Chapter 8

Graphs and networks

Chapter questions

- What is a graph?
- How do we identify the features of a graph?
- How do we draw a graph?
- How do we apply graphs in practical situations?
- How do we construct an adjacency matrix from a graph?
- How do we define and draw a planar graph?
- How do we identify the type of walk on a graph?
- How do we find the shortest path between two vertices of a graph?
- How do we find the minimum distance required to connect all vertices of a graph?

In this chapter, we show how **graphs** and **networks** can model and analyse everyday networks, maps and many kinds of organisational charts and diagrams.

ISBN 978-1-009-11034-1

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8A What is a graph?

Learning intentions

- ► To be able to define and identify a graph, vertex, edge and loop.
- ► To be able to find the degree of a vertex.
- ► To be able to find the sum of degrees.

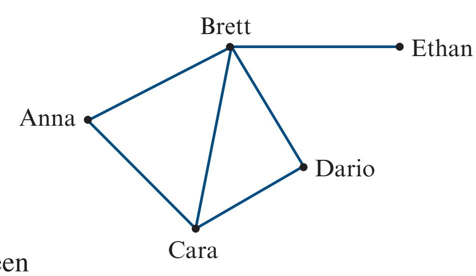
There are many situations in everyday life that involve connections between people or objects. Towns are connected by roads, computers are connected to the internet and people connect to each other through being friends on social media. A diagram that shows these connections is called a **graph**.

Graph elements: definitions

Graphs, vertices, edges

A **graph** is a diagram that consists of a set of points, called **vertices** (plural of vertex), and a set of lines, called **edges**.

A graph can be used to show how five people - Anna, Brett, Cara, Dario and Ethan - are connected on a social media website.



In the graph:

- the 5 vertices represent the 5 people
- the 6 edges represent the 6 connections between the people on the social media website
- Anna is a friend of Brett and Cara
- Brett is a friend of Anna, Cara, Dario and Ethan
- Cara is a friend of Anna, Brett and Dario
- Dario is a friend of Brett and Cara
- Ethan is a friend of Brett.

We will now explore the properties of graphs in more detail.

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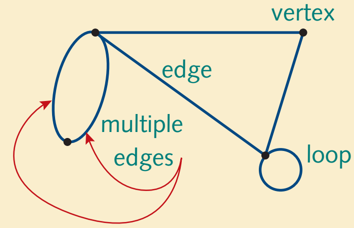
Graph elements: definitions

A **graph** is a diagram that consists of a set of points, called **vertices**, that are joined by a set of lines, called **edges**.

- A **loop** is an edge in a graph that joins a **vertex** in a graph to itself.
- Two or more edges that connect the same vertices are called **multiple edges**.



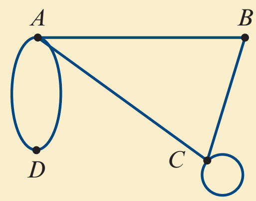
The **degree of a vertex** is the number of edges attached to the vertex.

The degree of a vertex, A, is written as: **deg(A)**.

For example, in the graph opposite,

deg(A) = 4, deg(B) = 2, deg(C) = 4 and deg(D) = 2.

Note: A loop contributes *two degrees* to a vertex because a loop is attached to its vertex at both ends.

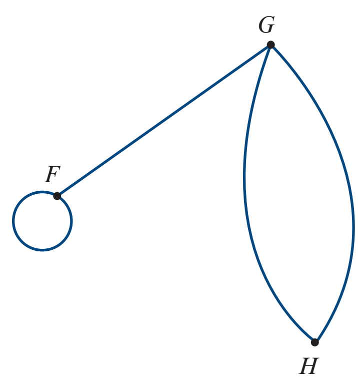


- In any graph, the **sum of degrees** of the vertices is equal to *twice the number of edges*.

  Note: A loop contributes *one edge* to the graph.
- We say a vertex is **even** if the degree of the vertex is even, and we say a vertex is **odd** if the degree of the vertex is odd.

For example, consider the graph opposite.

- There are 3 vertices
- There are 4 edges
- The vertex F has a loop
- There are multiple edges between vertices G and H
- The degree of vertex F is 3, so we write deg(F) = 3
- The degree of vertex G is 3, so we write deg(G) = 3
- The degree of vertex H is 2, so we write deg(H) = 2
- The sum of degrees for this graph is 3 + 3 + 2 = 8, which is *twice* the number of edges. (In this case there are 4 edges, and $2 \times 4 = 8$.)



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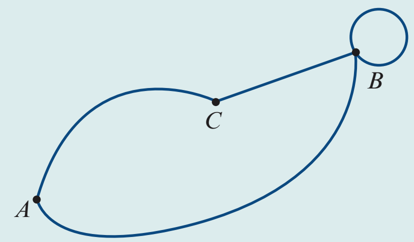
Example 1 Properties of a graph

Consider the graph shown.

a How many vertices does this graph have?
b How many edges does this graph have?
c What is the degree of vertex A?
d What is the degree of vertex B?
e What is the *sum of degrees* for this graph?

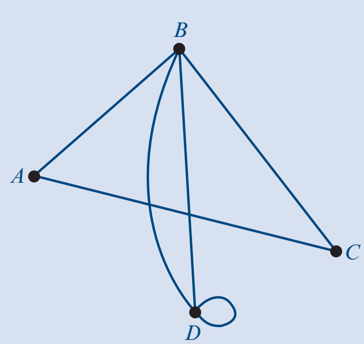


Explanation	Solution
a Vertices are the set of points that make up a graph. There are three points: A, B and C.	There are 3 vertices.
b Edges are the lines that connect the points. There are four lines. Note: A loop is considered one edge that connects a vertex to itself.	There are 4 edges.
c The degree of a vertex is the number of edges attached to it. There are two edges attached to vertex A.	$\deg(A) = 2$
d Vertex B has one edge connected to vertex A, one edge connected to vertex C and a loop. Note: A loop contributes two degrees to a vertex.	$\deg(B) = 4$
e This graph has four edges. The *sum of degrees* is equal to *twice the number of edges*. Note: $\deg(A) = 2$, $\deg(B) = 4$, $\deg(C) = 2$, therefore $2 + 4 + 2 = 8$	Sum of degrees $= 2 \times 4 = 8$

Now try this 1 Properties of a graph (Example 1)

Consider the graph shown.

a How many vertices does this graph have?
b How many edges does this graph have?
c What is the degree of vertex A?
d What is the degree of vertex D?
e What is the *sum of degrees* for this graph?



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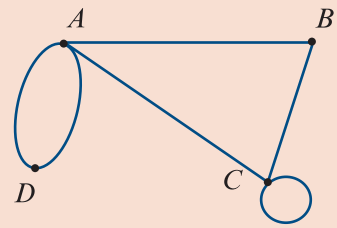
Section Summary

- A **graph** is a diagram that consists of a set of points, called **vertices**, and a set of lines, called **edges**.
- A **loop** is an edge that connects a vertex to itself.
- In this graph, there are 4 **vertices**: A, B, C, and D.
- In this graph, there are 6 **edges**: AB, AC, AD *(twice)*, BC and the loop at C.



- The **degree of vertex** A, written deg(A), is the *number of edges attached to the vertex*. Loops contribute two degrees to a vertex. For example, in the graph above: deg(B) = 2 and deg(C) = 4.
- The **sum of degrees** is equal to *twice the number of edges*.
- We say a vertex is **even** if the degree of the vertex is even, and we say a vertex is **odd** if the degree of the vertex is odd.

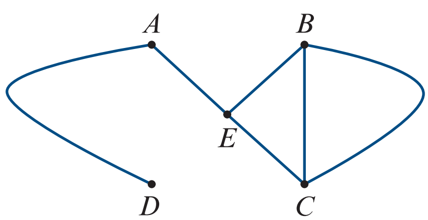
Exercise 8A

Building understanding

Example 1

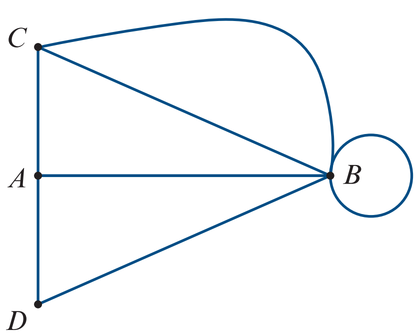
1 For each graph shown, copy and complete the statements by filling in the boxes.

a



- **i** The graph has ☐ vertices.
- **ii** The graph has ☐ edges.
- **iii** The graph has ☐ loops.
- **iv** deg(A) = ☐
- **v** deg(E) = ☐
- **vi** The graph has ☐ odd vertices.
- **vii** The graph has ☐ even vertices.

b



- **i** The graph has ☐ vertices.
- **ii** The graph has ☐ edges.
- **iii** The graph has ☐ loops.
- **iv** deg(B) = ☐
- **v** deg(D) = ☐
- **vi** The graph has ☐ odd vertices.
- **vii** The graph has ☐ even vertices.

