bmatrix}=\begin{bmatrix}3&1\\3&-1\\-2&-7\end{bmatrix}$

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Now try this 2 **Addition and subtraction of matrices (Example 2)**

Complete the following addition and subtraction of matrices.

a $\begin{bmatrix} 3 & -2 & 6 \\ 8 & 4 & 12 \end{bmatrix} + \begin{bmatrix} 11 & 5 & 7 \\ 1 & -3 & -5 \end{bmatrix}$ **b** $\begin{bmatrix} 10 & -1 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ -3 & 10 \end{bmatrix}$

Hint 1 Write the addition or subtraction.

Hint 2 Add (or subtract) the elements that are in the same position.

Hint 3 Evaluate each element.

The zero matrix, 0

In a **zero matrix**, every element is zero. The zero matrix is sometimes called the null matrix.

The following are examples of zero matrices.

$$\begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Just as in standard arithmetic, adding or subtracting a zero matrix does not make any change to the original matrix. For example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Subtracting any matrix from itself gives a zero matrix. For example:

$$\begin{bmatrix} 9 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Section Summary

- Matrices of the same order can be **added** by adding elements in the same position.
- Matrices of the same order can be **subtracted** by subtracting the elements in the same position.
- A **zero matrix** is any matrix where every element is zero.

Exercise 4B

Building understanding

1 Which of the following matrices can be added together?

$$A = \begin{bmatrix} 9 & -5 \\ -3 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 14 \\ -3 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 6 & 4 & -2 \\ 8 & 1 & 9 \end{bmatrix}$$

$$E = \begin{bmatrix} 5 & 6 \\ 12 & 2 \\ 7 & -2 \end{bmatrix} \quad F = \begin{bmatrix} 5 & 13 & -7 \\ 8 & -2 & 6 \end{bmatrix} \quad G = \begin{bmatrix} -4 & 8 \\ 10 & 2 \end{bmatrix} \quad H = \begin{bmatrix} 9 & 6 \\ -8 & 3 \\ 2 & 1 \end{bmatrix}$$

2 True or false?

$$\begin{bmatrix}3 & -1\\6 & 2\end{bmatrix}+\begin{bmatrix}8 & 5\\-7 & 4\end{bmatrix}=\begin{bmatrix}11 & 6\\-1 & 6\end{bmatrix}$$

3 Calculate:

$$\begin{bmatrix}-3 & 3\\9 & 12\end{bmatrix}-\begin{bmatrix}-3 & 3\\9 & 12\end{bmatrix}$$

Developing understanding

Example 2

4 Complete the following addition and subtraction of matrices.

a $\begin{bmatrix}4 & 3\\0 & 2\end{bmatrix}+\begin{bmatrix}5 & 7\\6 & 1\end{bmatrix}$ **b** $\begin{bmatrix}8 & 6\\9 & 3\end{bmatrix}+\begin{bmatrix}-1 & 2\\4 & 0\end{bmatrix}$ **c** $\begin{bmatrix}3 & 5\\7 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

d $\begin{bmatrix}9\\8\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}$ **e** $\begin{bmatrix}8 & 6\\2 & 9\end{bmatrix}-\begin{bmatrix}5 & 2\\1 & 3\end{bmatrix}$ **f** $\begin{bmatrix}7 & 4\\5 & 1\end{bmatrix}-\begin{bmatrix}3 & 6\\2 & -8\end{bmatrix}$

g $\begin{bmatrix}4 & 2\end{bmatrix}+\begin{bmatrix}8 & 5\end{bmatrix}$ **h** $\begin{bmatrix}7 & -5\end{bmatrix}-\begin{bmatrix}7 & -5\end{bmatrix}$ **i** $\begin{bmatrix}4 & -3\end{bmatrix}+\begin{bmatrix}-4 & 3\end{bmatrix}$

5 Using the matrices given:

$$A=\begin{bmatrix}3 & -2\\2 & 4\end{bmatrix}\quad B=\begin{bmatrix}5 & 7\\1 & 3\end{bmatrix}\quad C=\begin{bmatrix}6 & 2\\1 & 0\\3 & -8\end{bmatrix}\quad D=\begin{bmatrix}-3 & 5\\4 & -2\\1 & 7\end{bmatrix}\quad E=\begin{bmatrix}8\\3\end{bmatrix}$$

find, where possible:

a $A+B$ **b** $B+A$ **c** $A-B$ **d** $B-A$

e $B+E$ **f** $C+D$ **g** $B+C$ **h** $D-C$

6 Two people conducted a telephone poll, surveying voters about their voting intentions. The results for each person's survey are given in matrix form.

Sample 1:

$$\begin{array}{r c}
 & \begin{array}{cccc}\textit{Liberal} & \textit{Labor} & \textit{Independent} & \textit{Greens}\end{array}\\
\begin{array}{r}\textit{Men}\\\textit{Women}\end{array} & \begin{bmatrix}19 & 21 & 7 & 3\\18 & 17 & 11 & 4\end{bmatrix}
\end{array}$$

Sample 2:

$$\begin{array}{r c}
 & \begin{array}{cccc}\textit{Liberal} & \textit{Labor} & \textit{Independent} & \textit{Greens}\end{array}\\
\begin{array}{r}\textit{Men}\\\textit{Women}\end{array} & \begin{bmatrix}24 & 21 & 3 & 2\\19 & 20 & 6 & 5\end{bmatrix}
\end{array}$$

Write a matrix showing the overall result of the survey.

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7 Three sports stores do a stocktake at the start of each month. The results for January and February are recorded in matrix form.

January:

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Basketballs} & \textit{Netballs} & \textit{Cricket balls} & \textit{Footballs} \end{array} \\ \begin{array}{c} \textit{Store 1} \\ \textit{Store 2} \\ \textit{Store 3} \end{array} & \begin{bmatrix} 32 & 10 & 82 & 41 \\ 29 & 17 & 75 & 44 \\ 22 & 12 & 103 & 61 \end{bmatrix} \end{array}$$

February:

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Basketballs} & \textit{Netballs} & \textit{Cricket balls} & \textit{Footballs} \end{array} \\ \begin{array}{c} \textit{Store 1} \\ \textit{Store 2} \\ \textit{Store 3} \end{array} & \begin{bmatrix} 26 & 10 & 25 & 26 \\ 12 & 12 & 21 & 31 \\ 22 & 5 & 30 & 18 \end{bmatrix} \end{array}$$

Write a matrix showing the overall change in the stock levels from January to February.

8 The weights and heights of four people were recorded and then checked again one year later.

2024 results:

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Arlo} & \textit{Beni} & \textit{Cal} & \textit{Dane} \end{array} \\ \begin{array}{c} \textit{Weight (kg)} \\ \textit{Height (cm)} \end{array} & \begin{bmatrix} 32 & 44 & 59 & 56 \\ 145 & 155 & 160 & 164 \end{bmatrix} \end{array}$$

2025 results:

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Arlo} & \textit{Beni} & \textit{Cal} & \textit{Dane} \end{array} \\ \begin{array}{c} \textit{Weight (kg)} \\ \textit{Height (cm)} \end{array} & \begin{bmatrix} 38 & 52 & 57 & 63 \\ 150 & 163 & 167 & 170 \end{bmatrix} \end{array}$$

a Write the matrix that gives the changes in each person's weight and height after one year.

b Who gained the most weight?

c Which person had the greatest height increase?

Testing understanding

9 Find the values of a, b, c and d in the following matrix addition.

$$\begin{bmatrix} 3 & 11 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 2 & 8 \end{bmatrix}$$

10 Find the values of a, b, c and d in the following matrix subtraction.

$$\begin{bmatrix} 8 & 4 \\ -2 & 12 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ -9 & 0 \end{bmatrix}$$

11 The number of kicks, goals and handballs for three footballers was recorded for the first three rounds of the AFL football season. Unfortunately, some numbers were missing. Use the information below to complete the matrices.

Round 1:

$$\begin{array}{r|ccc|} & \textit{Kicks} & \textit{Goals} & \textit{Handballs} \\ \textit{Jack} & 10 & 4 & 2 \\ \textit{Nick} & \ldots & 0 & 26 \\ \textit{Mykola} & 18 & 1 & 12 \end{array}$$

Round 2:

$$\begin{array}{r|ccc|} & \textit{Kicks} & \textit{Goals} & \textit{Handballs} \\ \textit{Jack} & 7 & \ldots & 8 \\ \textit{Nick} & 13 & 2 & 12 \\ \textit{Mykola} & 9 & \ldots & \ldots \end{array}$$

Round 3:

$$\begin{array}{r|ccc|} & \textit{Kicks} & \textit{Goals} & \textit{Handballs} \\ \textit{Jack} & \ldots & 3 & \ldots \\ \textit{Nick} & 20 & 1 & 19 \\ \textit{Mykola} & \ldots & 4 & 11 \end{array}$$

Totals:

$$\begin{array}{r|ccc|} & \textit{Kicks} & \textit{Goals} & \textit{Handballs} \\ \textit{Jack} & 23 & \ldots & \ldots \\ \textit{Nick} & \ldots & \ldots & \ldots \\ \textit{Mykola} & \ldots & 7 & 34 \end{array}$$

- Jack kicked a total of 9 goals.
- Mykola had 6 more kicks in Round 1 than Round 3.
- Nick had a total of 43 kicks.
- In the third round, Jack had twice as many handballs than the first two rounds combined.

4C Scalar multiplication

Learning intentions

- To be able to perform scalar multiplication.
- To be able to apply scalar multiplication with addition and subtraction of matrices.

A *scalar* is just a number. Multiplying a matrix by a number is called **scalar multiplication**.

Multiplying a matrix by a scalar

Scalar multiplication is the process of multiplying a matrix by a number (a scalar).

In scalar multiplication, each element is multiplied by that scalar (number).

Example 3 Scalar multiplication

If $A = \begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$, find 3$A$.

Explanation	Solution
1 If $A = \begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$, then $3A = 3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$.	$3A = 3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$
2 Multiply each number in the matrix by 3.	$= \begin{bmatrix} 3 \times 5 & 3 \times 1 \\ 3 \times -3 & 3 \times 0 \end{bmatrix}$
3 Evaluate each element.	$= \begin{bmatrix} 15 & 3 \\ -9 & 0 \end{bmatrix}$

Now try this 3 Scalar multiplication (Example 3)

If $B = \begin{bmatrix} -7 & 2 \\ 6 & -3 \end{bmatrix}$, find 5$B$.

Hint 1 Write down the scalar multiplication.
Hint 2 Multiply each number in the matrix by the scalar.
Hint 3 Evaluate each element.

Scalar multiplication can also be used in conjunction with addition and subtraction of matrices.

Example 4 Scalar multiplication and subtraction of matrices

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find the matrix equal to $2A - 3B$.

Explanation	Solution
1 Write $2A - 3B$ in expanded matrix form.	$2A - 3B = 2 \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 3 \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
2 Multiply the elements in A by 2 and the elements in B by 3.	$= \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$
3 Subtract the elements in corresponding positions and evaluate.	$= \begin{bmatrix} 2-0 & 2-3 \\ 0-3 & 2-0 \end{bmatrix}$ $= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

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Now try this 4 Scalar multiplication and subtraction of matrices (Example 4)

If $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find the matrix equal to $3A - 2B$.

Hint 1 Write $3A - 2B$ in expanded matrix form.

Hint 2 Multiply the elements in A by 3 and the elements in B by 2.

Hint 3 Subtract the elements in corresponding positions.

Scalar multiplication has many practical applications. It is particularly useful in scaling up the elements of a matrix, for example, adding the GST to the cost of the prices of all items in a shop by multiplying a matrix of prices by 1.1 (adding GST of 10% is the same as multiplying by 1.1).

Example 5 Application of scalar multiplication

A gymnasium has the enrolments in courses shown in this matrix.

$$\begin{array}{r c} & \begin{array}{ccc} \textit{Body building} & \textit{Aerobics} & \textit{Fitness} \end{array} \\ \begin{array}{r} \textit{Men} \\ \textit{Women} \end{array} & \begin{bmatrix} 70 & 20 & 80 \\ 10 & 50 & 60 \end{bmatrix} \end{array}$$

The manager wishes to double the enrolments in each course. Construct a matrix showing the new enrolments for men and women in each course.

Explanation

1 Each element in the matrix is multiplied by 2.

2 Evaluate each element.

Solution

$$2 \times \begin{bmatrix} 70 & 20 & 80 \\ 10 & 50 & 60 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 70 & 2 \times 20 & 2 \times 80 \\ 2 \times 10 & 2 \times 50 & 2 \times 60 \end{bmatrix}$$

$$\begin{array}{r c} & \begin{array}{ccc} \textit{Body building} & \textit{Aerobics} & \textit{Fitness} \end{array} \\ \begin{array}{r} = \textit{Men} \\ \textit{Women} \end{array} & \begin{bmatrix} 140 & 40 & 160 \\ 20 & 100 & 120 \end{bmatrix} \end{array}$$

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Now try this 5 Application of scalar multiplication (Example 5)

A popular burger chain has the following prices for its products, as shown in this matrix.

$$\begin{array}{r|ccc|}
 & \textit{Hamburger} & \textit{Chips} & \textit{Can of Softdrink} \\
\textit{Sydney} & 13 & 4.50 & 3 \\
\textit{Melbourne} & 12 & 5 & 2.50
\end{array}$$

The manager wishes to know how much to charge after 10% GST is added, which she knows is the same as multiplying the matrix by 1.1. Show this in a matrix.

Hint 1 Write down the scalar multiplication.

Hint 2 Multiply each number in the matrix by the scalar.

Hint 3 Evaluate each element.

How to add, subtract and scalar multiply matrices using the TI-Nspire CAS

If $A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $9A$ **d** $15A - 11B$

Steps

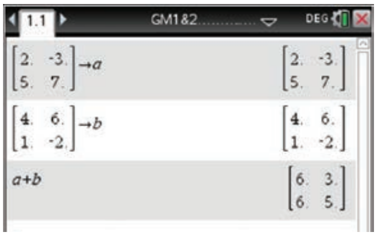
1 Press ⌂on **>New Document>Add Calculator**.

2 Enter the matrices A and B into your calculator.

$\begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix} \to a$ $\quad \begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix}$

$\begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix} \to b$ $\quad \begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix}$

a To calculate $A + B$, type $A + B$ and then press enter to evaluate.



$\begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix} \to a$ $\quad \begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix}$

$\begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix} \to b$ $\quad \begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix}$

$a+b$ $\quad \begin{bmatrix} 6. & 3. \\ 6. & 5. \end{bmatrix}$

$$A + B = \begin{bmatrix} 6 & 3 \\ 6 & 5 \end{bmatrix}$$

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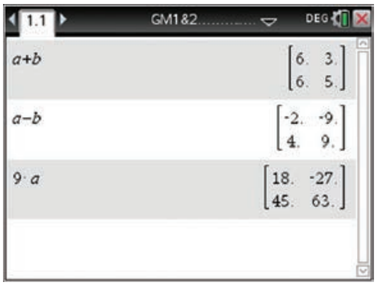
b To calculate $A - B$, type A – B and then press `enter` to evaluate.

c To calculate $9A$, type $9A$ and then press `enter` to evaluate.