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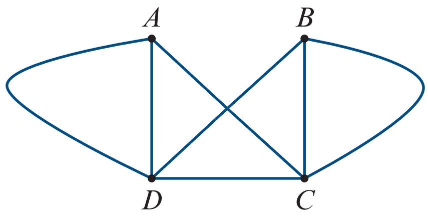
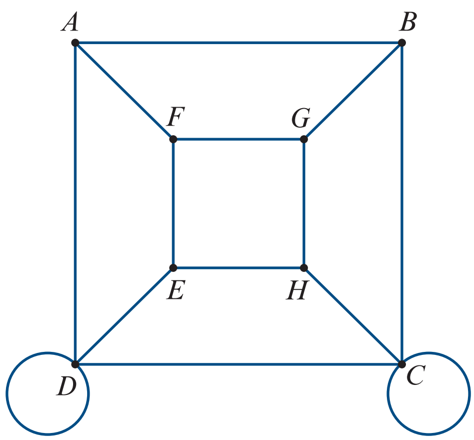
c



i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv $\deg(A) =$ ☐
v $\deg(C) =$ ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

d



i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv $\deg(C) =$ ☐
v $\deg(F) =$ ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

2 What is the sum of the degrees of the vertices of a graph with:

a five edges? **b** three edges? **c** one edge?

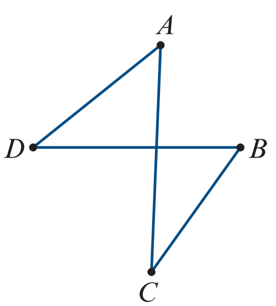
In each case, draw an example of the graph and check your answer.

Developing understanding

3 Will the sum of the vertex degrees of a graph always equal twice the number of edges? Explain your reasoning.

4 Consider the graph opposite.
A loop is added at vertex A:

a how will this change the degree of vertex A?
b how many edges are added to the graph?



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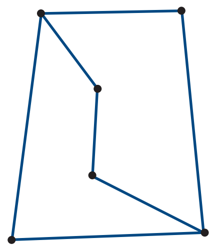
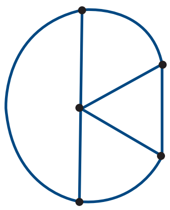
5 In the graph shown below, the sum of the degrees of the vertices is:

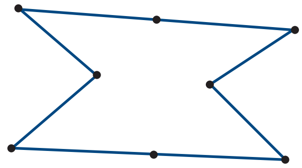


A 6 **B** 7 **C** 12 **D** 13 **E** 14

6 Two graphs, labelled Graph 1 and Graph 2, are shown below.



Graph 1

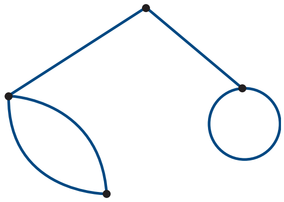


Graph 2

The sum of degrees of the vertices of Graph 1 is:

A two less than the sum of the degrees of the vertices of Graph 2.
B one less than the sum of the degrees of the vertices of Graph 2.
C equal to the sum of the degrees of the vertices of Graph 2.
D one more than the sum of the degrees of the vertices of Graph 2.
E two more than the sum of the degrees of the vertices of Graph 2.

7 The graph below has four vertices and five edges.



How many of the vertices in this graph have an odd degree?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

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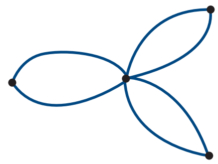
8 Consider the following graph.



Which one of the following statements is **not** true for this graph?

A There are four vertices.
B There are three loops.
C All vertices have an even degree.
D Three of the vertices have the same degree.
E The sum of the degrees of the vertices is twelve.

The Königsberg bridge problem

The problem that began the scientific study of graphs and networks is known as the *Königsberg bridge problem*. The problem began as follows.

The centre of the old German city of Königsberg was located on an island in the middle of the Pregel River. The island was connected to the banks of the river and to another island by five bridges. Two other bridges connected the second island to the banks of the river, as shown.

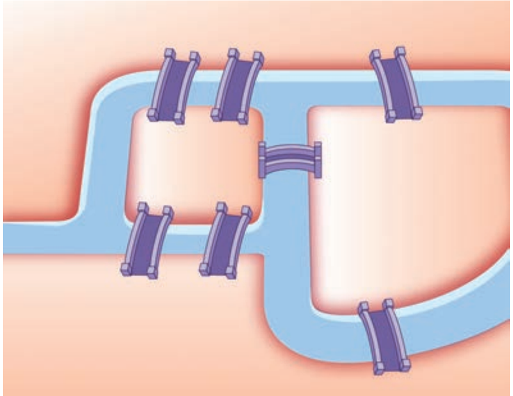
The seven bridges of Königsberg.

The problem simplified

A simplified view of the situation is shown in the drawing below.

Can a continuous walk be planned so that all bridges are crossed only once?

Whenever someone tried to walk the route, they either ended up missing a bridge or crossing one of the bridges more than once. Two such walks are marked on the diagrams that follow. See if you can trace out a walk on the diagram that crosses every bridge, but only once.



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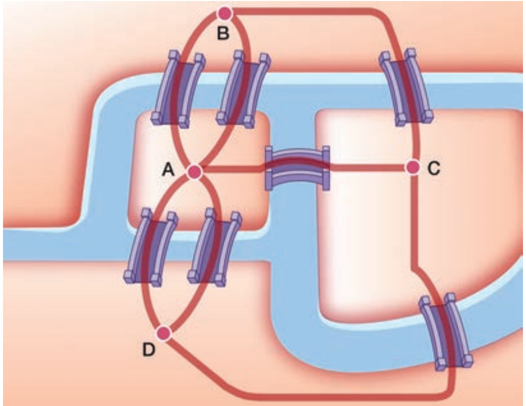
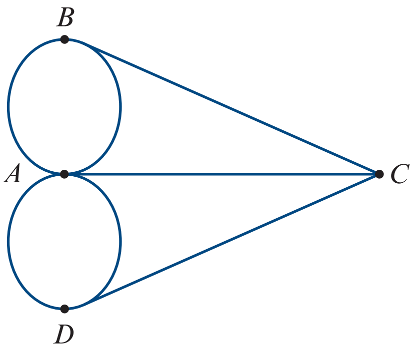
Enter the mathematician

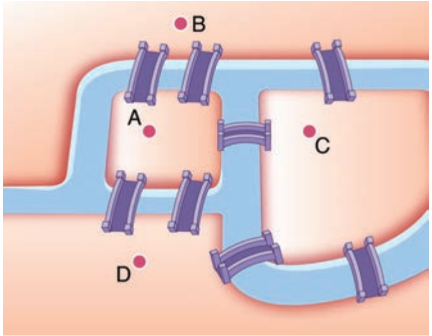
The Königsberg bridge problem was well-known in 18th century Europe and attracted the attention of the Swiss mathematician Euler (pronounced 'Oil-er'). He started analysing the problem by drawing a simplified diagram to represent the situation, as shown below. We now call this type of simplified diagram a *graph*.





Euler's diagram

9 **a** The picture opposite shows a situation in which an eighth bridge has been added.



i With a pencil or the tip of your finger, see whether you can trace out a continuous walk that crosses each of the bridges only once. Such a walk exists.

ii Construct a graph to represent this new situation with eight bridges. Labelled dots have been placed on the picture to help you draw your graph.

iii Your graph should have only two odd vertices. Which vertices have an odd degree?

iv As you will learn later, when the graph has only two odd vertices, you can only complete the task if you start at the places represented by the odd vertices. You will then finish at the place represented by the other odd vertex. Check to see.

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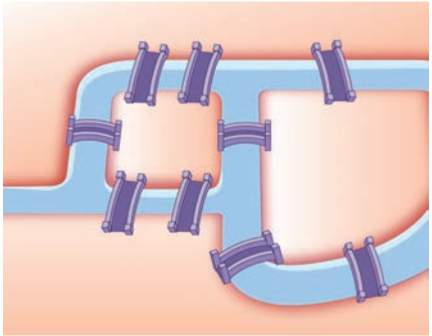
b A ninth bridge has been added, as shown opposite.



i With a pencil or the tip of your finger, see whether you can trace out a continuous walk that crosses each of the bridges only once. Such a walk exists.

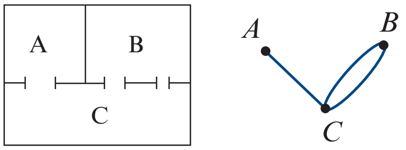
ii Construct a graph to represent this situation.

iii Your graph should *not* have any odd vertices; that is, they should all be even. Check to see.

iv As you will learn later, when the graph has only even vertices, you can start your walk from any island or any riverbank and still complete the task. Check to see.

Testing understanding

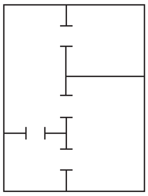
A graph is used to represent the floor plan of a house. The vertices *A*, *B* and *C* represent the rooms, and the edges represent the doorways connecting the rooms. See the diagram of a floor plan and a graph that represents it below.

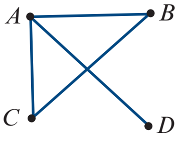


From the graph, we can see that:

- The single edge in the graph between vertices *A* and *C* shows that there is one doorway connecting room *A* to room *C*.
- The two edges between vertices *B* and *C* show that there are two doorways connecting room *C* to room *B*.
- There is no edge between vertices *A* and *B*, showing that there is no doorway connecting room *A* to room *B*.

10 A graph is used to represent the floor plan of a second house.
In the graph, the vertices *A*, *B*, *C* and *D* represent the rooms, and the edges represent the doorways connecting the rooms. The room labels are missing from the floor plan.
Use the information in the graph to correctly label the rooms in the plan.





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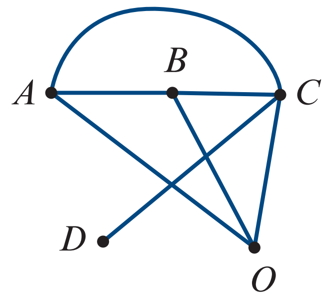
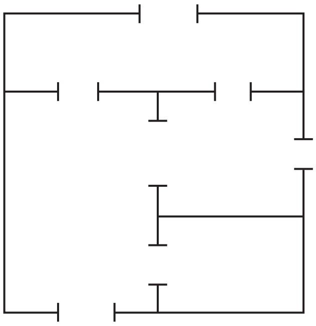
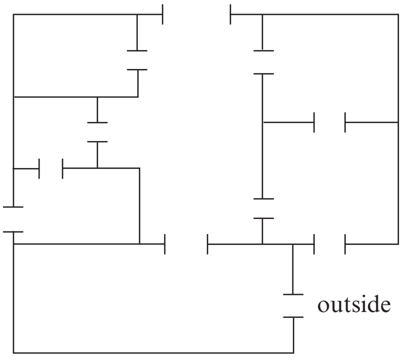
11 A graph is used to represent the floor plan of a third house. In this house, the doorways enable people to move between the rooms of the house, and for some of the rooms, between the room and the outside. In the graph, the vertices labelled A, B, C and D represent the rooms of the house, and the vertex labelled O represents the outdoor area surrounding the house.

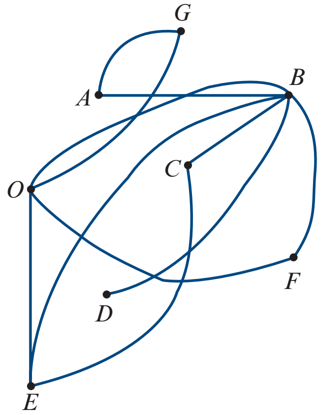




The room labels are missing from the floor plan. Use the information in the graph to correctly label the rooms in the floor plan.

12 A graph is used to represent the floor plan of a fourth house. In the graph, the vertices A, B, C, D, E, F and G represent the rooms, and edges represent the doorways connecting the rooms. The label, O, is used to label the outside area of the house which can be accessed via a doorway from some rooms. The room labels are missing from the plan.
Use the information in the graph to correctly label the rooms in the plan.





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8B Isomorphic (equivalent) connected graphs and adjacency matrices

Learning intentions

- ▶ To be able to identify connected graphs and bridges.
- ▶ To be able to identify isomorphic graphs.
- ▶ To be able to use an adjacency matrix to represent a graph.

Connected graphs and bridges

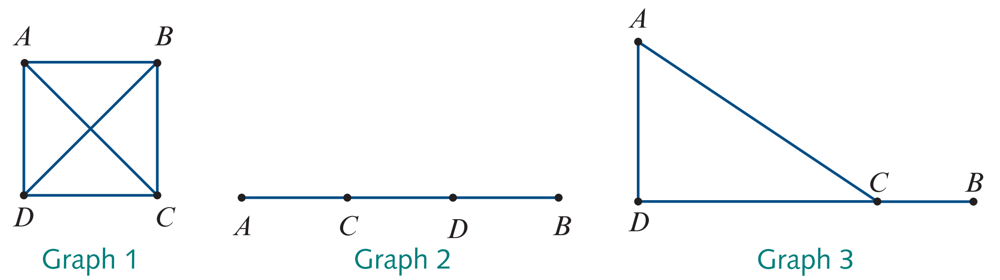
Connected graphs

So far, all the graphs we have encountered have been **connected**.

Connected graphs

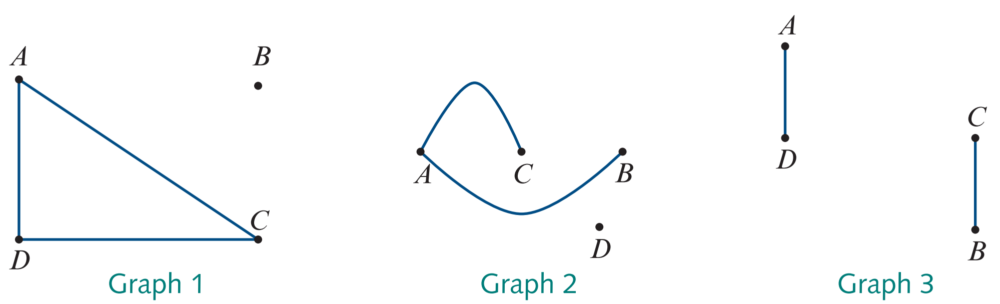
In a **connected graph**, every vertex is connected to every other vertex, either directly or indirectly. That is, every vertex in the graph can be reached from every other vertex in the graph.

For example, the three graphs shown below are all connected.



The graphs are connected because, starting at any vertex, say A, you can always find a path along the edges of the graph to take you to every other vertex.

However, the three graphs below are *not connected*, because there is not a path along the edges that connects vertex A (for example) to every other vertex in the graph.



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Bridges

Connected graphs have applications in a range of problems, such as planning airline routes, communication systems and computer networks, where a single missing connection can lead to an inoperable system. Such critical connections are called *bridges*.

Bridge

A **bridge** is an edge in a connected graph that, if removed, leaves the graph disconnected.

In Graph 1 below, edge *CD* is a bridge because removing *CD* from the graph leaves it disconnected, as shown in Graph 2.

Graph 1 Graph 2

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Isomorphic graphs

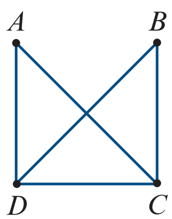
Different-looking graphs can contain the same information.

Isomorphic (equivalent) graphs

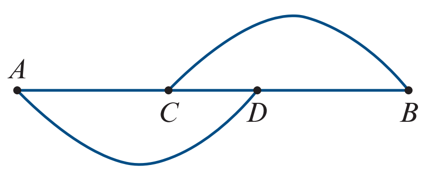
Isomorphic graphs are graphs that contain identical information. They have the

- same number of vertices
- same number of edges
- and the same connections between vertices.

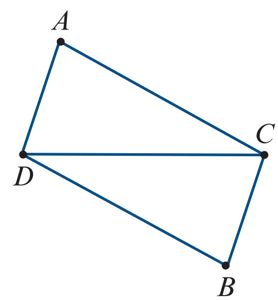
For example, the three graphs below look quite different, but in graphical terms, they are equivalent.



Graph 1



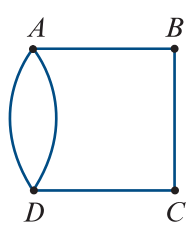
Graph 2



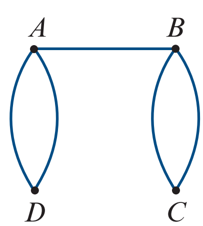
Graph 3

This is because they contain the same information. Each graph has the same number of edges (5) and vertices (4), corresponding vertices have the same degree (e.g. deg(A) = 2 for each graph) and the edges join the vertices in the same way (A to C, A to D, B to C, B to D, and D to C).

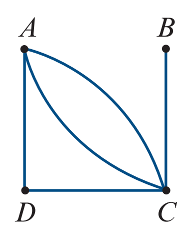
However, the three graphs below, although having the same numbers of edges and vertices, are not isomorphic. This is because corresponding vertices do *not* have the same degree and the edges do *not* connect the same vertices.



Graph 1



Graph 2



Graph 3

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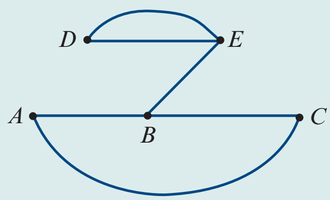
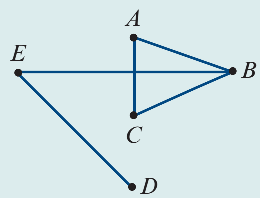
Example 2 Identifying bridges and isomorphic graphs

Consider the following three graphs.

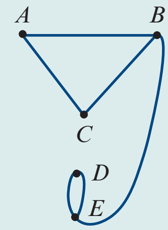
Graph 1



Graph 2



Graph 3



a Which of the three graphs above are connected?

b Graph 2 contains two bridges. Identify the bridges.

c Two of the graphs are isomorphic. Identify the isomorphic graphs.

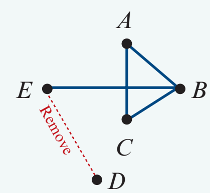
Explanation

a In each of the three graphs, every vertex is connected to every other vertex, directly or indirectly, through another vertex.

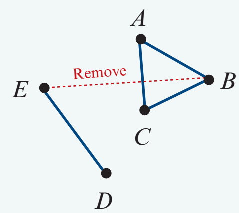
Solution

All three graphs are connected.

b **1** One bridge is *DE*. If you remove the edge between the vertices *D* and *E*, the graph will now be disconnected *(the vertex D will no longer be connected to any vertices).*



2 Likewise, if you remove the edge between the vertices *E* and *B*, the graph will also be disconnected *(the vertices D and E will no longer be connected to the vertices A, B and C).*



The two bridges are *DE* and *EB*.

c **1** First, check if all three graphs contain the same number of vertices and edges. All three graphs have five vertices, *however*, Graphs 1 and 3 have six edges, whereas Graph 2 only has five edges.