$a+b$ $\begin{bmatrix} 6. & 3. \\ 6. & 5. \end{bmatrix}$

$a-b$ $\begin{bmatrix} -2. & -9. \\ 4. & 9. \end{bmatrix}$

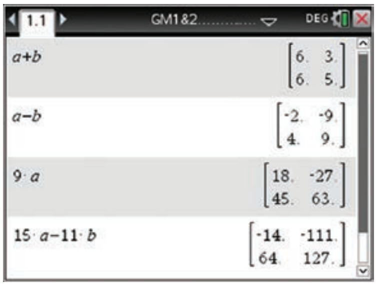
$9 \cdot a$ $\begin{bmatrix} 18. & -27. \\ 45. & 63. \end{bmatrix}$

$$A - B = \begin{bmatrix} -2 & -9 \\ 4 & 9 \end{bmatrix}$$

$$9A = \begin{bmatrix} 18 & -27 \\ 45 & 63 \end{bmatrix}$$

d To calculate $15A - 11B$, type $15A - 11B$ and then press `enter` to evaluate.

$$15A - 11B = \begin{bmatrix} -14 & -111 \\ 64 & 127 \end{bmatrix}$$



$a+b$ $\begin{bmatrix} 6. & 3. \\ 6. & 5. \end{bmatrix}$

$a-b$ $\begin{bmatrix} -2. & -9. \\ 4. & 9. \end{bmatrix}$

$9 \cdot a$ $\begin{bmatrix} 18. & -27. \\ 45. & 63. \end{bmatrix}$

$15 \cdot a - 11 \cdot b$ $\begin{bmatrix} -14. & -111. \\ 64. & 127. \end{bmatrix}$

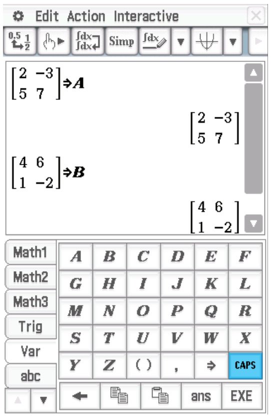
How to add, subtract and scalar multiply matrices using the ClassPad

If $A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $9A$ **d** $15A - 11B$

Steps

1 Enter the matrices A and B into your calculator.



$\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix} \Rightarrow A$

$\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$

$\begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix} \Rightarrow B$

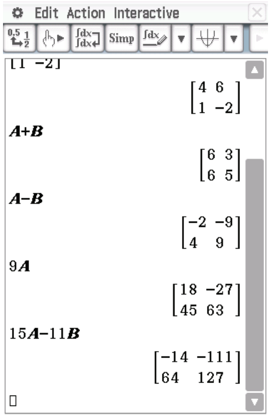
$\begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$

a To calculate $A + B$, type $\boldsymbol{A + B}$ and then press EXE to evaluate.

b To calculate $A - B$, type $\boldsymbol{A - B}$ and then press EXE to evaluate.

c To calculate $9A$, type $\boldsymbol{9A}$ and then press EXE to evaluate.

d To calculate $15A - 11B$, type $\boldsymbol{15A - 11B}$ and then press EXE to evaluate.



$\begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$

$A+B$

$\begin{bmatrix} 6 & 3 \\ 6 & 5 \end{bmatrix}$

$A-B$

$\begin{bmatrix} -2 & -9 \\ 4 & 9 \end{bmatrix}$

$9A$

$\begin{bmatrix} 18 & -27 \\ 45 & 63 \end{bmatrix}$

$15A-11B$

$\begin{bmatrix} -14 & -111 \\ 64 & 127 \end{bmatrix}$

$$A + B = \begin{bmatrix} 6 & 3 \\ 6 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 & -9 \\ 4 & 9 \end{bmatrix}$$

$$9A = \begin{bmatrix} 18 & -27 \\ 45 & 63 \end{bmatrix}$$

$$15A - 11B = \begin{bmatrix} -14 & -111 \\ 64 & 127 \end{bmatrix}$$

Section Summary

- **Scalar multiplication** is the multiplication of a matrix by a number (the scalar). When multiplying by a scalar, each element of the matrix is multiplied by the scalar (number).

Skill-sheet

Exercise 4C

Building understanding

1 Evaluate:

$$\begin{bmatrix} 3 \times 2 & 3 \times -1 \\ 3 \times 8 & 3 \times 7 \end{bmatrix}$$

2 Complete the following:

$$5\begin{bmatrix} -6 & 7 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 \times -6 & \ldots \times \ldots \\ 5 \times 3 & \ldots \times \ldots \end{bmatrix}$$

3 Complete the following:

$$2\begin{bmatrix} -2 & 5 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 \times -2 + 3 \times \ldots & \ldots \times \ldots + \ldots \times \ldots \\ 2 \times 1 + 3 \times \ldots & \ldots \times \ldots + \ldots \times \ldots \end{bmatrix}$$

Developing understanding

Example 3

4 Calculate the values of the following.

a $2\begin{bmatrix} 7 & -1 \\ 4 & 9 \end{bmatrix}$ **b** $5\begin{bmatrix} 0 & -2 \\ 5 & 7 \end{bmatrix}$ **c** $-4\begin{bmatrix} 16 & -3 \\ 1.5 & 3.5 \end{bmatrix}$ **d** $1.5\begin{bmatrix} 1.5 & 0 \\ -2 & 5 \end{bmatrix}$

e $3\begin{bmatrix} 6 & 7 \end{bmatrix}$ **f** $6\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ **g** $\frac{1}{2}\begin{bmatrix} 4 & 6 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ **h** $-1\begin{bmatrix} 3 & 6 & -8 \end{bmatrix}$

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Example 4

5 Given the matrices:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 6 \\ 1 & -4 \end{bmatrix} \quad C = \begin{bmatrix} -3 & 4 \\ -2 & -5 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

find:

a $3A$ **b** $2B + 4C$ **c** $5A - 2B$ **d** $2O$ **e** $3B + O$

6 Enter the matrices $A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$ into your CAS calculator and evaluate:

a $17A - 14B$ **b** $29B - 21A$ **c** $9A + 7B$ **d** $3(5A - 4B)$

7 For the matrices:

$$A = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$

find the matrix for:

a $3A + 4B$ **b** $5C - 2D$ **c** $2(3A + 4B)$ **d** $3(5C - 2D)$

Example 5

8 A chemical plant uses four different chemicals: A, B, C and D, in particular amounts to make 1 litre each of Product 1 and Product 2. The amount of each chemical (in millilitres) to be used is given in the matrix below.

$$\begin{array}{l} \\ \textit{Product 1} \\ \textit{Product 2} \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 300 & 150 & 125 & 425 \\ 250 & 170 & 260 & 320 \end{bmatrix} \end{array}$$

The chemical plant needs to make 3 litres of each product for a customer.

a Perform scalar multiplication to find the total amount of each chemical required to make 3 litres of each product.

b Determine the total amount of chemical B required to make 3 litres of each product.

9 The expenses arising from costs and wages for each section of three department stores: A, B and C, are shown in the Costs matrix. The Sales matrix shows the money from the sale of goods in each section of the three stores. Figures represent the nearest million dollars.

Costs:

$$\begin{array}{c|ccc} & \textit{Clothing} & \textit{Furniture} & \textit{Electronics} \\ \hline A & 12 & 10 & 15 \\ B & 11 & 8 & 17 \\ C & 15 & 14 & 7 \end{array}$$

Sales:

$$\begin{array}{c|ccc} & \textit{Clothing} & \textit{Furniture} & \textit{Electronics} \\ \hline A & 18 & 12 & 24 \\ B & 16 & 9 & 26 \\ C & 19 & 13 & 12 \end{array}$$

a Write a matrix showing the profits in each section of each store.

b If 30% tax must be paid on profits, show the amount of tax that must be paid by each section of each store. No tax needs to be paid for a section that has made a loss.

10 Zoe competed in the gymnastics rings and parallel bars events in a three-day gymnastics tournament. A win was recorded as 1 and a loss as 0. The three column matrices show the results for Saturday, Sunday and Monday.

$$\begin{array}{lccc} & Sat & Sun & Mon \\ \textit{Gymnastics rings} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \textit{Parallel bars} & & & \end{array}$$

a Create a 2×1 column matrix which records her total wins for each of the two types of events.

b Zoe received \$50 for each win. Create a 2×1 matrix which records her total prize money for each of the two types of events.

Testing understanding

11 Consider the following matrices:

$$A = \begin{bmatrix} 6 & 3 \\ -2 & 14 \end{bmatrix} \quad B = \begin{bmatrix} 15 & -7 \\ 8 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -3 & 15 \\ 14 & -1 \\ -1 & 8 \end{bmatrix} \quad D = \begin{bmatrix} -18 \\ 2 \end{bmatrix}$$

Determine if the following are defined, and if so, evaluate the answer.

a $5A - 2B$ **b** $C - 2D$ **c** $2B - D$

12 Find the values of the pronumerals in the following matrix equations.

a

$$3\begin{bmatrix} 5 & a \\ -2 & 6 \end{bmatrix} + 2\begin{bmatrix} b & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 19 & 3 \\ c & d \end{bmatrix}$$

b

$$5\begin{bmatrix} -7 & 10 \\ 9 & a \end{bmatrix} - 3\begin{bmatrix} b & -4 \\ d & 6 \end{bmatrix} = \begin{bmatrix} -47 & 2d \\ c & 2 \end{bmatrix}$$

4D Matrix multiplication

Learning intentions

- To be able to determine if matrix multiplication is possible for two matrices.
- To be able to determine the order of the resulting matrix formed under matrix multiplication.
- To be able to perform matrix multiplication by hand.
- To be able to perform matrix multiplication using a CAS calculator.

Matrix multiplication is the multiplication of a matrix by another matrix.

The matrix multiplication of two matrices, A and B, can be written as $A \times B$ or just AB. Matrix multiplication is not the simple multiplication of the numbers but instead requires the use of an algorithm involving the sum of pairs of numbers that have been multiplied.

The method of matrix multiplication can be demonstrated by using a practical example. The numbers of Books and Puzzles sold by Fatima and Gaia are recorded in matrix N. The selling prices of the Books and Puzzles are shown in matrix P.

$$N = \begin{array}{r} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \textit{Books} & \textit{Puzzles} \end{array} \\ \begin{bmatrix} 7 & \quad 4 \\ 5 & \quad 6 \end{bmatrix} \end{array} \qquad P = \begin{array}{r} \\ \textit{Books} \\ \textit{Puzzles} \end{array} \!\!\begin{array}{c} \$ \\ \begin{bmatrix} 20 \\ 30 \end{bmatrix} \end{array}$$

We want to make a matrix, S, that shows the value of the sales made by each person.

Fatima sold: **7** Books at **\$20** + **4** Puzzles at **\$30**.

Gaia sold: **5** Books at **\$20** + **6** Puzzles at **\$30**.

$$S = \begin{array}{r} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \!\!\begin{array}{c} \$ \\ \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 6 \times 30 \end{bmatrix} \end{array}$$

The steps used in this example follow the algorithm for the matrix multiplication of $N \times P$.

As we move across the *first row* of matrix N, we move down the *column* of matrix P, adding the products of the pairs of numbers as we go.

Then we move across the *second row* of matrix N and down the *column* of matrix P, adding the products of the pairs of numbers as we go.

$$N \times P$$

$$\begin{bmatrix} \overrightarrow{7 \quad 4} \\ 5 \quad 6 \end{bmatrix} \begin{bmatrix} \downarrow 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ \boxed{} \ \boxed{} \end{bmatrix}$$

$$\begin{bmatrix} 7 \quad 4 \\ \overrightarrow{5 \quad 6} \end{bmatrix} \begin{bmatrix} \downarrow 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 6 \times 30 \end{bmatrix}$$

$$= \begin{bmatrix} 140 + 120 \\ 100 + 180 \end{bmatrix}$$

$$= \begin{array}{c} \$ \\ \begin{bmatrix} 260 \\ 280 \end{bmatrix} \end{array} \begin{array}{l} \\ \textit{Fatima} \\ \textit{Gaia} \end{array}$$

Rules for matrix multiplication

Given the way the products are formed, the number of columns in the first matrix must equal the number of rows in the second matrix. Otherwise, matrix multiplication is not defined.

Matrix multiplication

For matrix multiplication to be defined:

order of 1st matrix $m \times n$ — order of 2nd matrix $n \times p$

n and n: must be the same

For the earlier example of the books and puzzles sales:

order of 1st matrix 2×2 — order of 2nd matrix 2×1

2 and 2: the same

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Order of the product matrix

The order of the product matrix is given by:

order of 1st matrix $m \times n$ order of 2nd matrix $n \times p$

order of answer

$m \times p$

For the earlier example of the books and puzzles sales:

order of 1st matrix 2×2 order of 2nd matrix 2×1

order of answer

2×1

Example 6 Rules for matrix multiplication

For the following matrices: $A = \begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$ $D = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

decide whether the matrix multiplication in each question below is defined. If matrix multiplication is defined, give the order of the answer matrix.

a AB **b** BA **c** CD

Explanation	Solution
a AB	
1 Write the order of each matrix.	$\begin{array}{cc} A & B \\ 3 \times 2 & 2 \times 1 \end{array}$
2 The inside numbers are the same.	Matrix multiplication is defined for $A \times B$.
3 The outside numbers give the order of $A \times B$.	The order of the product, AB, is 3×1.
b BA	
1 Write the order of each matrix.	$\begin{array}{cc} B & A \\ 2 \times 1 & 3 \times 2 \end{array}$
2 The inside numbers are not the same.	Multiplication is not defined for $B \times A$.
c CD	
1 Write the order of each matrix.	$\begin{array}{cc} C & D \\ 1 \times 3 & 3 \times 1 \end{array}$
2 The inside numbers are the same.	Multiplication is defined for $C \times D$.
3 The outside numbers give the order of $C \times D$.	The order of the product, CD, is 1×1.

Now try this 6 **Rules for matrix multiplication (Example 6)**

For the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & -1 & 5 \\ 1 & 4 & 10 \end{bmatrix}$$

decide if matrix multiplication, AB, is defined, and if so, give the order of the answer matrix.

Hint 1 Write the order of each matrix.

Hint 2 Are the inside numbers the same?

Hint 3 If so, write down the order of the answer matrix.

Some people think of the matrix multiplication of $A \times B$ using a *run and dive* description.

Matrix multiplication of $A \times B$

The *run and dive* description of matrix multiplication is to add the products of the pairs made as you:

- *run* along the first row of A and *dive* down the first column of B,
- repeat running along the first row of A and diving down the next column of B until all columns of B have been used,
- now start running along the next row of A and repeat diving down each column of B, entering your results in a new row,
- repeat this routine until all rows of A have been used.

This procedure can be very tedious and error prone, so we will only do simple cases by hand. A CAS calculator will generally be used to do matrix multiplication.

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Example 7 Matrix multiplication

For the following matrices: $A = \begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$ $D = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

determine the following matrix multiplications by hand:

a AB **b** CD

Explanation	Solution
a AB	
1 Move across the first row of A and down the column of B, adding the products of the pairs.	$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ \\ \\ \end{bmatrix}$
2 Move across the second row of A and down the column of B, adding the products of the pairs.	$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \\ \\ \end{bmatrix}$
3 Move across the third row of A and down the column of B, adding the products of the pairs.	$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \\ 1 \times 8 + 3 \times 9 \end{bmatrix}$
4 Tidy up by doing some arithmetic.	$= \begin{bmatrix} 40 + 18 \\ 32 + 54 \\ 8 + 27 \end{bmatrix}$
5 Write your answer.	So $A \times B = \begin{bmatrix} 58 \\ 86 \\ 35 \end{bmatrix}$
b CD	
1 Move across the row of C and down the column of D, adding the products of the pairs.	$\begin{bmatrix} 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 8 + 4 \times 6 + 7 \times 5 \end{bmatrix}$
2 Tidy up by doing some arithmetic.	$= [16 + 24 + 35]$
3 Write your answer.	So $C \times D = [75]$

Now try this 7 **Matrix multiplication (Example 7)**

For the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & -1 & 5 \\ 1 & 4 & 10 \end{bmatrix}$$

perform matrix multiplication to find: $C = AB$.

Hint 1 Move across the first row of A and down the first column of B, adding the products of pairs to find the entry for c_{11}.

Hint 2 Move across the first row of A and down the second column of B, adding the products of pairs to find the entry for c_{12}.

Hint 3 Continue to find the other entries in the same way.

Hint 4 Tidy up by doing some arithmetic.

Hint 5 State the final answer.

In the previous example, $AB \neq BA$. Usually, when we reverse the order of the matrices in matrix multiplication, we get a different answer. This differs from ordinary arithmetic, where multiplication gives the same answer when the terms are reversed, for example, $3 \times 4 = 4 \times 3$.

Matrix multiplication

Remember: In general, matrix multiplication is not commutative. That is: $AB \neq BA$.

Matrix powers

Now that we can multiply matrices, we can also determine the **power of a matrix**. This is an important tool that will be useful when we meet communication and dominance matrices later in the next section and transition matrices later in the chapter.

Just as we define:

2^2 as 2×2,
2^3 as $2 \times 2 \times 2$,
2^4 as $2 \times 2 \times 2 \times 2$ and so on,

we define the various powers of matrices as:

A^2 as $A \times A$,
A^3 as $A \times A \times A$,
A^4 as $A \times A \times A \times A$ and so on.

Only square matrices can be raised to a power.

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Example 8 Evaluating matrix expressions involving powers

If $A = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$, determine:

a A^2

b AB^2

Explanation

1 Write down the matrices.

Solution

$$A = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix},\ B = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$$

2 Enter the matrices A and B into your calculator.

$$\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \rightarrow a \qquad \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix} \rightarrow b \qquad \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$$

3 Type in each of the expressions as written, and press enter to evaluate. Write down your answer.

$$a^2 \qquad \begin{bmatrix} 13 & 2 \\ 8 & 5 \end{bmatrix}$$

$$a \cdot b^2 \qquad \begin{bmatrix} -6 & -11 \\ -29 & -52 \end{bmatrix}$$

a $A^2 = \begin{bmatrix} 13 & 2 \\ 8 & 5 \end{bmatrix}$

b $AB^2 = \begin{bmatrix} -6 & -11 \\ -29 & -52 \end{bmatrix}$

Note: For CAS calculators, you must use a multiplication sign between a and b^2 in the last example, otherwise it will be read as variable $(ab)^2$.

Now try this 8 Evaluating matrix expressions involving powers (Example 8)

For the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -7 & -1 & 5 \\ 1 & 4 & 10 \end{bmatrix}$$

Calculate A^2B.

Hint 1 Enter each matrix in your calculator.

Hint 2 Remember to use the multiplication symbol between a^2 and b on your calculator.

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Identity matrix

A special type of square matrix is called the **identity matrix**.

Identity matrix

The identity matrix (which is known by the letter I) is a square matrix of any size with 1s along the *leading diagonal* and 0s in all the other positions. For example,

$$[1] \qquad \begin{bmatrix}1&0\\0&1\end{bmatrix} \qquad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

When the identity matrix, I, is multiplied by a matrix, for example, matrix A, the answer is A. For matrix A, $AI = A = IA$, meaning that the identity matrix multiplied by another matrix acts in the same way as the number 1 multiplied by another number. Thus, the identity matrix is a *multiplicative identity element*.

Example 9 The identity matrix

Consider the following matrices. $A = \begin{bmatrix}5&2\\8&3\end{bmatrix}$ and $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$.

a Find AI.

b Find IA.

Explanation	**Solution**
a **1** Write the order of each matrix. The inside numbers are the same, so matrix multiplication is defined. The outside numbers tell us that the answer is a 2×2 matrix.	$\begin{array}{cc} A & I \\ 2\times 2 & 2\times 2 \end{array}$ $A \times I = \begin{bmatrix}5&2\\8&3\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$
2 Do the matrix multiplication by hand or using your calculator.	$= \begin{bmatrix}5\times 1+2\times 0 & 5\times 0+2\times 1\\ 8\times 1+3\times 0 & 8\times 0+3\times 1\end{bmatrix}$ $= \begin{bmatrix}5&2\\8&3\end{bmatrix}$
b **1** Write the order of each matrix. The inside numbers are the same, so matrix multiplication is defined. The outside numbers tell us that the answer is a 2×2 matrix.	$\begin{array}{cc} I & A \\ 2\times 2 & 2\times 2 \end{array}$ $I \times A = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}5&2\\8&3\end{bmatrix}$
2 Do the matrix multiplication by hand or using your calculator.	$= \begin{bmatrix}1\times 5+0\times 8 & 1\times 2+0\times 3\\ 0\times 5+1\times 8 & 0\times 2+1\times 3\end{bmatrix}$ $= \begin{bmatrix}5&2\\8&3\end{bmatrix}$

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Now try this 9 The identity matrix (Example 9)

$$A = \begin{bmatrix} -3 & 9 \\ 2 & -7 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a Find AI. **b** Find IA.

Hint 1 Write the order of each matrix.

Hint 2 Perform the matrix multiplication by hand.

How to multiply two matrices using the TI-Nspire CAS

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, find the matrix CD.

Steps

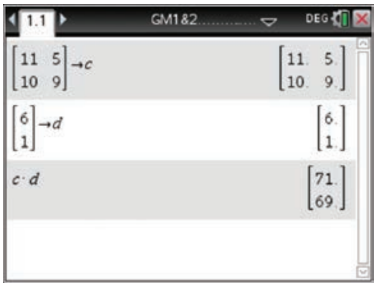
1 **⌂ on>New Document>Add Calculator**.

2 Enter the matrices C and D into your calculator.

3 To calculate matrix CD, type in $c \times d$. Press **enter** to evaluate.

Note: You must put a multiplication sign between the c and d.

Check: Since C has dimension 2×2 and D has dimension 2×1, matrix CD should be a 2×1 matrix.



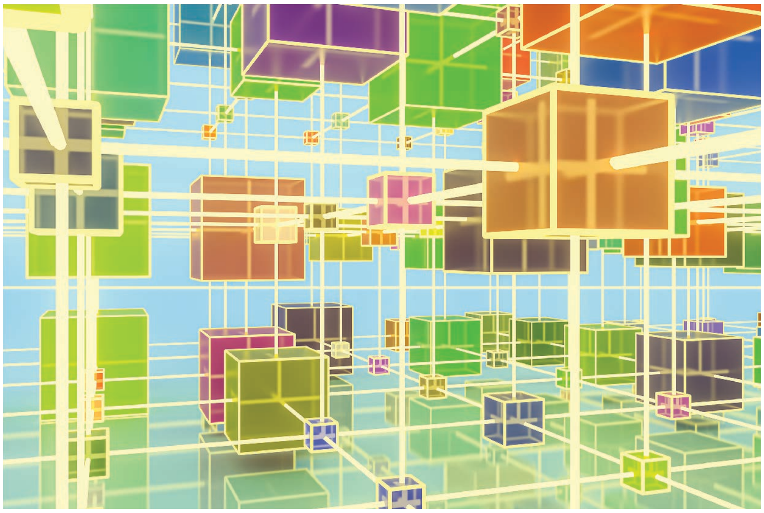
$\begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix} \to c$ $\begin{bmatrix} 11. & 5. \\ 10. & 9. \end{bmatrix}$

$\begin{bmatrix} 6 \\ 1 \end{bmatrix} \to d$ $\begin{bmatrix} 6. \\ 1. \end{bmatrix}$

$c \cdot d$ $\begin{bmatrix} 71. \\ 69. \end{bmatrix}$

4 Write your answer.

$$CD = \begin{bmatrix} 71 \\ 69 \end{bmatrix}$$



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How to multiply two matrices using the ClassPad

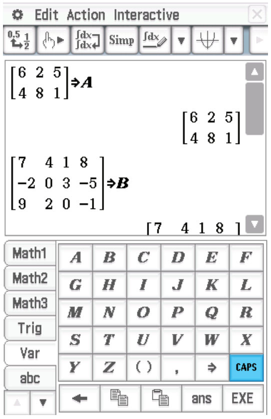
Find $A \times B$: $A = \begin{bmatrix} 6 & 2 & 5 \\ 4 & 8 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 7 & 4 & 1 & 8 \\ -2 & 0 & 3 & -5 \\ 9 & 2 & 0 & -1 \end{bmatrix}$

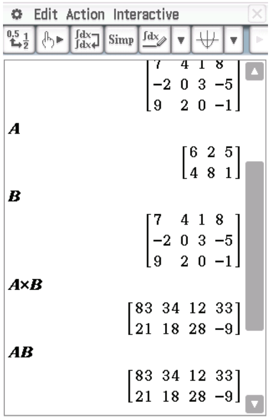
Steps

1 Enter the matrices A and B into your calculator.

2 To calculate $A \times B$, type $\boldsymbol{A} \times \boldsymbol{B}$ or $\boldsymbol{AB}$ and then press **EXE** to evaluate.

3 *Check:* Since A has dimensions 2×3 and B has dimensions 3×4, matrix AB should be a 2×4 matrix.





4 Write your answer.

$$AB = \begin{bmatrix} 83 & 34 & 12 & 33 \\ 21 & 18 & 28 & -9 \end{bmatrix}$$

Section Summary

- **Matrix multiplication** is the process of multiplying a matrix by another matrix. The entry for b_{ij} is found by adding the products of the pairs from row i and column j.

  For a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

  For example,

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 \times 4 + 2 \times 0 & 3 \times 7 + 2 \times 6 \\ -1 \times 4 + 5 \times 0 & -1 \times 7 + 5 \times 6 \end{bmatrix}$$

- An **identity** matrix is a square matrix with 1s along the leading diagonal and 0s in all other positions. The identity matrix behaves like the number 1 in arithmetic. Any matrix multiplied by I remains unchanged. That is, for matrix A, $AI = A = IA$.

Skill-sheet

Exercise 4D

Building understanding

1 Complete the following:

$$\begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \times \cdots + (-5) \times \ldots \\ 3 \times \cdots + 1 \times \ldots \end{bmatrix}$$

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2 Determine the order of each matrix, and decide if the two matrices can be multiplied.

$$A = \begin{bmatrix} -3 & -7 & 2 \\ 5 & 9 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -6 & 4 \\ 11 & 8 \\ -1 & -3 \end{bmatrix}$$

3 Determine the order of the resulting matrix when matrix A and matrix B are multiplied together to form AB.

$$A = \begin{bmatrix} -2 & 6 & 1 \\ 8 & -12 & 6 \\ 0 & 5 & 6 \\ 1 & 4 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 20 & -6 & 2 & 1 \\ 0 & 5 & 8 & -1 \\ 7 & -9 & 3 & 2 \end{bmatrix}$$

4 Write down the 3×3 identity matrix.

Developing understanding

Example 6

5 For the following matrices: $A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 \\ 0 & 8 \\ 2 & -5 \end{bmatrix}$

- decide whether the matrix multiplication in each question below is defined.
- If matrix multiplication is defined, give the order of the answer matrix.

a AB **b** BA **c** CB **d** BC

e AA **f** BB **g** AC **h** CA

6 Write the orders of each pair of matrices and decide if matrix multiplication is defined.

a $\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

d $\begin{bmatrix} 8 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Example 7

7 Perform the following matrix multiplications by hand, using the rule for multiplying matrices.

a $\begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 4 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix}$

d $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} 7 & 4 \\ 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **f** $\begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \end{bmatrix}$

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g $\begin{bmatrix}4 & 1 & 2\end{bmatrix}\begin{bmatrix}0\\6\\1\end{bmatrix}$ **h** $\begin{bmatrix}6 & 2\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}$ **i** $\begin{bmatrix}3 & 2 & 1\end{bmatrix}\begin{bmatrix}6\\5\\8\end{bmatrix}$

8 Consider the following matrices: $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ $B = \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}$

a Find AB. **b** Find BA. **c** Does $AB = BA$?

9 Perform the following matrix multiplications using your CAS calculator.

a $\begin{bmatrix}3 & -2\\1 & 0\end{bmatrix}\begin{bmatrix}4 & 2\\6 & 7\end{bmatrix}$ **b** $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}9 & 6\\7 & 8\end{bmatrix}$ **c** $\begin{bmatrix}5 & 8\end{bmatrix}\begin{bmatrix}7\\6\end{bmatrix}$

d $\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}4\\5\\0\end{bmatrix}$ **e** $\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\\1 & 1\end{bmatrix}$ **f** $\begin{bmatrix}2 & 5\\4 & 1\\0 & 7\end{bmatrix}\begin{bmatrix}8\\3\end{bmatrix}$

Example 8
Example 9

10 Noting that $A^2 = A \times A$, $A^3 = A \times A \times A$, etc., calculate:

i A^2 **ii** A^3 **iii** A^4

for each of the following matrices.

a $A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ **b** $A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ **c** $A = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Testing understanding

11 Consider the following list of matrices.

$$A = \begin{bmatrix}7\\3\\-2\end{bmatrix} B = \begin{bmatrix}8\\-5\end{bmatrix} C = \begin{bmatrix}-1 & 0 & 6\end{bmatrix} \text{ and } D = \begin{bmatrix}4 & 0\\-1 & 2\end{bmatrix}$$

Identify which matrices can be multiplied together, and determine the order of the resulting matrix.

12 Perform the following matrix multiplication:

$$\begin{bmatrix}1 & 7\\3 & 0\end{bmatrix}\begin{bmatrix}6 & 5 & -1\\3 & 0 & 2\end{bmatrix}$$

13 Two matrices were multiplied together to give a third matrix, as shown below.

$$\begin{bmatrix}-6 & 3\\-2 & 5\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}-27 & -30\\-37 & -18\end{bmatrix}$$

Find the values of a, b, c and d to identify the second matrix.

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4E Inverse matrices and solving simultaneous equations using matrices

Learning intentions

- To be able to find the inverse of a matrix.
- To be able to solve simultaneous equations using matrices.

The inverse of a matrix

Written as A^{-1}, the **inverse** of matrix A is a matrix that multiplies A to make the identity matrix, I.

$$A \times A^{-1} = I = A^{-1} \times A$$

Only square matrices have inverses, but not all square matrices have inverses.

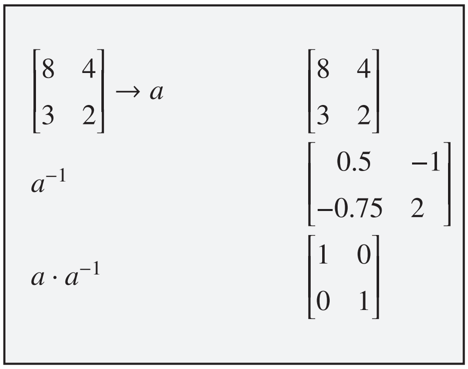
Finding the inverse of a matrix is best done using your CAS calculator. If the inverse does not exist, your CAS will give you an error message.

How to find the inverse of a matrix using a CAS calculator

Find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

Steps

1. Enter matrix A into your calculator.
2. Type in $A\boxed{\wedge}-1$ and evaluate.
3. Form the product AA^{-1}. It should give you the identity matrix, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



$\begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix} \rightarrow a$ $\quad\begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$

a^{-1} $\quad\begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$

$a \cdot a^{-1}$ $\quad\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Write your answer.

The inverse of A is

$$A^{-1} = \begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$$

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How to find the inverse of a matrix using the ClassPad

Find the inverse of matrix $A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

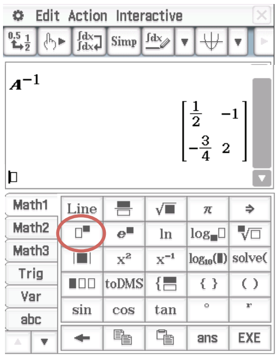
Steps

1 To calculate the inverse matrix, A^{-1}:

- Type in $\boldsymbol{A}\boxed{\wedge}-1$ or Type $\boldsymbol{A}\ \boxed{\square^{\blacksquare}}-1$.
- Press $\boxed{\text{EXE}}$ to evaluate.

2 Write your answer.

The inverse of A is: $A^{-1} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{3}{4} & 2 \end{bmatrix}$.



A^{-1}

$\begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{3}{4} & 2 \end{bmatrix}$

The determinant of a matrix

The determinant is used in finding the inverse of a matrix, and in fact, determining if an inverse exists.

The determinant of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$$

Example 10 Finding the determinant of a 2 × 2 matrix

Find the determinant of each of the following matrices:

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 7 \\ 2 & -2 \end{bmatrix}.$$

Explanation	Solution
1 Write down the matrix and use the rule: $\det(A) = a \times d - b \times c$.	$\det(A) = 5 \times 6 - 2 \times 3 = 24$
2 Evaluate.	$\det(B) = (-3) \times (-2) - 7 \times 2 = -8$

Now try this 10 **Finding the determinant of a 2 × 2 matrix (Example 10)**

Find the determinant of the following matrix.

$$G = \begin{bmatrix} -3 & -9 \\ 2 & -7 \end{bmatrix}$$

Hint 1 Use the rule: $\det(G) = a \times d - b \times c$.

Hint 2 Be careful when multiplying negatives.

While we would typically use a calculator to find the inverse of a matrix, we can find the inverse of a 2 × 2 matrix by hand, using the determinant.

The rule for finding the inverse of a 2 × 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse, A^{-1}, is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $\dfrac{1}{a \times d - b \times c} \neq 0$. That is, provided the determinant is not equal to 0.