2 Next, check that all edges connect to the same vertices. All edges in Graphs 1 and 3 connect to the same vertices *(e.g. DE twice, BE, BC, AB and AC).*

The two isomorphic graphs are Graph 1 and Graph 3.

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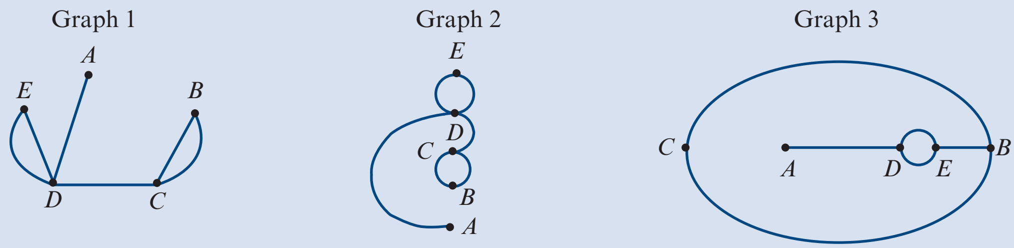
Now try this 2 Identifying bridges and isomorphic graphs (Example 2)

Consider the following three graphs.



a Which of the graphs above are connected?

b Graph 1 contains two bridges. Identify the bridges.

c Two of the graphs are isomorphic. Identify the isomorphic graphs.

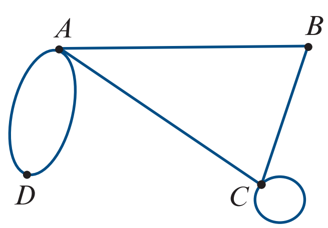
Adjacency matrices

Matrices are a compact way of communicating the information in a graph. There are various types of matrices that can be used to represent the information in a graph. We will only consider one type, the **adjacency matrix**.

Adjacency matrix

An **adjacency matrix** is a square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

Adjacency matrices are useful when the information in a graph needs to be entered into a computer. For example:



$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

On the left, we have a graph with four vertices, A, B, C and D.

On the right, we have a 4×4 adjacency matrix (four rows and four columns).

The rows and columns are labelled A to D, as shown, to match the vertices in the graph.

The numbers in the matrix refer to the number of edges joining the corresponding vertices.

For example, in this matrix:

- the '0' in row A, column A, indicates that no edges connect vertex A to vertex A.
- the '1' in row B, column A, indicates that one edge connects vertex A to vertex B.
- the '2' in row D, column A, indicates that two edges connect vertex A to vertex D.
- the '1' in row C, column C, indicates that one edge connects vertex C to vertex C (a loop) and so on until the matrix is complete.

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Example 3 Drawing a graph from an adjacency matrix

Draw the graph that is represented by the following adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

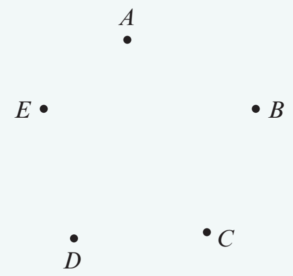
Explanation

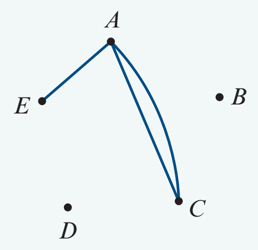
1 Draw a dot for each vertex, and label *A* to *E*.

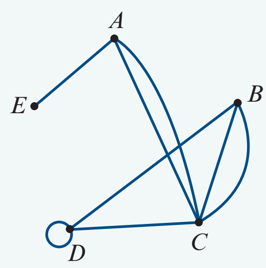
2 Starting with row *A*, there is a '2' in column *C* and a '1' in column *E*. This means vertex *A* has three edges (two connecting to vertex *C* and one connecting to vertex *E*). Draw these edges.

3 Continue this process, row by row, until all edges are drawn. Note the '1' in row *D*, column *D*, refers to a *loop*, because it's an edge that connects a vertex to itself.

Solution







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Now try this 3 Drawing a graph from an adjacency matrix (Example 3)

Draw the graph that is represented by the following adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Section Summary

- A graph is **connected** if every vertex in the graph is accessible from every other vertex in the graph along a series of adjacent edges.
- A **bridge** is a single edge in a connected graph that, if removed, leaves the graph disconnected. A graph can have more than one bridge.
- Graphs are said to be **isomorphic** if:
  - they have the same numbers of edges and vertices
  - corresponding vertices have the same degree, and the edges connect the same vertices.
- An **adjacency matrix** summarises the information in a graph. There is one row and one column for each vertex. It uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph. A loop (an edge that connects a vertex to itself) is counted as one edge in the matrix.

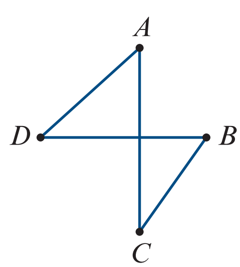
Exercise 8B

Building understanding

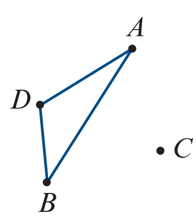
Example 2

1 Which of the following graphs are connected?

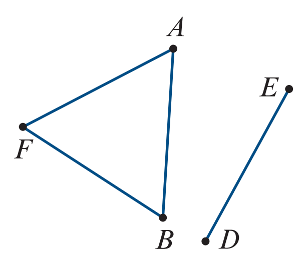
A



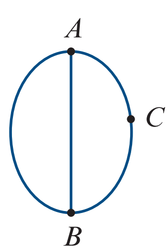
B



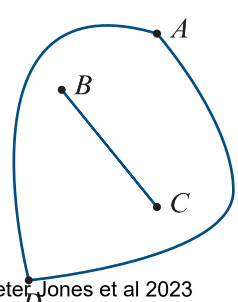
C



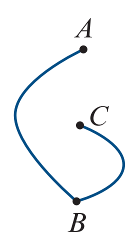
D



E



F



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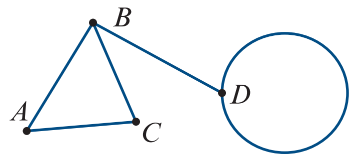
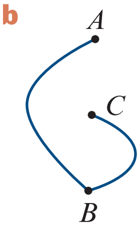
2 Identify the bridge (or bridges) in the graphs below.

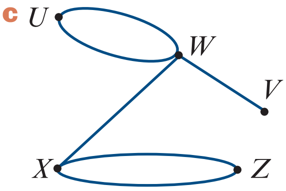
a



b

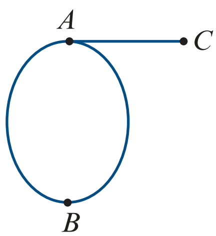


c

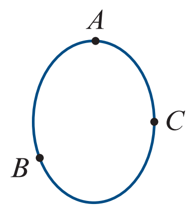


3 In each of the following sets of three graphs, two of the graphs are isomorphic. In each case, identify the isomorphic graphs.

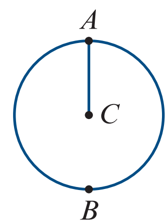
a



Graph 1

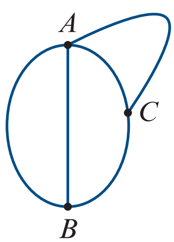


Graph 2

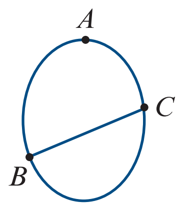


Graph 3

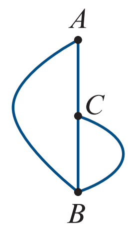
b



Graph 1

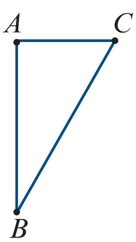


Graph 2

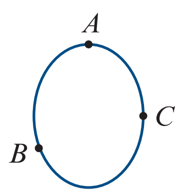


Graph 3

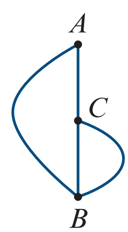
c



Graph 1

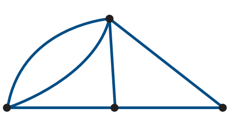


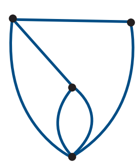
Graph 2

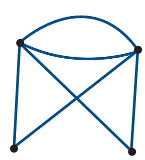


Graph 3

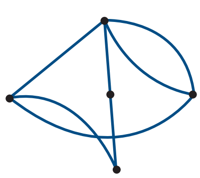
d

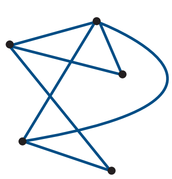

Graph 1

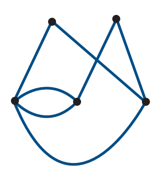

Graph 2


Graph 3

e


Graph 1


Graph 2


Graph 3

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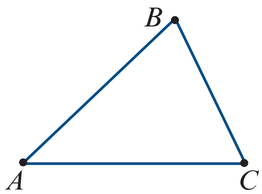
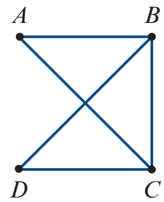
4 Construct an adjacency matrix for each of the following graphs.

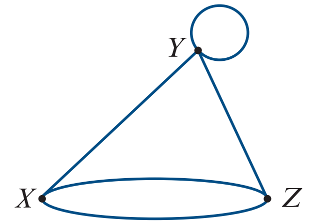
a



b



c



Developing understanding

5 Draw a connected graph with:

a three vertices and three edges
b three vertices and five edges
c four vertices and six edges
d five vertices and five edges

6 Draw a graph that is *not* connected with:

a three vertices and two edges
b four vertices and three edges
c four vertices and four edges
d five vertices and three edges

7 What is the smallest number of edges that can form a connected graph with four vertices?

8 Draw a graph with four vertices in which every edge is a bridge.

Example 3

9 Construct a graph for each of the following adjacency matrices.

a
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ W \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

d
$$\begin{array}{c} \\ F \\ G \\ H \\ I \\ J \end{array}\begin{array}{c} \begin{array}{ccccc} F & G & H & I & J \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

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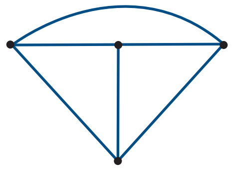
Testing understanding

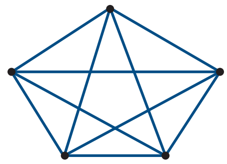
10 A graph has six vertices and only one bridge. What is the minimum number of edges that this graph must have if it is a connected graph?

11 The following graphs are connected. How many edges in each graph can be removed so that the graph will have the minimum number of edges to remain connected?

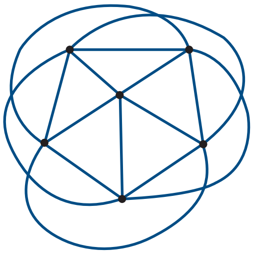
a



b

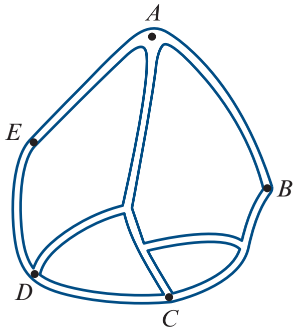


c

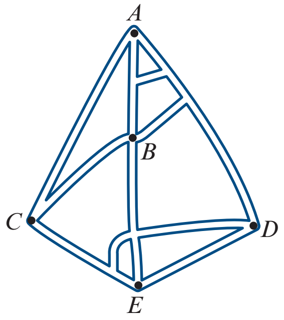


12 The following maps below show all the road connections between five towns, A, B, C, D and E. Construct an adjacency matrix for each of the maps below.

a



b



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8C Planar graphs and Euler's formula

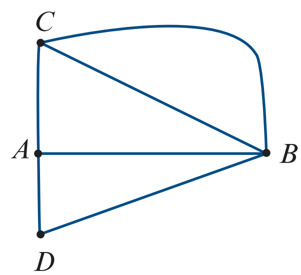
Learning intentions

- To be able to classify a graph as a **planar graph**.
- To be able to redraw a graph in an equivalent planar form.
- To be able to identify the number of faces of a graph.
- To be able to verify Euler's formula.
- To be able to apply Euler's formula to find unknown properties of a planar graph.

Planar graphs

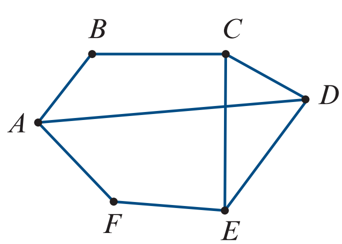
Planar graphs

A **planar graph** can be drawn on a plane (page surface) so that no edges intersect (cross), except at the vertices.

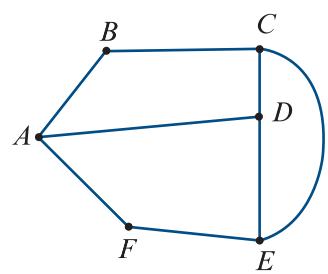


A planar graph: no intersecting edges

Some graphs do not initially appear to be planar; for example, Graph 1, shown below left. However, Graph 2 (below right) is equivalent (isomorphic) to Graph 1. Graph 2 is clearly planar, so Graph 1 is also planar.



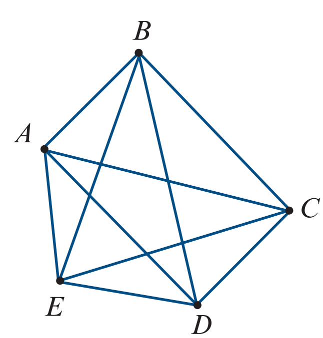
Graph 1



Graph 2

Not all graphs are planar.

For example, the graph opposite cannot be redrawn in an equivalent planar form, no matter how hard you try.



Non-planar graph

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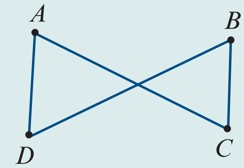
Example 4 Redrawing a graph in planar form

Redraw the graph shown opposite in planar form.



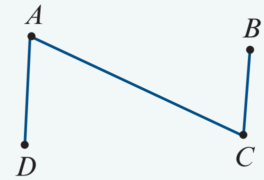
Explanation

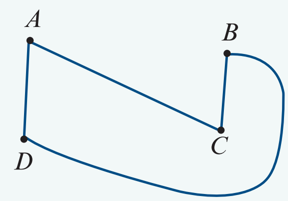
1 Redraw the graph with edge *DB* removed.

Note: We have removed edge *DB* because it intersects edge *AC*.

2 Replace edge *DB* as a curved line that avoids intersecting with the other three edges. The graph is now in an equivalent planar form: no edges intersect, except at vertices.

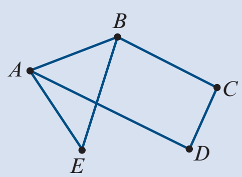
Solution (there are others)





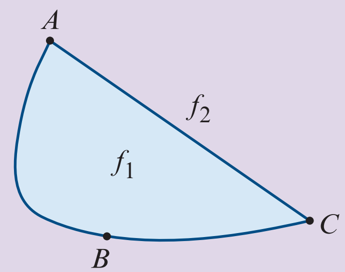
Now try this 4 Redrawing a graph in planar form (Example 4)

Redraw the graph below in planar form.



Faces of a graph

The graph opposite can be regarded as dividing the paper it is drawn on into two regions. These regions are called **faces**.



Faces

A **face** is a region of space. The space may be enclosed within the connected edges of a graph or the space surrounding the graph.

One face, f_1, is bounded by the graph.

The other face, f_2, is the region surrounding the graph. This 'outside' face is infinite.

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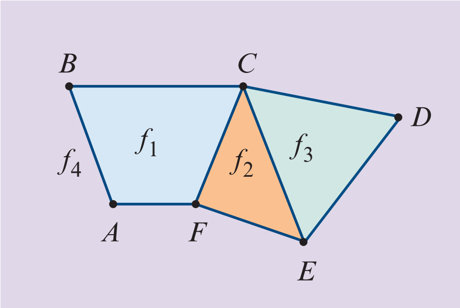
The graph opposite divides the paper into four regions, so we say that it has four faces: f_1, f_2, f_3 and f_4. Here f_4 is an infinite face.



Euler's formula

Euler discovered that, for connected planar graphs, there is a relationship between the *number of vertices*, v, the *number of edges*, e, and the *number of faces*, f.

Euler's formula

For a connected planar graph:

number of vertices + number of faces = number of edges + 2

or

$$v + f = e + 2$$

where v = number of vertices, e = number of edges and f = number of faces.

For example, for the graph opposite:

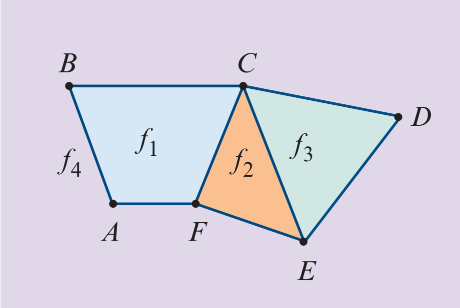
$v = 6$, $f = 4$ and $e = 8$.

$v + f = e + 2$

$6 + 4 = 8 + 2$

$10 = 10$

confirming Euler's formula.

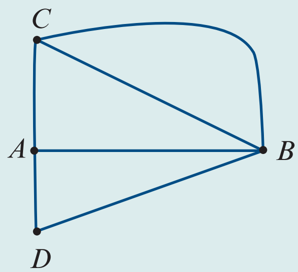


Example 5 **Verifying Euler's formula**

Consider the connected planar graph shown.

a Write down the number of vertices, v, the number of edges, e, and the number of faces, f.

b Verify Euler's formula.



Explanation	Solution
a **1** There are four vertices: A, B, C, D, so $v = 4$.	Number of vertices: $v = 4$

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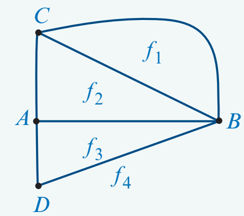
2 There are six edges: *AB, AC, AD, BC* (× 2) and *BD*, so $e = 6$.

3 There are four faces, so $f = 4$.

Tip: Mark the faces on the diagram. Do not forget the infinite face, f_4, that surrounds the graph.

Number of edges: $e = 6$



Number of faces: $f = 4$

b **1** Write down Euler's formula.