For example, if $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$ then the inverse, $A^{-1} = \dfrac{1}{24} \begin{bmatrix} 6 & -2 \\ -3 & 5 \end{bmatrix}$

The rule shows that the inverse of a matrix cannot be found if the determinant of the matrix is equal to zero.

Solving simultaneous equations using matrices

Matrices can be used to solve simultaneous equations. If we have two equations in terms of x and y, we can write them out in matrix form and then use a CAS calculator to solve them.

Example 11 **Solving simultaneous equations**

Express the following simultaneous equations as matrices.

$$8x + 2y = 46$$
$$5x - 3y = 19$$

Explanation	Solution
1 Write each side of the equation as a matrix.	$\begin{bmatrix} 8x + 2y \\ 5x - 3y \end{bmatrix} = \begin{bmatrix} 46 \\ 19 \end{bmatrix}$
2 Write the left-hand side of the matrix equation as the product of two matrices.	$\begin{bmatrix} 8 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 46 \\ 19 \end{bmatrix}$

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Now try this 11 **Solving simultaneous equations (Example 11)**

Express the following simultaneous equations as matrices.

$$\begin{aligned} 6x - y &= 15 \\ -2x + 5y &= -19 \end{aligned}$$

Hint 1 Write each side of the equation as a matrix.

Hint 2 Write the matrix on the left-side of the equation as a product of two matrices

How to solve simultaneous equations using a CAS calculator

Solve to find x and y:

$$\begin{aligned} 5x + 2y &= 21 \\ 7x + 3y &= 29 \end{aligned}$$

Steps

1 The two simultaneous equations can be represented by the matrix equation shown.

$$\begin{bmatrix} 5x + 2y \\ 7x + 3y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

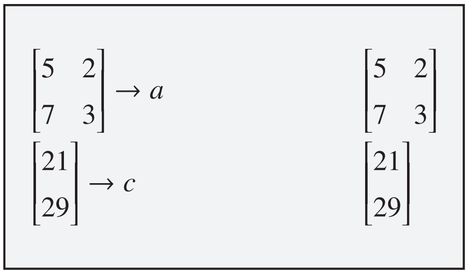
2 The left-hand side of the matrix equation in step **1** can be written as the product of two matrices.

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

3 Name the matrices as shown. Matrix X contains the solutions to the simultaneous equations.

$$A \times X = C$$

4 Enter matrix A and matrix C.



$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \to a$ $\qquad \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$\begin{bmatrix} 21 \\ 29 \end{bmatrix} \to c$ $\qquad \begin{bmatrix} 21 \\ 29 \end{bmatrix}$

5 We want to find the values of matrix X.

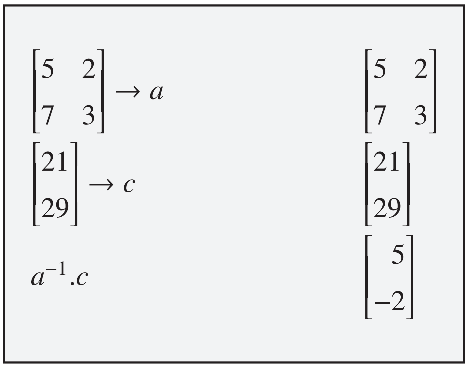
$X = A^{-1} \times C$

Since:

$$\begin{aligned} A \times X &= C \\ A^{-1} \times A \times X &= A^{-1} \times C \\ I \times X &= A^{-1} \times C \\ X &= A^{-1} \times C \end{aligned}$$

6 Enter the matrix product $A^{-1} \times C$.

Note: Order is critical here: $X = A^{-1}C$, *not* CA^{-1}.



$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \rightarrow a$ $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$\begin{bmatrix} 21 \\ 29 \end{bmatrix} \rightarrow c$ $\begin{bmatrix} 21 \\ 29 \end{bmatrix}$

$a^{-1}.c$ $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$

7 Write matrix X.

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

8 Write the solutions to the equations.

So $x = 5$ and $y = -2$.

9 Substitute the values for x and y into the equations to check if they are correct.

Section Summary

- The **inverse matrix**, A^{-1}, of matrix A is defined such that $A \times A^{-1} = I$.
- **Simultaneous equations** can be solved by writing the equations in matrix form, $A \times X = C$, and then solved to find X: $X = A^{-1}C$. Use a CAS calculator for this step.

Skill-sheet

Exercise 4E

Building understanding

1 True or false: If $A \times B = I$, then B is the inverse of matrix A.

2 Confirm that D is the inverse of C by using your CAS calculator to find CD for:

$$C = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & -1 \\ -2.5 & 1.5 \end{bmatrix}$$

3 Convert the following simultaneous equations into matrix form by completing the following:

a $3x + 7y = 27$
$5x + 6y = 28$
$$\begin{bmatrix} 3 & \dots \\ \dots & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ \dots \end{bmatrix}$$

b $2x + 8y = -2$
$3x + 20y = 5$
$$\begin{bmatrix} \dots & 8 \\ \dots & \dots \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ \dots \end{bmatrix}$$

Developing understanding

4 Use your CAS calculator to find the inverse of each matrix. Check by showing that $AA^{-1} = I$ for each matrix.

a $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$ **g** $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & 8 \\ 1 & 2 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

5 Use your CAS calculator and the matrices shown below to find the following:

$$A = \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -7 & 2 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -4 & 9 \\ 6 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 8 \\ 4 & -1 \end{bmatrix}$$

a $(A + B)^{-1}$ **b** $(3C - D)^{-1}$ **c** $(CD)^{-1}$ **d** $AB + D^{-1}$

Example 10

6 Calculate the determinant for each of the matrices in the previous question.

7 Use your CAS calculator and the matrices shown below to find the following:

$$A = \begin{bmatrix} -1 & 7 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix}$$

a A^{-1} **b** B^{-1} **c** $B^{-1}A^{-1}$ **d** $(AB)^{-1}$

Example 11

8 Express the following simultaneous equations in matrix form.

a $5x + y = 13$
$3y + 2y = 12$

b $x + 2y = 10$
$4x + y = 5$

c $7x - 2y = -31$
$-3x + 2x = -1$

d $6x + 5y = 38$
$9x + 3y = 66$

9 Use matrix methods on your CAS calculator to solve the following simultaneous equations.

a $3x + 2y = 12$
$5x + y = 13$

b $4x + 3y = 10$
$x + 2y = 5$

c $4x - 3y = 10$
$3x + y = 1$

d $8x + 3y = 50$
$5x + 2y = 32$

e $6x + 7y = 68$
$4x + 5y = 46$

f $6x - 5y = -27$
$7x + 4y = -2$

Testing understanding

10 Evaluate $A \times (AA^{-1}) \times (A^{-1}A) \times A^{-1}$ for

$$A = \begin{bmatrix} 6 & 2 \\ 5 & -1 \end{bmatrix}$$

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11 Express the following as two simultaneous equations.

$$\begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 23 \\ 55 \end{bmatrix}$$

12 Use matrix multiplication to find the inverse matrices below, by first finding the values of a, b, c and d.

a

$$\begin{bmatrix} -1 & 7 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & -6 \\ 11 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4F Using matrices to model road and communication networks

Learning intentions

- To be able to summarise relationships in a network in a matrix.
- To be able to represent a social network in a communication matrix.

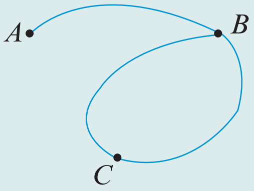
A **network** is a set of objects which are connected together. Examples of networks include towns which may be connected by roads, or people who may be connected by knowing one another. Networks can be illustrated by network diagrams where the objects are points (vertices) and the connections are lines (edges).

Example 12 Using a matrix to represent connections

The network diagram on the right shows road connections between three towns, A, B and C.

a Use a matrix to represent the road connections. Each element should describe the number of ways to travel *directly* from one town to another.

b What information is given by the sum of the elements in column B?



Explanation

a As there are three towns: A, B and C, use a 3 × 3 matrix to show the direct connections.

There are 0 roads directly connecting any town to itself. So enter 0 in row A, column A, and so on.

Note: If there were a road directly connecting town A to itself, it would be a loop from A back to A, and 1 would be added to that element in the matrix.

Solution

$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0 & & \\ & 0 & \\ & & 0 \end{matrix}\right. & & \left.\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

There is one road directly connecting *B* to *A* (or *A* to *B*). So enter 1 in row *B*, column *A*, and in row *A*, column *B*.

$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{matrix}\right] & & & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

There are no direct roads between *C* and *A*. So enter 0 in row *C*, column *A*, and row *A*, column *C*.

$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & \\ 0 & & 0 \end{matrix}\right] & & & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

There are 2 roads between *C* and *B*. Enter 2 in row *C*, column *B*, and row *B*, column *C*.

$$\begin{array}{ccc|c} A & B & C & \\ 0 & 1 & 0 & A \\ 1 & 0 & 2 & B \\ 0 & 2 & 0 & C \end{array}$$

b The second column, *B*, shows the number of roads directly connected to town *B*.

The sum of the second column, *B*, is the total number of roads directly connected to town *B*.

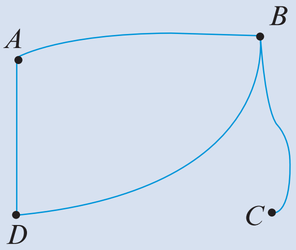
$1 + 0 + 2 = 3$

Now try this 12 Using a matrix to represent connections (Example 12)

The network diagram on the right shows the road connections between four towns, *A*, *B*, *C* and *D*.

a Use a matrix to represent the road connections. Each element should describe the number of ways to travel *directly* from one town to another.

b What information is given by the sum of the elements in column *D*?



Hint 1 Determine how many rows and columns should be in the matrix.

Hint 2 If there is no road between two towns, place a 0 in the corresponding entry. If there is one road, place a 1.

Hint 3 Sum up the numbers in the second column. Think about what each entry represents so that you can understand what the sum of the column represents.

A particular type of network is a social network where the objects are individuals, and the connections between them indicate that they communicate with each other.

Diagrams can be used to show the communication links between people in a social network. A line between two people in a diagram indicates that there is a direct communication link between the people in the network. This is represented by a ‘1’ in the associated matrix.

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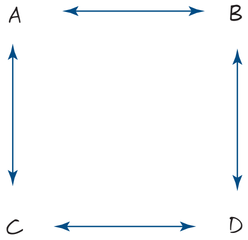
The absence of a line between two people in a network indicates that they cannot communicate with each other directly. This is shown by a ‘0’ in the associated matrix.

When people communicate directly with each other, we say that there is a **‘one step’ communication link**. When people can communicate indirectly with each other via another person, we say that there is a **‘two-step’ communication link**.

Consider the communication matrix, S, representing the network shown below.

$$S = \begin{array}{c@{}c} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$



The matrix S can be multiplied by itself, S^2, to determine the number of ways that pairs of people in a network can communicate with each other via a third person.

$$S^2 = \begin{array}{c@{}c} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$

For example, this tells us that A can communicate with D through two different people. Looking at the network above, we can see that A can communicate with D through B (ABD) or through C (ACD). We can also see that A can communicate with themselves through two different people: B (ABA) or with C (ACA). It also tells us that A cannot communicate with B through one other person.

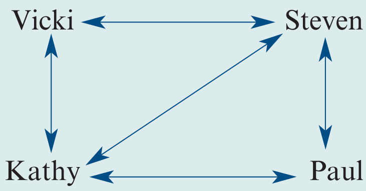
The matrix S^3 would give the three-step communications between people, which is the number of ways of communicating with someone via two people.

The matrix method of investigating communications can be applied to friendships, travel between towns and other types of two-way connections.

Example 13 Using matrices to model the communication links in a social network

The diagram shows the communications within a group of friends. In this diagram:

a Use a matrix, N, to record the presence or absence of direct communication links between the people in the network. Use the first letter of each name to label the columns and rows. Explain how the matrix can be read, using the labels.



b Why is the matrix symmetric?

c What information is given by the sum of column K?

d N^2 gives the number of two-step communications between people. Namely, how many ways one person can communicate with someone via another person. Find the matrix N^2, the square of matrix N.

e Use the matrix N^2 to find the number of two-step ways Kathy can communicate with Steven, and write the connections.

f In the N^2 matrix, there is a 3 where the S column meets the S row. This indicates that there are three two-step communications which Steven can have with himself. Explain how this can be given a sensible interpretation.

Solution

a

$$N = \begin{array}{c} \begin{array}{cccc} V & S & K & P \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ V \\ S \\ K \\ P \end{array}$$

Reading from column S and row K, a 1 indicates that Steven communicates with Kathy. The number 0 is used where there is no communication, for example, between Vicki and Paul.

b The symmetry occurs because the communication is two-way. For example, Vicki communicates with Steven and Steven communicates with Vicki.

c The sum of a column gives the total number of people that a given person can communicate with.

For example, Kathy can communicate with: $1 + 1 + 0 + 1 = 3$ people.

d

$$N^2 = \begin{array}{c} \begin{array}{cccc} V & S & K & P \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ V \\ S \\ K \\ P \end{array}$$

e Reading down the K column to the S row, there are 2 two-step communications between Kathy and Steven. These can be found in the arrows diagram.

Kathy → Vicki → Steven
Kathy → Paul → Steven

f There are three ways Steven can communicate with himself via another person.

Steven → Vicki → Steven
Steven → Kathy → Steven
Steven → Paul → Steven

For example, using the first case above, Steven might ring Vicki and ask her to ring him back later to remind him of an appointment.

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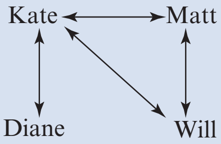
Now try this 13 Using matrices to model the communication links in a social network (Example 13)

The diagram shows the communications within a group of friends.



a Record the social links in a matrix, A, using the first letter of each name to label the columns and rows. Explain how the matrix should be read.

b Explain why there is a symmetry about the leading diagonals of the matrix.

c What information is given by the sum of a column or a row?

d Find the matrix A^2, the square of A.

e Using the matrix A^2, find how many ways that Matt can communicate with Kate, and write the connections.

Hint 1 Determine how many rows and columns should be in the matrix.

Hint 2 If there is no communication between individuals, then place a zero. If two individuals communicate directly, then place a 1.

Hint 3 To square a matrix, multiply the matrix by itself.

Section Summary

- A matrix can model communication or connection between two or more towns or people. Connection matrices are always square matrices.
- Matrices and matrix techniques can be used to model and analyse the properties of road, communication and other real-world networks.

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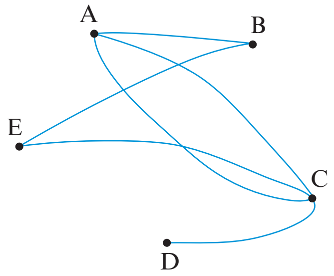
Exercise 4F

Building understanding

Consider the following network of roads between five towns.



1 List the towns that are directly connected to B.

2 Which two towns are directly connected to each other by two roads?

3 Complete the matrix to represent this network:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 2 & 0 & 0 \\ \dots & 0 & 0 & \dots & 1 \\ \dots & 0 & 0 & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 0 \end{array}\right] \end{array}$$

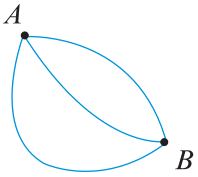
Developing understanding

Example 12

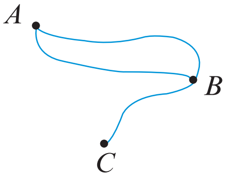
4 The road networks below show roads connecting towns.

a In each case, use a matrix to record the number of ways of travelling *directly* from one town to another.

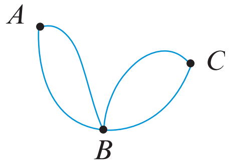
i



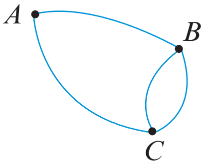
ii



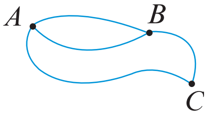
iii



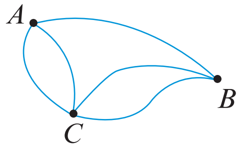
iv



v



vi



b What does the sum of the second column of each matrix represent?