2 Substitute the values of v, e, and f. Evaluate.

3 Write your conclusion.

Euler's formula:

$v + f = e + 2$

$4 + 4 = 6 + 2$

$8 = 8$

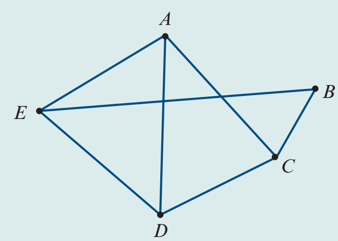
∴ Euler's formula is verified.

Example 6 Verifying Euler's formula

Consider the connected planar graph shown.

a Write down the number of vertices, v, the number of edges, e, and the number of faces, f.

b Verify Euler's formula.



Explanation

a **1** There are five vertices: *A, B, C, D, E*, so $v = 5$.

2 There are seven edges: *AC, AD, AE, BC, BE, CD, DE*, so $e = 7$.

3 The original graph had two edges crossing, therefore it must be redrawn in planar form to determine how many faces. There are four faces, so $f = 4$.

Tip: Mark the faces on the diagram. Do not forget the face, f_4, that surrounds the graph.

b **1** Write down Euler's formula.

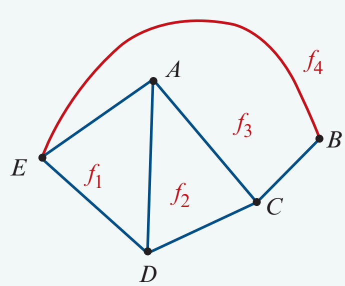
2 Substitute the values of v, e, and f. Evaluate.

3 Write your conclusion.

Solution

Number of vertices: $v = 5$

Number of edges: $e = 7$



Number of faces: $f = 4$

Euler's formula: $v + f = e + 2$

$5 + 4 = 7 + 2$

$9 = 9$

∴ Euler's formula is verified.

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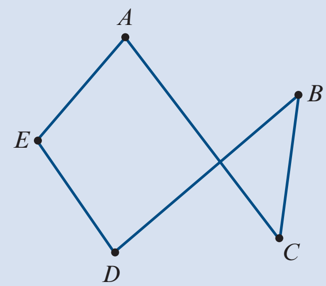
Now try this 6 Verifying Euler's formula (Examples 5 and 6)

Consider the connected graph opposite.
Verify Euler's formula.



Example 7 Using Euler's formula to find unknown characteristics of a graph

For a planar connected graph, find:

a the number of edges, given it has four vertices and six faces.

b the number of faces, given it has four vertices and five edges.

Explanation	Solution
a **1** Write down v and f.	$v = 4, f = 6$
2 Write down Euler's formula.	Euler's formula:
3 Substitute the values of v and f.	$v + f = e + 2$
4 Solve for e.	$4 + 6 = e + 2$
	$10 = e + 2$
	$e = 8$
5 Write your answer.	Therefore this graph has 8 edges.
b **1** Write down v and e.	$v = 4, e = 5$
2 Write down Euler's formula.	Euler's formula:
3 Substitute the values of v and e.	$v + f = e + 2$
4 Solve for f.	$4 + f = 5 + 2$
	$4 + f = 7$
	$f = 7 - 4$
	$f = 3$
5 Write your answer.	Therefore this graph has 3 faces.

Now try this 7 Using Euler's formula to find unknown characteristics of a graph (Example 7)

For a planar connected graph, find the number of vertices given it has 5 faces and 8 edges.

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Section Summary

- **Planar graphs** are graphs that can be drawn so that no two edges cross or intersect, except at the vertices.
- A **face** is an area in a graph that is enclosed by the edges. The space that surrounds a graph is also always counted as a face.
- The number of faces can only be determined when the graph is drawn in planar form (no crossing edges).
- **Euler's formula**: $v + f = e + 2$
  where:
  v = number of vertices,
  f = number of faces (including the surrounding space)
  and e = number of edges.
- Euler's formula only applies to planar graphs.

Skill-sheet

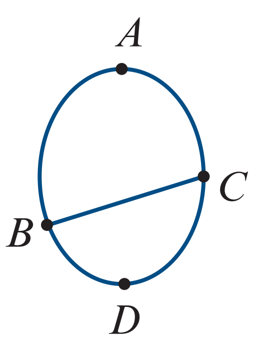
Exercise 8C

Building understanding

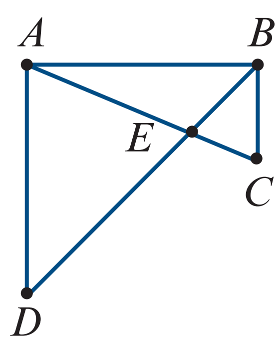
Example 4

1 Which of the following graphs are drawn in planar form?

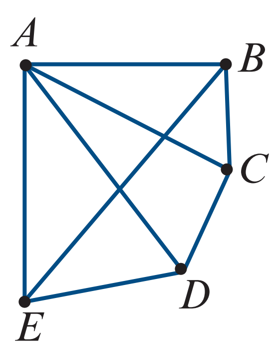
A



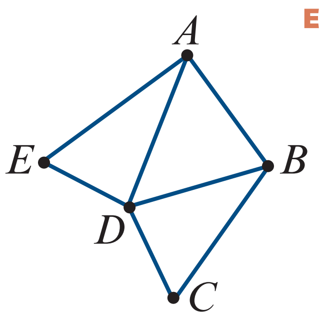
B



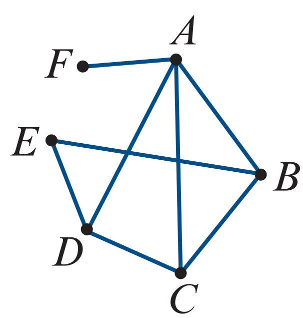
C



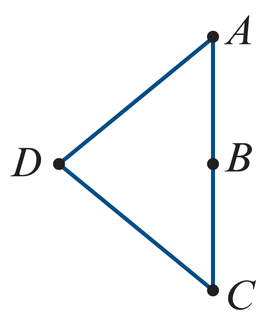
D



E



F



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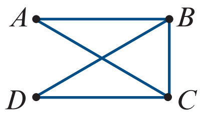
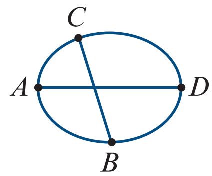
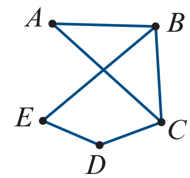
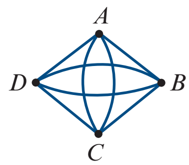
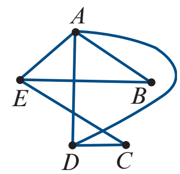
2 Redraw each graph in an equivalent planar form.

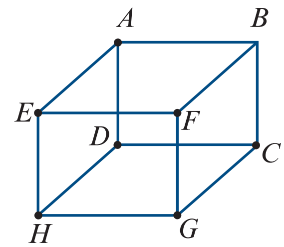
a


b


c


d


e


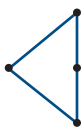
f


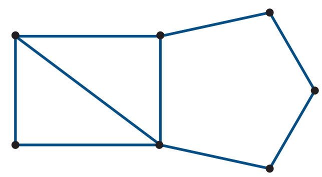
Developing understanding

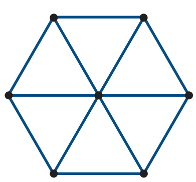
Example 5
Example 6

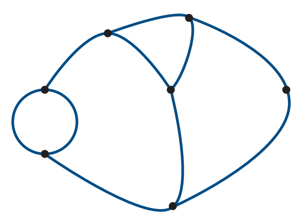
3 For each of the following graphs:

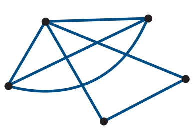
i state the values of v, e and f **ii** verify Euler's formula.

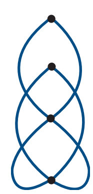
a


b


c


d


e


f


Example 7

4 For a planar connected graph, find:

a f given $v = 4$ and $e = 4$
b v given $e = 3$ and $f = 2$
c e given $v = 3$ and $f = 3$
d v given $e = 6$ and $f = 4$
e f given $v = 4$ and $e = 6$
f f given $v = 6$ and $e = 11$
g e given $v = 10$ and $f = 11$

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5 The following adjacency matrices represent planar graphs. Find the number of faces for each graph.

a
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ V \\ W \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{ccccc} V & W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

d
$$\begin{array}{c} \\ F \\ G \\ H \\ I \\ J \end{array}\begin{array}{c} \begin{array}{ccccc} F & G & H & I & J \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Testing understanding

6 A connected planar graph has six vertices and nine edges. A further three edges were added to the graph. The number of faces increased by:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

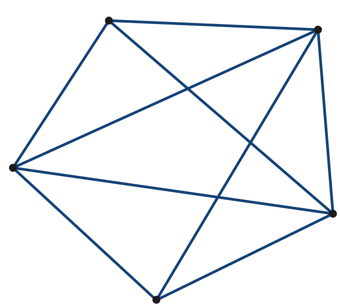
7 A planar graph has four faces. This graph could have:

A Seven vertices and seven edges
B Seven vertices and four edges
C Seven vertices and five edges
D Four vertices and seven edges
E Five vertices and seven edges

8 A planar graph has five vertices. Three vertices have degree four and two have degree three. The number of faces of this graph is:

A 6 **B** 9 **C** 11 **D** 12 **E** 18

9 Consider the graph shown opposite.
Which one of the following statements is not true for the graph?



A The graph has even and odd vertices.
B The graph has five faces.
C Euler's formula is verified for this graph.
D The graph is planar with five vertices.
E The graph is connected.