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5 The matrices shown below record the number of ways of going directly from one town to another.

a In each case, represent each matrix with a road network between towns A, B and C.

i
$$\begin{array}{cc} \begin{matrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{matrix} \end{array}$$

ii
$$\begin{array}{cc} \begin{matrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{matrix} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{matrix} \end{array}$$

iii
$$\begin{array}{cc} \begin{matrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{matrix} \end{array}$$

iv
$$\begin{array}{cc} \begin{matrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{matrix} & \\ \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{matrix} \end{array}$$

b State the information that is given by the sum of the first column in the matrices of part **a**.

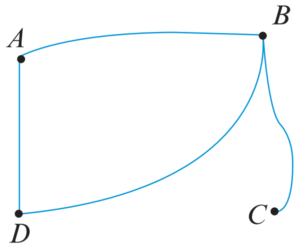
6 The network diagram opposite has lines showing which people from the four people, A, B, C and D, have met.

a Represent the network using a matrix. Use 0 when two people have *not* met and 1 when they have met.

b How can the matrix be used to tell who has met the most people?

c Who has met the most number of people?

d Who has met the fewest number of people?



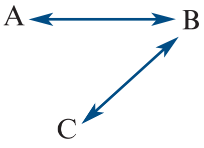
Example 13

7 Consider the following communication matrix where the letters represent the names of people, a '1' indicates that there is direct communication between the people represented in the respective row and column, and a '0' indicates that there is no direct communication. Given that communication is a two-way process, find the error in the communication matrix.

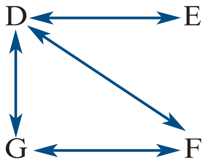
$$\begin{array}{cc} \begin{matrix} A & B & C & D \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \\ D \end{matrix} \end{array}$$

8 Write the matrix for each communication diagram. Use the number 1 when direct communication between two people exists and 0 for no direct communication.

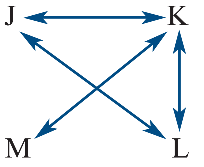
a



b



c



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9 Road connections between towns are recorded in the matrices below. The letters represent towns. The number 1 indicates that there is a road directly connecting the two towns. The number 0 indicates that there is no road directly connecting the two towns.

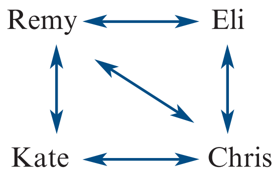
Draw a diagram corresponding to each matrix showing the roads connecting the towns.

a
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

b
$$\begin{array}{cc} \begin{matrix} P & Q & R & S \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} P \\ Q \\ R \\ S \end{matrix} \end{array}$$

c
$$\begin{array}{cc} \begin{matrix} T & U & V & W \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} T \\ U \\ V \\ W \end{matrix} \end{array}$$

10 Communication connections between Chris, Eli, Kate and Remy are shown in the diagram.



a Write a matrix, Q, to represent the connections. Label the columns and rows in alphabetical order using the first letter of each name. Enter 1 to indicate that two people communicate directly or 0 if they do not.

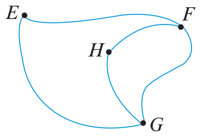
b What information is given by the sum of column R?

c **i** Find Q^2.

ii Using the matrix Q^2, find the total number of ways that Eli can communicate with a person via another person.

iii Write the chain of connections for each way that Eli can communicate to a person via another person.

11 Roads connecting the towns Easton, Fields, Hillsville and Gorges are shown in the diagram. The first letter of each town is used.



a Use a matrix, R, to represent the road connections. Label the columns and rows in alphabetical order using the first letter of each town's name. Write 1 when two towns are directly connected by a road and write 0 if they are not connected.

b What does the sum of column F reveal about the town Fields?

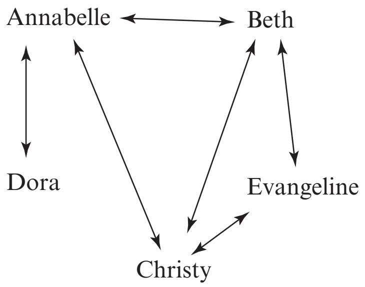
c **i** Find R^2.

ii How many ways are there to travel from Fields to a town via another town? Include ways of starting and ending at Fields.

iii List the possible ways of part **ii**.

Testing understanding

12 Communication between five people is shown below:



a Construct a matrix, F, to represent the connections between the five people. Label each row and column with the first letter of their name. Enter 1 if the two individuals *directly* communicate and 0 if they do not.

b What is the minimum number of steps required to ensure that all individuals can communicate with everyone? Hint: Calculate F^2, F^3 etc.

c List all the possible paths of communication between Evangeline and Dora in 3-step communication.

13 The Hampden football league is made up of ten teams: North Warrnambool (N), South Warrnambool (S), Warrnambool (W), Hamilton (H), Camperdown (Ca), Cobden (Co), Portland (Pl), Port Fairy (PF), Terang (T) and Koroit (K).

The first three rounds were as follows:

1 N vs. S, W vs. H, Ca vs. Co, Pl vs. PF, T vs. K

2 N vs. W, S vs. Ca, Co vs. Pl, PF vs. T, K vs. H

3 N vs H, S vs. Co, W vs. Pl, PF vs. K, T vs. Ca

Given this information:

a Construct a matrix, labelling each row and column with each team. Place a 1 in the matrix if the two teams play each other in the first three rounds and a 0 if they do not play each other.

b State the sum of each row and each column.

c What would the sum of row 6 be after 8 rounds of football?

d What should the matrix look like at the end of round 9?

e What happens to the matrix in round 10?

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4G Introduction to transition matrices

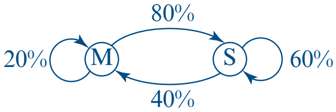
Learning intentions

- To be able to set up a transition matrix.
- To be able to interpret a transition matrix.

Suppose that in a certain town you have two choices of activity each Saturday night: going to the movies (M) or staying at home (H). Suppose further that it has been established that:

- 20% of people who go to the movies this Saturday will go again next Saturday (and hence, 80% of people who go to the movies this Saturday will stay home next Saturday).
- 40% of people who stay home this Saturday will go to the movies next Saturday (and hence, 60% of people who stay home this Saturday will stay home again next Saturday.)

The diagram below describes this information.



The information can also be summarised in a matrix as shown below. Note that the percentages have been written as decimals.

$$\begin{array}{l} \quad\text{this Saturday} \\ \begin{array}{ccccc} & M & S & & \\ & \left[\begin{array}{c} 0.2 \\ 0.8 \end{array}\right. & \left.\begin{array}{c} 0.4 \\ 0.6 \end{array}\right] & \begin{array}{c} M \\ S \end{array} & \text{next Saturday} \end{array} \end{array}$$

This matrix is an example of a **transition matrix**.

Properties of a transition matrix

A transition matrix is always a square matrix.

An important feature of a transition matrix is that each column total of the proportions (or percentages) must equal 1 (100%).

Setting up a transition matrix

Example 14 Setting up a transition matrix

An amusement park has two rides for pre-school children. These are the Ferris wheel (F) and the Merry-go-round (M). After each ride, children can choose which ride they want to go on next. The park observes the following:

- 70% of children on the Ferris wheel will go on the Ferris wheel again.
- 40% of children on the Merry-go-round will go on the Merry-go-round again.

a Find the percentage of children on the Ferris wheel who will go on the Merry-go-round next time.

b Find the percentage of children on the Merry-go-round who will go on the Ferris wheel next time.

c Construct a transition matrix that describes the transition from one step to the next for this situation.

Explanation	Solution
a Children on the Ferris wheel either go on the Ferris wheel again or the Merry-go-round. Thus, the percentages must add up to 100%.	30% of children will go on the Merry-go-round next time.
b Children on the Merry-go-round either go on the Merry-go-round again or the Ferris wheel. Thus, the percentages must add up to 100%.	60% of children will go on the Ferris wheel next time.
c Set up the transition matrix with F and M representing the Ferris wheel and Merry-go-round. The first column represents the percentages (written as decimals) for children currently on the Ferris wheel, so place 0.7 and 0.3 in these positions. The second column represents the percentages for children currently on the Merry-go-round, so place 0.6 and 0.4 in these positions.	this ride $\begin{array}{ccc} F & M & \\ \left[\begin{array}{c} 0.7 \\ 0.3 \end{array}\right. & \left.\begin{array}{c} 0.6 \\ 0.4 \end{array}\right] & \begin{array}{c} F \\ M \end{array} \text{next ride} \end{array}$

Now try this 14 **Setting up a transition matrix (Example 14)**

An observant sales assistant has noted whether customers buy hot food (H) or cold food (C) in the store for lunch. They notice that:

- 80% of customers who buy hot food for lunch will buy hot food again the next day.
- 60% of customers who buy cold food for lunch will buy cold food again the next day.

Construct a transition matrix that describes the percentages of customers choosing hot or cold food for lunch from one day to the next.

Hint 1 Calculate the percentage of people buying cold food tomorrow who purchased hot food today.

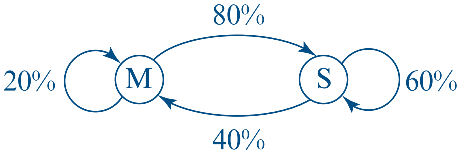
Hint 2 Calculate the percentage of people buying hot food tomorrow who purchased cold food today.

Hint 3 Set up a transition matrix with labels H and C for the rows and columns. Fill in the percentages - remember that 80% should be written as 0.8.

Applying a transition matrix

Considering our earlier example of going to the movies or staying home, we saw that the transition matrix, T, and its transition diagram could be used to describe the weekly pattern of behaviour.

$$T = \begin{array}{cc} \text{this week} & \\ \begin{array}{cc} M & S \end{array} & \\ \left[\begin{array}{cc} 0.2 & 0.4 \\ 0.8 & 0.6 \end{array}\right] & \begin{array}{c} M \\ S \end{array} \end{array} \text{ next week}$$



Using this information alone, a number of predictions can be made.
For example, if 200 people go to the movies this week, the transition matrix predicts that:

- 20% or 40 of these people will go to the movies next week ($0.2 \times 200 = 40$)
- 80% or 160 of these people will stay at home next week ($0.8 \times 200 = 160$)

Further, if 150 people stayed at home this week, the transition matrix predicts that:

- 40% or 60 of these people will go to the movies next week ($0.4 \times 150 = 60$)
- 60% or 90 of these people will stay at home next week ($0.6 \times 150 = 90$)

Thus, we can predict that if 200 people went to the movies this week, and 150 stayed home, then the number of people going to the movies next week is $0.2 \times 200 + 0.4 \times 150 = 100$, and the number of people who stay home next week is $0.8 \times 200 + 0.6 \times 150 = 250$.

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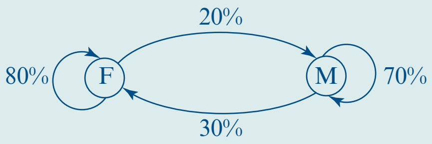
Example 15 Interpreting a transition matrix

The following transition matrix, T, and its transition diagram can be used to describe the movement of children between a Ferris wheel and a Merry-go-round.

$$\begin{array}{rcl} & \text{this ride} & \\ & \begin{array}{cc} F & M \end{array} & \\ T = & \left[\begin{array}{cc} 0.8 & 0.3 \\ 0.2 & 0.7 \end{array}\right] & \begin{array}{c} F \\ M \end{array}\ \text{next ride} \end{array}$$



a What percentage of children on the Ferris wheel are predicted to go on the:

i Ferris wheel next time? **ii** Merry-go-round next time?

b 140 children went on the Ferris wheel. How many of these children do you expect to go on the:

i Ferris wheel next time? **ii** Merry-go-round next time?

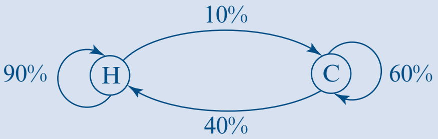
c If 140 children went on the Ferris wheel and 80 children went on the Merry-go-round, how many children do you predict will go on the Ferris wheel next time?

Explanation	Solution
a You should read from the first column of the transition matrix.	
i Consider the first row.	80%
ii Consider the second row.	20%
b You should read from the first column of the transition matrix.	
i Since 80% of children on the Ferris wheel are expected to go on the Ferris wheel next time, then we multiply 0.8 by the number of children on the Ferris wheel.	$0.8 \times 140 = 112$ We expect 112 children from the Ferris wheel to go on the Ferris wheel next time.
ii Since 20% of children on the Ferris wheel are expected to go on the Merry-go-round next time, then we multiply 0.2 by the number of children on the Ferris wheel.	$0.2 \times 140 = 28$ We expect 28 children from the Ferris wheel to go on the Merry-go-round next time.
c Since 30% of children on the Merry-go-round are expected to go on the Ferris wheel next time, we multiply 0.3 by the number of children on the Merry-go-round. This is added to our answer from above.	$0.3 \times 80 = 24$ $112 + 24 = 136$ We predict that 136 students will go on the Ferris wheel next time.

Now try this 15 Interpreting a transition matrix (Example 15)

The following transition matrix, T, and its transition diagram can be used to describe customers choosing between a hot and a cold meal, given their previous choice.

$$T = \begin{array}{c} \text{today} \\ \begin{array}{cc} H & C \end{array} \\ \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ H \\ C \end{array} \text{ tomorrow}$$



a Of the customers who purchased hot food today, what percentage of people do you expect to buy hot meals and cold meals tomorrow?

b If the store sold 400 hot meals on one day, how many of those who purchased hot meals do you expect to purchase hot meals the next day?

c If the store sold 400 hot meals and 500 cold meals on one day, how many hot meals in total do you expect them to sell the next day?

Section Summary

- A transition matrix is always a square matrix.
- An important feature of a transition matrix is that each column total of the proportions (or percentages) must equal 1 (100%).

Exercise 4G

Building understanding

1 Consider the following matrix.

$$T = \begin{array}{c} \text{this time} \\ \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.8 & 0.3 \\ \square & \square \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \end{array} \text{ next time}$$

Fill in the blanks so that the matrix is a transition matrix.

2 An amusement park has two rides: a Ferris wheel and a Merry-go-round. Experience has shown that:

- 60% of children on the Ferris wheel go on the Ferris wheel again.
- 50% of children on the Merry-go-round go on the Merry-go-round again.

a What percentage of children on the Ferris wheel go on the Merry-go-round next time?

b What percentage of children on the Merry-go-round go on the Ferris wheel next time?

3 Consider the following transition matrix.

$$\begin{array}{ccc} & \text{this time} & \\ & \begin{array}{cc} A & \quad B \end{array} & \\ T = & \left[\begin{array}{cc} 0.9 & 0.3 \\ 0.1 & 0.7 \end{array}\right] & \begin{array}{c} A \\ B \end{array} \text{next time} \end{array}$$

Using the transition matrix above, if the initial population at two different locations, *A* and *B*, is 50 people each:

a How many people at location *A* are predicted to stay at location *A* next period?

b How many people at location *B* are predicted to move to location *A* next period?

c How many people are predicted to be at location *A* at the start of the next period?

Developing understanding

Example 14

4 The local football club kiosk sells pies and dim sims. They observe that:

- 60% of customers who buy a pie, choose to buy a pie next week.
- 80% of customers who buy dim sims, choose to buy dim sims next week.

Find the percentage of customers who:

a bought a pie this week who are predicted to buy dim sims next week.

b bought dim sims this week who are predicted to buy a pie next week.

c Construct a transition matrix that describes the transition between the two foods.

5 A new resident to an area is trying to work out the weather patterns. They observe:

- it rains 70% of the time tomorrow if it rains today.
- it rains 60% of the time tomorrow if it does not rain today.

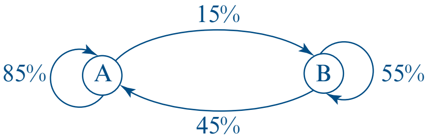
a Find the percentage of the time that it will:

  i not rain tomorrow if it rained today.

  ii not rain tomorrow if it did not rain today.

b Construct a transition matrix that describes the transition between raining and not raining.

6 The following diagram describes the transition between two states, A and B.



Construct a transition matrix that can be used to represent the diagram.