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10 A graph is connected and has five vertices and four edges. Consider the following four statements.

- The graph is planar.
- The graph can have multiple edges between two vertices.
- All the vertices of the graph can have an even degree.
- Two vertices of the graph can have an odd degree.

The number of these statements which are true for this graph is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 A connected planar graph has an equal number of vertices and edges. The number of faces in this graph is equal to:

A 1 **B** 2 **C** 3 **D** 4

E Cannot determine, more information is required.

12 A connected planar graph has an equal number of vertices and faces. The number of edges in this graph is equal to:

A the number of faces

B twice the number of faces

C half the number of faces

D two less than twice the number of faces

E two more than half the number of faces

8D Walks, trails, paths, circuits and cycles

Learning intentions

- ► To be able to identify a walk as a trail, path, circuit or cycle.

Many practical problems that can be modelled by graphs involve moving around a graph, for example, designing a postal delivery route or solving the Königsberg bridge problem. To solve such problems, you will need to know about a number of concepts that we use to describe the different ways we can move around a graph. We will use the following graph to explore these ideas.

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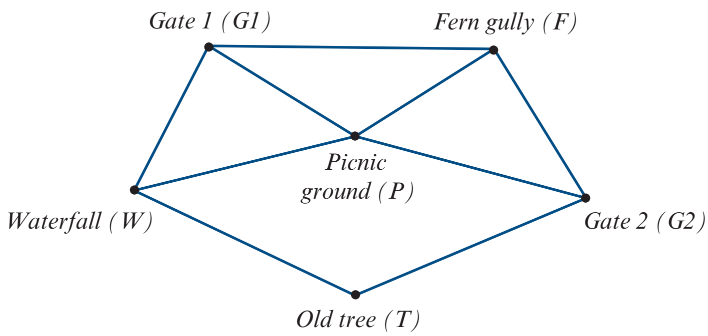


The graph above represents a series of tracks in Sherbrooke Forest that connect a picnic ground (vertex *P*), a waterfall (vertex *W*), a very old tree (vertex *T*) and a fern gully (vertex *F*).

People can enter and leave the forest through either Gate 1 (vertex *G1*) or Gate 2 (vertex *G2*). The edges in the graph represent tracks that connect these places, for example, the edge *WP* represents the track between the waterfall and the picnic ground.

Walks, trails and paths

Informally, a walk is any route through a graph that moves from one vertex to another along the joining edges. When there is no ambiguity, a walk in a graph can be specified by listing the vertices visited on the walk.

Walk

A **walk** is a sequence of edges linking successive vertices, that connects two different vertices in a graph.

A walk in the forest

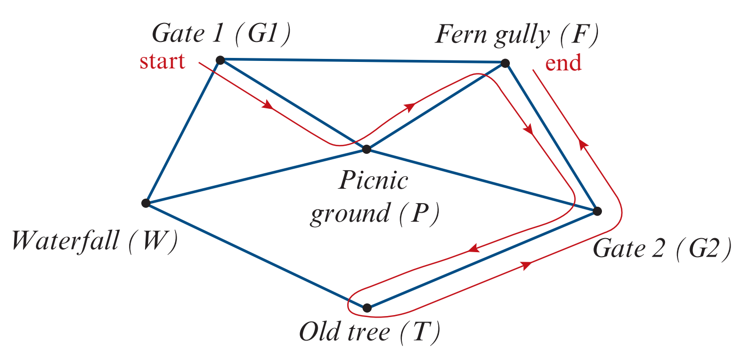
Using the forest track graph (shown in blue), an example of a *walk* is:

G1-P-F-G2-T-G2-F

The red arrows on the graph trace out a walk.

Note1: The double red arrows on the graph indicate that this track is walked along in both directions.

Note2: A walk does not require all of its edges or vertices to be different.



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Trail

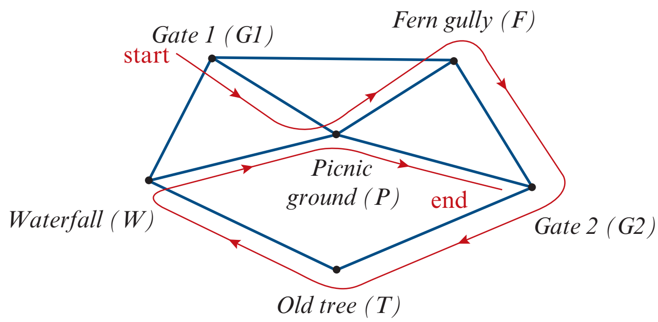
A **trail** is a walk with no repeated edges.

A forest trail

Using the forest track graph, an example of a *trail* is:

G1-P-F-G2-T-W-P-G2

The red arrows on the graph trace out this trail. It has *no repeated edges*. However, there are two repeated vertices, *P* and *G*2. This is permitted on a trail.



Path

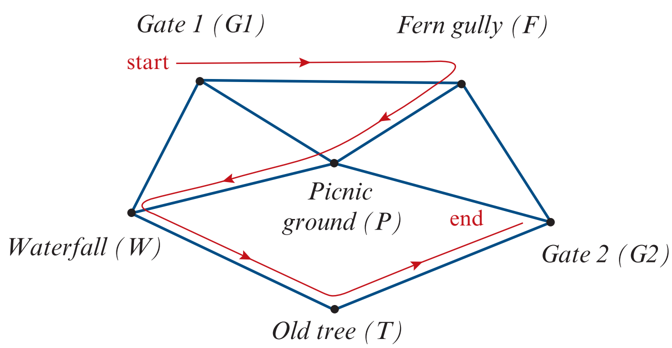
A **path** is a walk with no repeated edges and no repeated vertices.

A path in the forest

Using the forest track graph, an example of a *path* is:

G1-F-P-W-T-G2

The red arrows on the graph trace out this path. There are *no repeated edges* or *vertices*.



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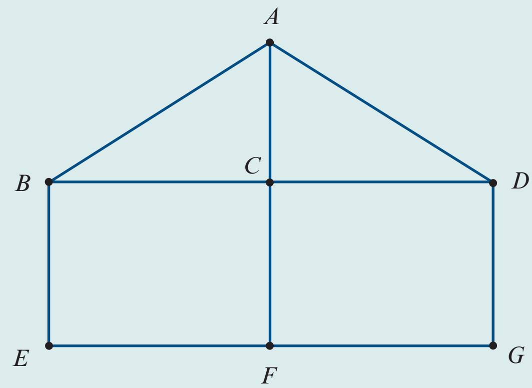
Example 8 Identifying types of walks

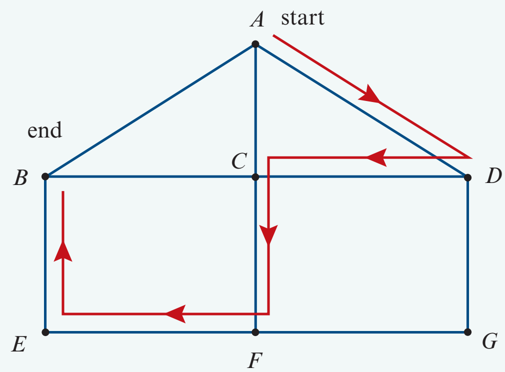
The graph opposite shows how seven vertices are connected. One walk is described as *A*-*D*-*C*-*F*-*E*-*B*. Does this walk represent a trail or a path?



Explanation

This walk starts at one vertex, ends at a different vertex, has no repeated edges and no repeated vertices, so it must be a path.

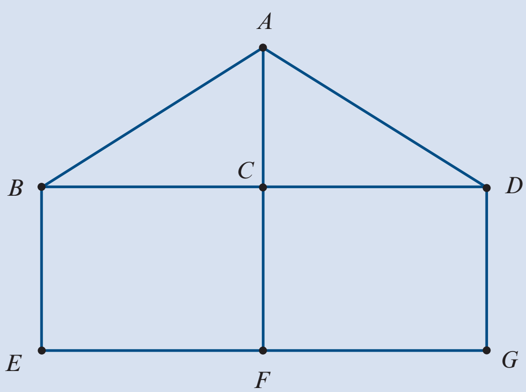
Solution



This walk is a path.

Now try this 8 Identifying types of walks (Example 8)

The graph below shows how seven vertices are connected. One walk is described as *B*-*E*-*F*-*C*-*B*-*A*. Does this walk represent a trail or a path?



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Circuits and cycles

There is nothing to stop a walk, trail or path starting and ending at the same vertex. When this happens, we say that the walk, trail or path is closed. Because *closed trails* and *closed paths* are so important in practice, we give them special names. We call them *circuits* and *cycles*.

Circuit

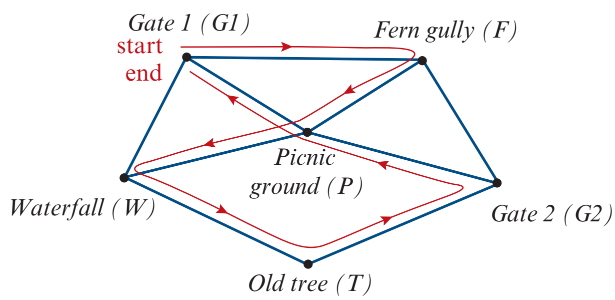
A **circuit** is a walk that starts and ends at the same vertex and has no repeated edges.