Example 15

7 The following matrix, T, is used to describe the daily pattern of morning coffee orders at a cafe, where L represents a latte and F represents a flat white. On Monday, the cafe sells 160 lattes and 100 flat whites.

$$\begin{array}{cc} & \text{Monday} \\ & \begin{array}{cc} L & F \end{array} \\ T = & \left[\begin{array}{cc} 0.75 & 0.4 \\ 0.25 & 0.6 \end{array}\right] \begin{array}{c} L \\ F \end{array} \text{Tuesday} \end{array}$$

a What percentage of customers who buy a latte on Monday are predicted to buy a:

  i latte on Tuesday?

  ii flat white on Tuesday?

b If the cafe sells 160 lattes on Monday, how many of the people who buy a latte on Monday are predicted to buy a latte on Tuesday?

c If the cafe sells 160 lattes on Monday, how many of the people who buy a latte on Monday are predicted to buy a flat white on Tuesday?

d If the cafe sells 160 lattes and 100 flat whites on Monday, in total, how many people are predicted to buy a latte on Tuesday?

e If the cafe sells 160 lattes and 100 flat whites on Monday, in total, how many people are predicted to buy a flat white on Tuesday?

8 The following matrix, Q, is used to describe whether students arrive on time to school each morning, where T represents being on time and L represents being late. On Monday, 170 students were on time and 30 students were late.

$$\begin{array}{cc} & \text{today} \\ & \begin{array}{cc} T & L \end{array} \\ Q = & \left[\begin{array}{cc} 0.9 & 0.7 \\ 0.1 & 0.3 \end{array}\right] \begin{array}{c} T \\ L \end{array} \text{tomorrow} \end{array}$$

a How many students who were on time on Monday are predicted to be on time on Tuesday?

b How many students who were late on Monday are predicted to be on time on Tuesday?

c In total, how many students do you expect to be on time on Tuesday?

d In total, how many students do you expect to be late on Tuesday?

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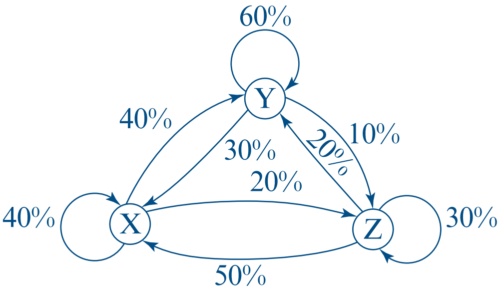
Testing understanding

9 The following diagram describes the transition between three states: X, Y and Z. Complete the following transition matrix to represent this diagram.



$$\begin{array}{cccl} & \text{this time} & & \\ X & Y & Z & \\ 0.4 & \square & 0.5 & X \\ \square & 0.6 & \square & Y \text{ next time} \\ 0.2 & \square & \square & Z \end{array}$$

4H Using recursion to answer questions that require multiple applications of a transition matrix

Learning intentions

- To be able to use recursion to generate a sequence of **state matrices**.
- To be able to determine what happens to the sequence of state matrices in the long term.

Consider the earlier example of going to the movies or staying home. We can now ask questions about the number of people we expect to be at the movies after 1 week, 2 weeks and so on.

This type of problem is similar to those considered in financial modelling. For example, if \$3000 is invested at 5% per annum, how much will we have after 1 year, 2 years, 3 years etc?

In the same way that financial problems were solved using a **recurrence relation** to model the change in growth of value in an investment or loan over time, a recurrence relation can be formed using matrices which can model the change in the number of people choosing to go to the movies from week to week.

The starting point for this example is a matrix that tells us how many people initially go to the movies and how many stay at home. This is called the **initial state matrix** and is denoted S_0.

$$S_0 = \begin{bmatrix} 200 \\ 150 \end{bmatrix}$$

To find the number of people who go to the movies after 1 week, we use the transition matrix, T, to generate the next **state matrix** in the sequence, S_1, as follows:

$$\begin{aligned} S_1 &= TS_0 \\ &= \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 200 \\ 150 \end{bmatrix} = \begin{bmatrix} 0.2 \times 200 + 0.4 \times 150 \\ 0.8 \times 200 + 0.6 \times 150 \end{bmatrix} \\ &= \begin{bmatrix} 100 \\ 250 \end{bmatrix} \end{aligned}$$

Thus, after one week, we predict that 100 people will go to the movies and 250 people will stay home.

A similar calculation can be performed to find out how many people we predict will go to the movies and will stay home after two weeks.

$$S_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 100 \\ 250 \end{bmatrix} = \begin{bmatrix} 0.2 \times 100 + 0.4 \times 250 \\ 0.8 \times 100 + 0.6 \times 250 \end{bmatrix} = \begin{bmatrix} 120 \\ 230 \end{bmatrix}$$

It is clear that there is a pattern. So far, we have seen that:

$S_1 = T \times S_0$

$S_2 = T \times S_1$

Continuing this pattern we have:

$S_3 = T \times S_2$

$S_4 = T \times S_3$

$S_5 = T \times S_4$

or or more generally, after n weeks (or n applications of the transition matrix, T),

$$S_{n+1} = T \times S_n.$$

Matrix recurrence relations

A matrix recurrence relation for generating a sequence of state matrices associated with a transition matrix is given by:

$$S_0 = \text{ initial state matrix,} \qquad S_{n+1} = T \times S_n$$

where T is the transition matrix and S_n is the state matrix after n applications of the transition matrix, T, or in practical situations, n time intervals.

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Example 16

Using a recursion relation to generate successive state matrices

Children at a fair can either go on the Ferris wheel or the Merry-go-round. At the start, 140 children went on the Ferris wheel and 80 children went on the Merry-go-round.

Use the recurrence relation:

$$S_0 = \text{ initial value}, \qquad S_{n+1} = TS_n$$

where:

$$S_0 = \begin{bmatrix} 140 \\ 80 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

to predict the number of children on the Ferris wheel and Merry-go-round after 1 turn and after 3 turns.

Explanation	Solution
1 Use the rule: $S_{n+1} = TS_n$, to predict the number of children on the Ferris wheel and Merry-go-round after one turn, by forming the product $S_1 = TS_0$ and evaluating.	$S_1 = TS_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 140 \\ 80 \end{bmatrix} = \begin{bmatrix} 136 \\ 84 \end{bmatrix}$ Thus, we predict that 136 children will go on the Ferris wheel and 84 will go on the Merry-go-round.
2 To predict the number of children on the Ferris wheel and Merry-go-round after three turns, first calculate the number after two turns and then three.	$S_2 = TS_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 136 \\ 84 \end{bmatrix} = \begin{bmatrix} 134 \\ 86 \end{bmatrix}$ $S_3 = TS_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 134 \\ 86 \end{bmatrix} = \begin{bmatrix} 133 \\ 87 \end{bmatrix}$

Now try this 16

Using a recursion relation to generate successive state matrices (Example 16)

A cafe keeps track of the percentage of people who purchase hot food and cold food each day. On the first day, 400 hot meals and 500 cold meals are chosen.

Use the recurrence relation:

$$S_0 = \text{ initial value} \qquad S_{n+1} = TS_n$$

where:

$$S_0 = \begin{bmatrix} 400 \\ 500 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}$$

to predict how many people chose a hot or cold meal after 1 day or after 3 days.

Hint 1 Use $S_1 = TS_0$, to find the number of hot and cold meals after 1 day.

Hint 2 Use the recurrence relation to find the number of hot and cold meals after 2 days, so you can then find the number after 3 days.

An explicit rule for determining the state matrix after *n* applications of the recursion rule: $S_{n+1} = TS_n$

While we can use the recurrence relation:

$$S_0 = \text{initial state matrix} \qquad S_{n+1} = TS_n$$

to generate a sequence of state matrices step-by-step, there is a more efficient way when we need to determine the state matrix, S_n, for a large value of n. For example, consider the following matrices, defining the initial population (10 000 and 35 000) of two regional towns, and a transition matrix that defines the proportion of each population that moves between the two towns each year.

$$S_0 = \begin{bmatrix} 10\,000 \\ 35\,000 \end{bmatrix} \qquad T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

Using the recurrence relation, the population state matrix for the two towns after one year is:

$$S_1 = TS_0 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 10\,000 \\ 35\,000 \end{bmatrix} = \begin{bmatrix} 10\,250 \\ 34\,750 \end{bmatrix}$$

After two years:

$$S_2 = TS_1 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 10\,250 \\ 34\,750 \end{bmatrix} = \begin{bmatrix} 10\,450 \\ 34\,550 \end{bmatrix}$$

Thus, $S_2 = T \times (TS_0) = T^2S_0$, so the rule for finding the population after n years is: $S_n = T^nS_0$.

An explicit rule for determining S_n after *n* transitions (or time intervals)

If the recurrence relation for determining matrices is:

$$S_0 = \text{initial state matrix}, \qquad S_{n+1} = TS_n$$

then an explicit rule for finding S_n is:

$$S_n = T^n \times S_0$$

where T is the transition matrix and S_0 is the initial state matrix.

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Example 17 Using matrices to model and analyse population growth and decay

A study was conducted to investigate the change in the populations of Geelong (G) and Ballarat (B) due to movement of people between the two cities.

At the start of the study, the populations of Geelong and Ballarat were respectively 250 000 and 100 000 people. Using a matrix approach to model future changes, the initial state matrix is:

$$S_0 = \begin{bmatrix} 250\,000 \\ 110\,000 \end{bmatrix}$$

The transition matrix that can be used to chart the population movements between the two cities from year to year is:

$$T = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$$

a Using the recursion rule: $S_{n+1} = TS_n$, predict the population of Geelong after one year.

b Using the recursion rule: $S_{n+1} = TS_n$, predict the population of Geelong after two, three and four years.

c Using the rule: $S_n = T^n S_0$, predict Geelong's population after 20, 30 and 40 years.

d What do you notice about the predicted population of Geelong? What do you think will happen beyond 40 years?

Explanation	Solution
a Calculate: $S_1 = TS_0$, and read off the top line.	$S_1 = \begin{bmatrix} 241\,500 \\ 118\,500 \end{bmatrix}$ The predicted population after one year in Geelong is 241 500.
b Use the rule: $S_{n+1} = TS_n$, and then read off the top line.	After two years: 235 125 After three years: 230 344 After four years: 226 758
c Use the rule: $S_n = T^n S_0$, and then read off the top line.	After twenty years: 216 108 After thirty years: 216 006 After forty years: 216 000
d Consider what is happening to the population over time - is it increasing or decreasing? Is it approaching a particular value?	Over time, the predicted population in Geelong is declining. Initially, the population is falling rapidly from 250 000 to 235 125 in one year, but then appears to stabilise at around 216 000 people.

Now try this 17 **Using matrices to model and analyse population growth and decay (Example 17)**

A study of the relative growth in duck populations at two nesting sites, *A* and *B*, is to be analysed using matrix methods.

At the start of the study, there were 8000 ducks at site *A* and 10 000 ducks at site *B*. For these populations, the initial population state matrix is:

$$S_0 = \begin{bmatrix} 8000 \\ 10\,000 \end{bmatrix}$$

Over time, ducks move from one site to the other site. The transition matrix that can be used to describe this movement is:

$$T = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

a What is the predicted population in each location after one, two and three years?

b What is the predicted population in each site after twenty, thirty and forty years?

c What do you notice about the predicted population, and what do you think will happen beyond forty years?

Hint 1 Use the rule: $S_n = T^n S_0$, to find the population each year.

Hint 2 Consider what is happening to the population over time - is it increasing or decreasing? Is it approaching a particular value?

From the Geelong and Ballarat population example above, the long-run populations of the two cities become stable and do not change from one year to the next. This does not mean that nobody is moving between the two cities, rather that the number of people moving from Geelong to Ballarat and Ballarat to Geelong is equal. We can see this by calculating the expected number of people moving between each city when $n = 41$(meaning 41 years). In the example, the population of Geelong when $n = 40$ was 216 000, and the population of Ballarat was 144 000. When $n = 41$, the number of people moving from Geelong to Ballarat will be:

$$0.1 \times 216\,000 = 21\,600$$

The number of people moving from Ballarat to Geelong will be:

$$0.15 \times 144\,000 = 21\,600$$

Since these two numbers are equal, we refer to this as the **steady state** or the **equilibrium state**.

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Section Summary

- A **transition matrix**, T, is a square matrix which represents the transition (movement) from one state in a sequence to the next state in a sequence.
- A **state matrix**, S_n, is a column matrix which lists the numbers in each state after n time periods.
- The **recurrence relation**:

  $$S_0 = \text{ initial state matrix} \qquad S_{n+1} = T \times S_n$$

  can be used to generate a sequence of state matrices, $S_0, S_1, S_2, \ldots$, where successive states only depend on their immediate predecessor.
- An explicit rule for finding S_n is: $S_n = T^n S_0$, where T is the transition matrix and S_0 is the initial state matrix.

Skill-sheet

Exercise 4H

Building understanding

1 A factory has a large number of machines. The machines can be in one of two states: operating (O) and broken (B). Broken machines can be repaired and go back into operation, and operating machines can break down. Construct an initial state matrix if there are initially 100 machines in operation and 25 broken machines.

Example 16

2 Consider the following matrices defining the number of students in a year level who get above a C on their maths test and those who get a C or lower, and the transition matrix for the next period. Assume getting above a C is listed first.

$$S_0 = \begin{bmatrix} 60 \\ 140 \end{bmatrix} \qquad T = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix}$$

How many students do you expect to get above a C on the next maths test?

3 For the initial state matrix, $S_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ and the transition matrix, $T = \begin{bmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{bmatrix}$:

a use the recursion relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to determine: S_1, S_2 and S_3

b determine the value of T^2

c use the rule: $S_n = T^n S_0$, to determine S_7.

Developing understanding

Example 17

4 A small town has two fish and chip shops (Bill's Barnacles and Sally's Seafood), and local residents choose which shop to use each week. Consider the following matrices defining the initial customers and a transition matrix that describes the movement between the two shops each week.

Assume that Bill is listed first.

$$S_0 = \begin{bmatrix} 200 \\ 150 \end{bmatrix} \qquad T = \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix}$$

a How many people are predicted to go to Bill's Barnacles next week?

b How many people are predicted to go to Bill's Barnacles after four weeks? What about Sally's Seafood?

c How many people are predicted to go to Bill's Barnacles after twelve weeks? What about Sally's Seafood?

d In the long term, how many customers do you expect to go to Bill's Barnacles and Sally's Seafood each week?

5 In a football club, players can either be available to play or are injured. 90% of players who are available in one week are available the next week, while 40% of injured players are available to play the next week.

a Construct a fully labelled transition matrix to describe this situation, where available players are listed first. Call the matrix T.

b The club has a total of 50 players on their list, all of whom are initially available. Construct a column matrix, S_0, that describes this situation.

c Using T and S_0, how many players do you expect will be available after one week?

d In the long run, how many players do you expect the club to have on its injury list each week?

6 A local town produces a weekly newspaper. Residents can either read the newspaper (R) or ignore the newspaper (I).

In a given week:

- 80% of people who read the paper will read it next week.
- 20% of people who read the paper will ignore it next week.
- 25% of people who ignore the paper will read it next week.
- 75% of people who ignore the paper will ignore it next week.

a Construct a fully labelled transition matrix to describe the situation, where reading the paper is listed first. Call the matrix T.

b Initially, 4000 residents read the paper and 2000 residents ignore it. Write down a column matrix to describe the situation. Call the matrix S_0.

c How many of the residents who read the paper this week do we expect to read it next week? How many of those do we expect to ignore it next week?

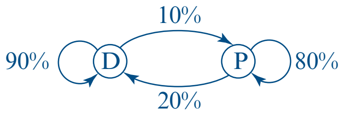
d How many of the residents who ignore the paper this week do we expect to read it next week? How many of those do we expect to ignore it next week?

e How many residents do we expect to read the paper after 6 weeks?

f In the long term, how many residents of the town do we expect to read the paper each week? How many residents do we expect to ignore the paper each week?

7 Residents in a large apartment block in the city have the option of either driving to work (D) or catching public transport (P).

Consider the diagram opposite that shows the way that people change between the two options on each day.



a Construct a fully labelled transition matrix to describe the situation, where driving to work is listed first. Call the matrix T.

b Initially, 40 residents drive to work and 160 residents get public transport. Write down a column matrix to describe the situation. Call the matrix S_0.

c How many of the residents who drive to work today do you expect will drive to work tomorrow? How many of those residents do you expect will get public transport tomorrow?

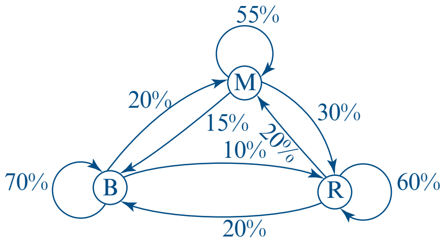
d How many of the residents who get public transport today do you expect will get public transport tomorrow? How many of those residents do you expect will drive to work tomorrow?

e How many residents do you expect to get public transport after 5 days?

f In the long term, how many residents do you expect to get public transport? How many residents do you expect to drive to work?

Testing understanding

8 A juice bar sells three types of juices: Banana Bonanza, Mango Magic and Radical Raspberry. The patterns of daily customers are recorded as shown in the diagram.



a Construct an appropriately labelled 3 by 3 transition matrix to describe this situation. Call the matrix T.

b On Monday, 300 people buy Banana Bonanza, 200 buy Mango Magic and 50 buy Radical Raspberry. Construct a column matrix, S_0, that describes this situation.

c Using matrix multiplication, calculate the number of each type of juice that you expect the store to sell on Tuesday.

d The store is concerned about stock levels. Calculate the total number of each type of drink that you expect the store to sell in the first week from Monday (based on S_0) through to the following Sunday.

e In the long run, how many of each juice do you expect the store will sell each day?

4I Applications of matrices

Learning intentions

- To be able to use matrix multiplication to solve application problems.
- To be able to use row and column matrices to extract information from matrices.

Data represented in matrix form can be multiplied to produce new useful information.

Example 18 Business application of matrices

Freddy and George's store has a sales promotion. One free cinema ticket is given with each DVD purchased. Two cinema tickets are given with the purchase of each game.

The number of DVDs and games sold by Freddy and George is given in matrix S.

The selling price of a DVD and a game, together with the number of free tickets, is given by matrix P.

$$S = \begin{array}{c} \\ \textit{Freddy} \\ \textit{George} \end{array}\begin{array}{c} \begin{array}{cc} \textit{DVDs} & \textit{Games} \end{array} \\ \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \\ \textit{DVDs} \\ \textit{Games} \end{array}\begin{array}{c} \begin{array}{cc} \$ & \textit{Tickets} \end{array} \\ \begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} \end{array}$$

Find the matrix product, $S \times P$, and interpret.

Explanation

Complete the matrix multiplication, $S \times P$.

Solution

$$\begin{array}{c|cc|} & \textit{DVDs} & \textit{Games} \\ \textit{Freddy} & 7 & 4 \\ \textit{George} & 5 & 6 \end{array} \times \begin{array}{c|cc|} & \$ & \textit{Tickets} \\ \textit{DVDs} & 20 & 1 \\ \textit{Games} & 30 & 2 \end{array}$$

$$= \begin{array}{c|cc|} & \$ & \textit{Tickets} \\ \textit{Freddy} & 7 \times 20 + 4 \times 30 & 7 \times 1 + 4 \times 2 \\ \textit{George} & 5 \times 20 + 6 \times 30 & 5 \times 1 + 6 \times 2 \end{array}$$

$$= \begin{array}{c|cc|} & \$ & \textit{Tickets} \\ \textit{Freddy} & 260 & 15 \\ \textit{George} & 280 & 17 \end{array}$$

Explanation

Interpret the matrix.

Solution

Freddy had sales of $260 and gave out 15 tickets.

George had sales of $280 and gave out 17 tickets.

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Now try this 18 **Business application of matrices (Example 18)**

Jacky and Peter's store has a special sales promotion. One free prize ticket is given with each drink purchased. Two prize tickets are given with the purchase of each hamburger.

The number of drinks and hamburgers sold by Jacky and Peter are given in matrix S.

The selling price of a drink and hamburger, together with the number of free prize tickets, is given by matrix P.

$$S = \begin{array}{r} \\ \text{Jacky} \\ \text{Peter} \end{array} \begin{array}{c} \begin{array}{cc} \text{Drinks} & \text{Hamburgers} \end{array} \\ \begin{bmatrix} 3 & 10 \\ 6 & 8 \end{bmatrix} \end{array} \qquad P = \begin{array}{r} \\ \text{Drinks} \\ \text{Hamburgers} \end{array} \begin{array}{c} \begin{array}{cc} \$ & \text{Tickets} \end{array} \\ \begin{bmatrix} 4 & 1 \\ 12 & 2 \end{bmatrix} \end{array}$$

Find the matrix product, $S \times P$, and interpret.

Hint 1 Write down the matrices to be multiplied.

Hint 2 Carry out the matrix multiplication using the algorithm.

Hint 3 Determine what each element in the matrix is telling you.

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Properties of row and column matrices

Row and column matrices provide efficient ways of extracting information from data stored in large matrices. Matrices of a convenient size will be used to explore some of the surprising and useful properties of row and column matrices.

Example 19 Using row and column matrices to extract information

Three rangers completed their monthly park surveys of feral animal sightings, as shown in Matrix S.

$$S = \begin{array}{r} \\ \text{Aaron} \\ \text{Barry} \\ \text{Chloe} \end{array}\begin{array}{c} \begin{array}{cccc} \textit{cats} & \textit{dogs} & \textit{foxes} & \textit{rabbits} \end{array} \\ \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix} \end{array} \qquad A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

a Evaluate $S \times B$.

b What information about matrix S is given in the product $S \times B$?

c Evaluate $A \times S$.

d What information about matrix S is given in the product $A \times S$?

Explanation

a Matrix multiplication of a 3×4 and a 4×1 matrix produces a 3×1 matrix.

b Look at the second last step in the working of $S \times B$.

c Matrix multiplication of a 1×3 and a 3×4 matrix produces a 1×4 matrix.

d In the second last step of part **c**, we see that each element is the sum of the sightings for each type of animal.

Solution

$$S \times B = \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 27 + 9 + 34 + 59 \\ 18 + 15 + 10 + 89 \\ 35 + 6 + 46 + 29 \end{bmatrix} = \begin{bmatrix} 129 \\ 132 \\ 116 \end{bmatrix}$$

Each row of SB gives the sum of the rows in S. Namely, the total sightings made by each ranger.

$$A \times S = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix}$$

$$A \times S = \begin{bmatrix} 27{+}18{+}35 & 9{+}15{+}6 & 34{+}10{+}46 & 59{+}89{+}29 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 30 & 90 & 177 \end{bmatrix}$$

Each column of AS gives the sum of the columns in S, which gives the sum of the sightings of each type of animal.

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Now try this 19 Using row and column matrices to extract information (Example 19)

Four students recorded the number of minutes they spent watching television each weekday and recorded their times in matrix T.

$$T = \begin{array}{c} \\ \textit{Beth} \\ \textit{Zara} \\ \textit{Aria} \\ \textit{Wanda} \end{array} \begin{array}{c} \begin{array}{ccccc} \textit{Mon} & \textit{Tue} & \textit{Wed} & \textit{Thur} & \textit{Fri} \end{array} \\ \begin{bmatrix} 45 & 25 & 80 & 0 & 140 \\ 0 & 20 & 0 & 20 & 120 \\ 30 & 30 & 0 & 30 & 110 \\ 40 & 0 & 0 & 20 & 120 \end{bmatrix} \end{array} \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

a Evaluate $T \times B$.

b What information about matrix T is given in the product, $T \times B$?

c Evaluate $A \times T$.

d What information about matrix T is given in the product, $A \times T$?

Hint 1 Write down the matrices to be multiplied.

Hint 2 Carry out the matrix multiplication using the algorithm.

Hint 3 Determine what each element in the matrix is telling you.

Section Summary

- Matrices can be used to store information. Matrix multiplication can allow key information from matrices to be extracted.

Exercise 4I

Building understanding

1 Joe and Stephen sell cans of softdrink and bottles of water at the senior school fair. The number of cans and bottles are given in matrix D, and the prices of cans and bottles are given in matrix P.

$$D = \begin{array}{c|cc|} & \textit{Cans} & \textit{Bottles} \\ \textit{Joe} & 214 & 103 \\ \textit{Stephen} & 162 & 189 \end{array} \qquad P = \begin{array}{c|c|} & \$ \\ \textit{Cans} & 3 \\ \textit{Bottles} & 2 \end{array}$$

a True or false: Matrix P is a column matrix.

b Complete the following matrix multiplication:

$$\begin{bmatrix} 214 & 103 \\ 162 & 189 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 214 \times 3 + 103 \times \ldots \\ \cdots \times 3 + \cdots \times 2 \end{bmatrix}$$

c What does the top line of the resulting matrix tell you?

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Developing understanding

Example 18

2 The matrix below shows the number of milkshakes and sandwiches that Helen had for lunch one week. The number of kilojoules (kJ) present in each food is given in the second matrix.

$$\begin{array}{c|cc|} & Milkshakes & Sandwiches \\ \hline Helen & 2 & 3 \end{array} \qquad \begin{array}{c|c|} & kJ \\ \hline Milkshakes & 1400 \\ Sandwiches & 1000 \end{array}$$

Use matrix multiplication to calculate how many kilojoules Helen had for lunch over the week.

3 Consider the following two matrices, where the first matrix shows the number of cars and bicycles owned by two families and the second matrix records the wheels and seats for cars and bicycles.

$$\begin{array}{c|cc|} & Cars & Bicycles \\ \hline Smith & 2 & 3 \\ Jones & 1 & 4 \end{array} \qquad \begin{array}{c|cc|} & Wheels & Seats \\ \hline Car & 4 & 5 \\ Bicycle & 2 & 1 \end{array}$$

Use matrix multiplication to find a matrix that gives the numbers of wheels and seats owned by each family.

4 Eve played a game of darts. The parts of the dartboard that she hit during one game are recorded in matrix H. The bull's eye is a small area in the centre of the dartboard. The points scored for hitting different regions of the dartboard are shown in matrix P.

$$H = \begin{array}{c|ccc|} & Bull's\ eye & Inner\ region & Outer\ region \\ \hline Hits & 2 & 13 & 5 \end{array}$$

$$P = \begin{array}{|c|c} Points & \\ \hline 20 & Bull's\ eye \\ 5 & Inner\ region \\ 1 & Outer\ region \end{array}$$

Use matrix multiplication to find a matrix giving her score for the game.

5 On a Saturday morning, Michael's cafe sold 18 quiches, 12 soups and 64 coffees. A quiche costs \$5, soup costs \$8 and a coffee costs \$3.

a Construct a row matrix to record the number of each type of item sold.

b Construct a column matrix to record the cost of each item.

c Use matrix multiplication of the matrices from parts **a** and **b** to find the total value of the morning sales.

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6 Han's stall at the football made the sales shown in the table.

Tubs of chips	Pasties	Pies	Sausage rolls
90	84	112	73

The selling prices were: chips \$4, pastie \$5, pie \$5 and a sausage roll \$3.

a Record the numbers of each product sold in a row matrix.

b Write the selling prices in a column matrix.

c Find the total value of the sales by using matrix multiplication of the row and column matrices found in parts **a** and **b**.

Example 19

7 The number of study hours completed by three students over four days is shown in matrix H.

$$H = \begin{array}{c} \\ Issie \\ Jack \\ Kaiya \end{array} \begin{array}{c} \begin{array}{cccc} Mon & Tues & Wed & Thur \end{array} \\ \begin{bmatrix} 2 & 3 & 2 & 3 \\ 1 & 4 & 0 & 2 \\ 3 & 4 & 3 & 2 \end{bmatrix} \end{array}$$

Using matrix multiplication with a suitable row or column matrix, complete the following.

a Produce a matrix showing the total study hours for each student.

b Hence, find a matrix with the average hours of study for each student.

c Obtain a matrix with the total number of hours studied on each night of the week.

d Hence, find a matrix with the average number of hours studied each night. Round your answers to one decimal place.

8 Matrix R records four students' results in five tests.

$$R = \begin{array}{c} \\ Ellie \\ Felix \\ Gavin \\ Hannah \end{array} \begin{array}{c} \begin{array}{ccccc} T1 & T2 & T3 & T4 & T5 \end{array} \\ \begin{bmatrix} 87 & 91 & 94 & 86 & 88 \\ 93 & 76 & 89 & 62 & 95 \\ 73 & 61 & 58 & 54 & 83 \\ 66 & 79 & 83 & 90 & 91 \end{bmatrix} \end{array}$$

Choose an appropriate row or column matrix, and use matrix multiplication to complete the following.

a Obtain a matrix with the sum of each student's results.

b Hence, give a matrix with each student's average test score.

c Derive a matrix with the sum of the scores for each test.

d Hence, give a matrix with the average score on each test.

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Testing understanding

9 Scalar multiplication occurs when a number multiplies a matrix. For a 2 × 2 matrix it has the general form:

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Suggest why the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ is called a *scalar matrix*.

10 The mobile phone bills of Anna, Boyd and Charlie for the four quarters of 2022 are recorded in matrix P.

$$P = \begin{array}{r} \\ Anna \\ Boyd \\ Charlie \end{array}\begin{array}{c} \begin{array}{cccc} Q1 & Q2 & Q3 & Q4 \end{array} \\ \begin{bmatrix} 47 & 43 & 52 & 61 \\ 56 & 50 & 64 & 49 \\ 39 & 41 & 44 & 51 \end{bmatrix} \end{array} \qquad E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

a Find $P \times E$, and comment on the result of that matrix multiplication.

b State the matrix, F, needed to extract the second quarter, $Q2$, costs.

To find the four quarterly costs on Charlie's phone bill, another matrix is required.

c What will be the order of the matrix that displays Charlie's quarterly costs?

d State the order of a matrix, G, that when it multiplies a 3 × 4, gives a 1 × 4 matrix as the result? Should matrix G pre-multiply (GP) or post-multiply (PG) matrix P?

e Suggest a suitable matrix G that will multiply matrix P and produce a matrix of Charlie's quarterly phone bills. Check that it works.

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Key ideas and chapter summary

Assign-ment

Matrix	A **matrix** is a rectangular array of numbers set out in rows and columns within square brackets. The rows are horizontal; the columns are vertical.
Order of a matrix	The **order (size) of a matrix** is the number of rows × number of columns. The number of rows is always given first.
Elements of a matrix	The **elements of a matrix** are the numbers within it. The position of an element is given by its row and column in the matrix. Element a_{ij} is in row i and column j. The row is always given first.
Connections	A matrix can be used to record various types of connections, such as social communications and roads directly connecting towns.
Equal matrices	Two matrices are equal when they have the same numbers in the same positions. They need to have the same order.
Adding matrices	Matrices of the same order can be *added* by adding numbers in the same positions.
Subtracting matrices	Matrices of the same order can be *subtracted* by subtracting numbers in the same positions.
Zero matrix, *O*	A **zero matrix** is any matrix with zeroes in every position.
Scalar multiplication	**Scalar multiplication** is the multiplication of a matrix by *a number*, where each element of the matrix is multiplied by the scalar.
Matrix multiplication	**Matrix multiplication** is the process of multiplying a matrix by another matrix. The entry for b_{ij} is found by adding the products of the pairs from row i and column j. For a 2 × 2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$ For example, $\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 \times 4 + 2 \times 0 & 3 \times 7 + 2 \times 6 \\ -1 \times 4 + 5 \times 0 & -1 \times 7 + 5 \times 6 \end{bmatrix}$

Identity matrix, *I*	An **identity matrix**, I, behaves like the number 1 in arithmetic. Any matrix multiplied by I remains unchanged. For 2×2 matrices, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ where $AI = A = IA$. Matrix multiplication by the identity matrix is commutative.
Transition matrix	A **transition matrix**, T, is a matrix which describes the transition (movement) from one step to the next of a sequence.
Inverse matrix, A^{-1}	When any matrix, A, is multiplied by its **inverse matrix**, A^{-1}, the answer is I, the identity matrix. That is: $A \times A^{-1} = I$
Solving simultaneous equations	Two **simultaneous equations**, for example: $5x + 2y = 21$ $7x + 3y = 29$ can be written in matrix form as: $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$ $A \times X = C$ The equations can be solved (for x and y) by finding the values in matrix X, as: $X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \times C$ Order is critical: $X = A^{-1}C$, *not* CA^{-1}. This is best done using a CAS calculator.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

4A **1** I can state the order of a given matrix. ☐

e.g. State the order of the following matrix:

$$\begin{bmatrix} 7 & -2 & 5 \\ 10 & 0 & 4 \\ -8 & 12 & 3 \end{bmatrix}$$

4A **2** I can describe the location of an element in a matrix. ☐

e.g. Considering the following matrix, B, state the value of the element in $b_{2\,3}$.

$$B = \begin{bmatrix} 57 & 63 & 19 \\ 48 & 54 & 6 \\ 39 & 45 & 22 \\ 37 & 59 & 31 \\ 42 & 22 & 19 \end{bmatrix}$$

4B **3** I can add and subtract matrices. ☐

e.g. Complete the following matrix addition.

$$\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$$

4B **4** I can identify a zero matrix. ☐

e.g. Identify the zero matrix from the matrices below.

$$\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4C **5** I can perform scalar multiplication. ☐

e.g. If $A = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$, then find $5A$.

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4C **6** I can apply scalar multiplication with addition and subtraction of matrices. ☐

e.g. If $A = \begin{bmatrix} 5 & 3 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, then find $2A + 3B$.

4D **7** I can determine if matrix multiplication is possible for two matrices. ☐

e.g. State whether it is possible to find AB or BA for the following matrices:

$$A = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 5 \\ -3 & 9 \end{bmatrix}$$

4D **8** I can determine the order of the resulting matrix formed under matrix multiplication. ☐

e.g. Determine the order of the answer matrix for the following matrix multiplication:

$$\begin{bmatrix} 9 & -11 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 7 & -6 & 2 & -2 \\ 4 & -3 & 1 & 5 \end{bmatrix}$$

4D **9** I can perform matrix multiplication by hand. ☐

e.g. Calculate the following matrix multiplication by hand:

$$\begin{bmatrix} 5 & 2 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 9 \\ -4 & -5 & 0 \end{bmatrix}$$

4D **10** I can perform matrix multiplication using a CAS calculator. ☐

e.g. Calculate the following matrix multiplication using a CAS calculator:

$$\begin{bmatrix} 6 & -11 & 3 & 7 \\ 0 & 8 & -2 & 8 \end{bmatrix}\begin{bmatrix} 12 & -3 \\ 1 & 5 \\ 4 & -6 \\ 10 & -2 \end{bmatrix}$$

4D **11** I can identify and construct an identity matrix of a given size. ☐

e.g. Construct an identity matrix of size 3×3.

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4E **12** I can find the inverse of a matrix. ☐

e.g. Find the inverse of the matrix $\begin{bmatrix} 7 & 2 \\ -1 & 3 \end{bmatrix}$

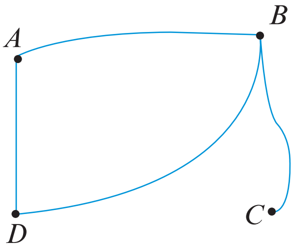
4E **13** I can solve simultaneous equations using matrices. ☐

e.g. Solve the following pair of simultaneous equations using matrices and a CAS calculator.

$$8x + 2y = 46 \qquad 5x - 3y = 19$$

4F **14** I can represent a road network in a matrix. ☐

e.g. Use a matrix to record the number of ways of travelling directly from one town to another.



4F **15** I can summarise relationships in a network in a matrix. ☐

e.g. Draw a network to show the direct connections between towns *A*, *B* and *C* given in the matrix below.

$$\begin{array}{cc} \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \end{array}$$

4G **16** I can set up a transition matrix. ☐

e.g. A charity sends out a monthly letter asking for donations. They notice that 60% of patrons who donated last month will donate this month, while 30% of patrons who did not donate last month will donate this month. Construct a transition matrix that describes how the behaviour of patrons who donate (D) or do not donate (N) changes from month to month.

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4G **17** I can interpret a transition matrix. ☐

e.g. The following transition matrix, T, can be used to describe whether customers bring their shopping bags to the supermarket (S) or leave them at home (H), based on their behaviour last week.

$$T = \begin{array}{c} \text{This week} \\ \begin{array}{cc} S & H \end{array} \\ \left[\begin{array}{cc} 0.8 & 0.3 \\ 0.2 & 0.7 \end{array}\right] \end{array} \begin{array}{l} \\ \\ S \\ H \end{array} \text{Next week}$$

If 150 people took their bags for their weekly shop, and 90 people left their bags at home this week, how many people do you expect to bring their own shopping bags next week?

4H **18** I can use recursion to generate a sequence state matrix. ☐

e.g. Consider the following matrices: one defining the initial populations in Geelong and Ballarat and the other, a transition matrix defining the proportion of the population moving between the two cities each year. Assume Geelong is listed first.

$$S_0 = \begin{bmatrix} 250\ 000 \\ 110\ 000 \end{bmatrix} \qquad T = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$$

How many people are expected to be in Geelong after five years?

4H **19** I can determine what happens to a sequence of state matrices in the long term. ☐

e.g. In the example of Geelong and Ballarat above, how many people are expected to be in Geelong in the long term?

4I **20** I can use matrix multiplication to solve application problems. ☐

e.g. The first matrix gives the hours for which Tom (row 1) and Louise (row 2) agreed to chop firewood (column 1) and mow the lawns (column 2). The second matrix gives the hourly rate of pay for chopping firewood (row 1) and mowing the laws (row 2).

$$\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Use matrix multiplication to generate a matrix giving the total earnings for Tom and Louise.

4I **21** I can use row and column matrices to extract specific information from matrices. ☐

e.g. On Saturday, Jack's coffee van sold 98 cups of coffee and 25 cups of tea. On Sunday, he sold 85 cups of coffee and 31 cups of tea. The price of coffee is \$4 and the price of a cup of tea is \$3. Construct a matrix for his sales. Using matrix multiplication with a suitable row or column matrix, produce a matrix showing the total number of drinks he sold each day.

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Multiple-choice questions

Use the matrix, F, in Questions 1 and 2.

$$F = \begin{bmatrix} 4 & 8 & 6 \\ 5 & 1 & 7 \end{bmatrix}$$

1 The order of matrix F is

A 6 **B** 2×3 **C** 3×2 **D** $2 + 3$ **E** $3 + 2$

2 The element f_{21} is:

A 3 **B** 2 **C** 8 **D** 1 **E** 5

3 Three students were asked the number of electronic devices their family owned. The results are shown in this matrix.

$$\begin{array}{r c} & \begin{array}{ccc} \textit{TVs} & \textit{DVD Players} & \textit{Laptops} \end{array} \\ \begin{array}{r} \textit{Caroline} \\ \textit{Delia} \\ \textit{Emir} \end{array} & \begin{bmatrix} 4 & 3 & 2 \\ 1 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix} \end{array}$$

The number of laptops owned by Emir's family is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

4 The matrix below gives the numbers of roads directly connecting one town to another. The total number of roads directly connecting town E to other towns is

$$\begin{array}{cc} \begin{array}{ccc} D & E & F \end{array} & \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix} & \begin{array}{c} D \\ E \\ F \end{array} \end{array}$$

A 0 **B** 2 **C** 3 **D** 5 **E** 12

5 For these two matrices to be equal, the required value of x is

$$\begin{bmatrix} 4 & 3x \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

A 2 **B** 3 **C** 4 **D** 6 **E** 18

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Use matrices M and N in Questions 6 to 10.

$$M = \begin{bmatrix}7 & 6\\4 & 3\end{bmatrix} \quad N = \begin{bmatrix}5 & -2\\1 & 0\end{bmatrix}$$

6 The matrix $M + N$ is

A $\begin{bmatrix}12 & 8\\5 & 3\end{bmatrix}$ **B** $\begin{bmatrix}12 & 4\\4 & 3\end{bmatrix}$ **C** $\begin{bmatrix}12 & 4\\4 & 0\end{bmatrix}$ **D** $\begin{bmatrix}12 & 4\\5 & 3\end{bmatrix}$ **E** $\begin{bmatrix}12 & 4\\5 & 0\end{bmatrix}$

7 The matrix $M - N$ is

A $\begin{bmatrix}2 & 8\\3 & 0\end{bmatrix}$ **B** $\begin{bmatrix}2 & 4\\3 & 0\end{bmatrix}$ **C** $\begin{bmatrix}2 & 4\\3 & 3\end{bmatrix}$ **D** $\begin{bmatrix}2 & 8\\3 & 3\end{bmatrix}$ **E** $\begin{bmatrix}2 & 8\\4 & 3\end{bmatrix}$

8 The matrix $N - N$ is

A 0 **B** $\begin{bmatrix}0\end{bmatrix}$ **C** $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ **D** $\begin{bmatrix}5 & -2\\1 & 0\end{bmatrix}$ **E** $\begin{bmatrix}-5 & 2\\-1 & 0\end{bmatrix}$

9 The matrix $2N$ is

A $\begin{bmatrix}10 & -4\\2 & 0\end{bmatrix}$ **B** $\begin{bmatrix}7 & 0\\3 & 2\end{bmatrix}$ **C** $\begin{bmatrix}10 & -4\\1 & 0\end{bmatrix}$ **D** $\begin{bmatrix}10 & -2\\2 & 0\end{bmatrix}$ **E** $\begin{bmatrix}10 & -4\\2 & 2\end{bmatrix}$

10 The matrix $2M + N$ is

A $\begin{bmatrix}14 & 10\\7 & 5\end{bmatrix}$ **B** $\begin{bmatrix}14 & 6\\7 & 5\end{bmatrix}$ **C** $\begin{bmatrix}24 & 8\\10 & 6\end{bmatrix}$ **D** $\begin{bmatrix}19 & 14\\9 & 6\end{bmatrix}$ **E** $\begin{bmatrix}19 & 10\\9 & 6\end{bmatrix}$

Use the matrices P, Q, R and S in Questions 11 to 14.

$$P = \begin{bmatrix}5 & 4 & 1\\7 & 6 & 8\end{bmatrix} \quad Q = \begin{bmatrix}2\\6\\0\end{bmatrix} \quad R = \begin{bmatrix}4 & 7\end{bmatrix} \quad S = \begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}$$

11 Matrix multiplication is not defined for

A PQ **B** SS **C** SP **D** PS **E** RS

12 The order of matrix QR is

A 1×1 **B** 3×2 **C** 2×3 **D** 6 **E** 5

13 Which of the following matrix multiplications gives a 1×3 matrix?

A QQ **B** RQ **C** PR **D** QR **E** RP

14 The matrix multiplication PQ gives the matrix

A $\begin{bmatrix}34\\50\end{bmatrix}$ **B** $\begin{bmatrix}10 & 24 & 0\\14 & 36 & 0\end{bmatrix}$ **C** $\begin{bmatrix}10 & 14\\24 & 0\\36 & 0\end{bmatrix}$ **D** $\begin{bmatrix}5 & 4 & 1\\7 & 6 & 8\\2 & 6 & 0\end{bmatrix}$ **E** $\begin{bmatrix}34 & 50\end{bmatrix}$

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15 The identity matrix for 2 × 2 matrices is

A $\begin{bmatrix} 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 The inverse of the matrix, $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ is

A $\begin{bmatrix} -4 & -5 \\ -2 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ **C** $\frac{1}{2}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$ **E** $\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$

Short-answer questions

Use matrix A in Questions 1 to 4.

$$A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 3 & 4 & 7 & 9 \end{bmatrix}$$

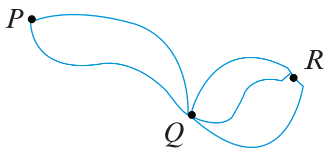
1 State the order of matrix A.

2 Identify the element $a_{1\,3}$.

3 If $C = \begin{bmatrix} 5 & 6 \end{bmatrix}$, find CA.

4 If the order of a matrix, B, was 4 × 1, what would be the order of the matrix resulting from AB?

5 Roads are shown joining towns P, Q and R. Use a matrix to record the number of roads directly connecting one town to another town.



6 Use the matrices below:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 5 \\ 7 & 6 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

to find

a $3A$ **b** $A + B$ **c** $B - A$ **d** $2A + B$

e $A - A$ **f** AB **g** BA **h** A^{-1}

i A^2 **j** AI **k** AA^{-1}

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Written-response questions

1 Farms A and B have their livestock numbers recorded in the matrix shown.

$$\begin{array}{c} \\ \text{Farm } A \\ \text{Farm } B \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Cattle} & \textit{Pigs} & \textit{Sheep} \end{array} \\ \begin{bmatrix} 420 & 50 & 100 \\ 300 & 40 & 220 \end{bmatrix} \end{array}$$

a How many pigs are on Farm B?

b What is the total number of sheep on both farms?

c Which farm has the largest total number of livestock?

2 A bakery recorded the sales for Shop A and Shop B of cakes, pies and rolls in a Sales matrix, S. The prices were recorded in the Prices matrix, P.

$$S = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Cakes} & \textit{Pies} & \textit{Rolls} \end{array} \\ \begin{bmatrix} 12 & 25 & 18 \\ 15 & 21 & 16 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \textit{Cakes} \\ \textit{Pies} \\ \textit{Rolls} \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \end{array}$$

a How many pies were sold by Shop B?

b What is the selling price of pies?

c Calculate the matrix product SP.

d What information is contained in matrix SP?

e Which shop had the largest income from its sales? How much were its takings?

3 Patsy and Geoff decided to participate in a charity fun run.

a Patsy plans to walk for 4 hours and jog for 1 hour. Geoff plans to walk for 3 hours and jog for 2 hours. Construct a matrix showing how many hours Patsy and Geoff spend walking and jogging.

b Walking raises \$2 per hour and consumes 1500 kJ/h (kilojoules per hour). Jogging raises \$3 per hour and consumes 2500 kJ/h. Construct a matrix showing how much is earned and how many kilojoules are consumed by walking and jogging.

c Use matrix multiplication to find a matrix that shows the money raised and the kilojoules consumed by Patsy and Geoff.

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4 An insurance company wants to be able to predict the number of accidents their customers will have in future years. Historically, they know that 60% of their customers who had an accident (A) in the last year will have another accident in the next year. They also know that 5% of customers who did not have an accident (N) last year will have an accident next year.

a What percentage of customers who had an accident last year are expected not to have an accident next year?

b Construct a transition matrix to describe this situation. Call the matrix, T.

c Last year, within one region, 4000 of the insurance company's customers had an accident while 26 000 did not have an accident. The company decided to use this as the starting point for making their predictions. Write down a column matrix, S_0, that describes this situation.

d Using T and S_0, how many people do we expect to have an accident in the next year?

e How many people do we expect to have an accident in five years' time?

f In the long term, how many people do we expect to have an accident each year and how many do we expect to not have an accident?

5 Supermarkets sell eggs in cartons of 12, apples in packets of 8 and yoghurt tubs in sets of 4. This is represented by matrix A. The cost for each type of packet is given by matrix B.

$$A = \begin{array}{c} \\ \textit{Items per packet} \end{array} \begin{array}{ccc} \textit{Eggs} & \textit{Apples} & \textit{Yoghurt} \\ \left[\begin{matrix} 12 \end{matrix}\right. & 8 & \left.\begin{matrix} 4 \end{matrix}\right] \end{array} \qquad B = \begin{array}{cc} & \$ \\ \textit{Eggs} & 8 \\ \textit{Apples} & 6 \\ \textit{Yoghurt} & 5 \end{array}$$

The sales of each type of packet are given by matrix C as a column matrix and by matrix D as a row matrix.

$$C = \begin{array}{cc} & \textit{Packets} \\ \textit{Eggs} & 100 \\ \textit{Apples} & 50 \\ \textit{Yoghurt} & 30 \end{array} \qquad D = \begin{array}{cccc} & \textit{Eggs} & \textit{Apples} & \textit{Yoghurt} \\ \textit{Packets} & 100 & 50 & 30 \end{array}$$

Choose the appropriate matrices and use matrix multiplication to find:

a the total number of items sold (counting each egg, apple or yoghurt tub as an item)

b the total value of all sales.

6 We are told that 2 apples and 3 bananas cost \$6. This can be represented by the equation $2x + 3y = 6$, where x represents the cost of an apple and y the cost of a banana.

a Write an equation for: 6 apples and 5 bananas cost \$14.

b With your equation and the equation given, use matrix methods on your CAS calculator to find the cost of an apple and the cost of a banana.

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