A circuit in the forest

Using the forest track graph, an example of a *circuit* is:

G1-F-P-W-T-G2-P-G1

This circuit *starts* and *ends* at the *same* vertex (*G1*). The red arrows on the graph trace out this circuit.



There are *no repeated edges*.
However, the circuit passes through vertex *P* twice.

Cycle

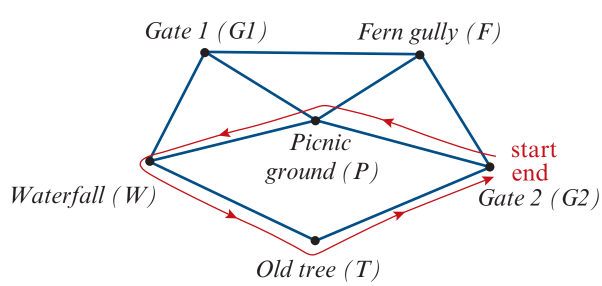
A **cycle** is a walk that starts and ends at the same vertex, has no repeated edges and has no repeated vertices.

A cycle in the forest

Using the forest track graph, an example of a *cycle* is:

G2-P-W-T-G2

The red arrows on the graph trace out this cycle. Except for the first vertex (and last) there are no repeated edges or vertices in a cycle.



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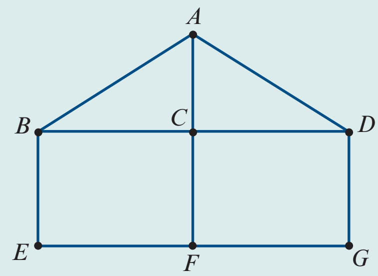
Example 9 Identifying types of walks

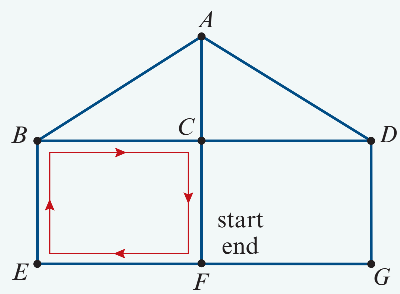
The graph opposite shows how seven vertices are connected. One walk is described as *F*-*E*-*B*-*C*-*F*. Does this walk represent a circuit or a cycle?



Explanation

This walk starts and ends at the same vertex, has no repeated edges and no repeated vertices, so it must be a cycle.

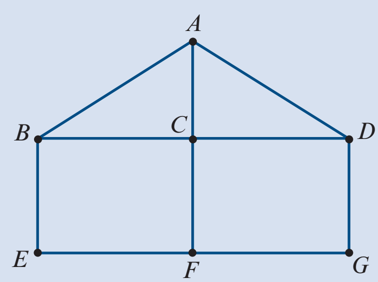
Solution



This walk is a cycle.

Now try this 9 Identifying types of walks (Example 9)

The graph opposite shows how seven vertices are connected. One walk is described as: *A*-*C*-*F*-*G*-*D*-*C*-*B*-*A*. Does this walk represent a circuit or a cycle?



Section Summary

- A **walk** is a sequence of edges linking successive vertices, that connects at least two different vertices in a graph.
- A **trail** is a walk with no repeated edges.
- A **path** is a walk with no repeated edges and no repeated vertices.
- A **circuit** is a walk that starts and ends at the same vertex and has no repeated edges.
- A **cycle** is a walk that starts and ends at the same vertex, has no repeated edges and no repeated vertices (except for the first/last vertex).

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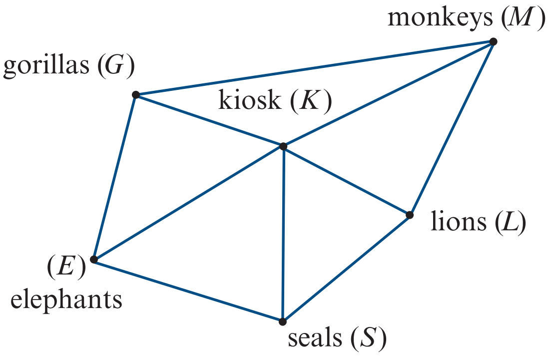
Skill-sheet

Exercise 8D

Building understanding

1 The graph below shows the pathways linking five animal enclosures in a zoo to each other and to the kiosk.



a Which of the following represents a trail in the graph?

i *S-L-K-M-K*

ii *G-K-L-S-E-K-M*

iii *E-K-L-K*

b Which of the following represents a path in the graph?

i *K-E-G-M-L* **ii** *E-K-L-M* **iii** *K-S-E-K-G-M*

c Which of the following represents a circuit in the graph?

i *K-E-G-M-K-L-K* **ii** *E-S-K-L-M-K-E* **iii** *K-S-E-K-G-K*

d Which of the following represents a cycle in the graph?

i *K-E-G-K* **ii** *G-K-M-L-K-G* **iii** *L-S-E-K-L*

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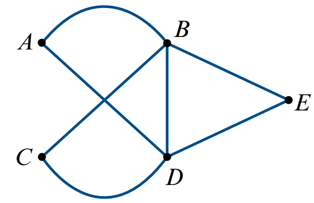
Developing understanding

Example 8

2 Using the graph opposite, identify the walks below as a trail, path, circuit, cycle or walk only.



a A-D-B-A

b A-D-E-B-D-C-B-A

c C-B-E-D-B

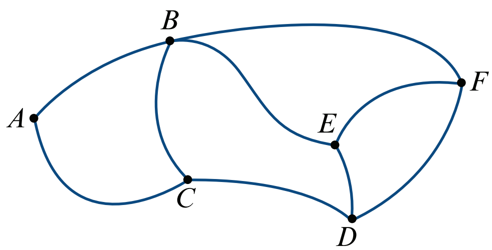
d C-D-E-B-A

e D-C-B-D-E-B-A-D

f E-B-D-C

Example 9

3 Using the graph opposite, identify the walks below as a trail, path, circuit, cycle or walk only.



a A-B-E-F-B

b B-C-D-E-B

c C-D-E-F-B-A

d A-B-E-F-B-E-D

e E-F-D-C-B-E

f B-C-A-B-F-E-B

Testing understanding

4 The road connections between different towns are represented below. For each:

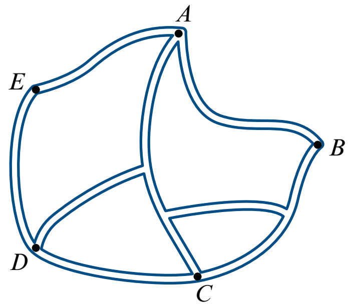
i How many different paths are there from Town A to Town D?

ii How many different cycles are there, starting at Town A?

a

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b



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8E Eulerian trails and circuits (extension)

Learning intentions

- ▶ To be able to identify a walk as an Eulerian trail.
- ▶ To be able to identify a walk as an Eulerian circuit.
- ▶ To be able to use the degrees of the vertices to identify when an Eulerian trail or circuit is possible.

Trails and circuits that follow every edge of a graph without duplicating any edge are called **Eulerian trails** and **Eulerian circuits**; named after the pioneering work done by Euler.

Eulerian trail

An **Eulerian trail** follows every edge of a graph with no repeated edges.
An Eulerian trail will exist if the graph:

- is connected
- has exactly *zero or two* vertices that have an *odd degree*.

If there are zero odd vertices, the Eulerian trail can start at any vertex in the graph. If there are two odd vertices, the Eulerian trail will start at one of the odd vertices and finish at the other.

Eulerian circuit

An **Eulerian circuit** is an Eulerian trail that starts and finishes at the same vertex.
An Eulerian circuit will exist if the graph:

- is connected
- has vertices that *all* have an *even degree*.

An Eulerian circuit can start at *any* of the vertices.

The reason we are interested in Eulerian trails and circuits is because of their practical significance. If, for example, a graph shows towns as vertices and roads as edges, then being able to identify a route through the graph that follows every road can be important for mail delivery or for checking the conditions of the roads.

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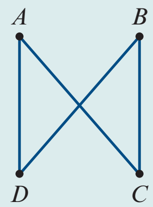
Example 10 Identifying Eulerian trails and circuits

For each of the following graphs:

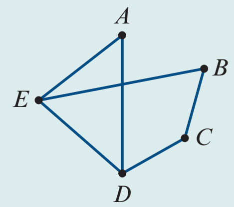
i Determine whether the graph has an Eulerian trail, an Eulerian circuit, both or neither, and state why.

ii If the graph has an Eulerian trail or an Eulerian circuit, show one example.

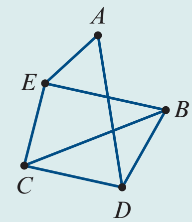
a



b



c



Solution

a Both: zero odd vertices.

A-C-B-D-A

b Eulerian trail: two odd vertices, the rest even.

E-A-D-E-B-C-D

c Neither: more than two odd vertices.

Note: In both cases, more than one solution is possible.

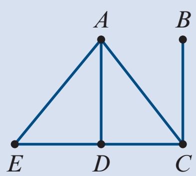
Now try this 10 Identifying Eulerian trails and circuits (Example 10)

For each of the following graphs:

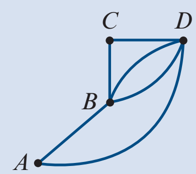
i Determine whether the graph has an Eulerian trail, an Eulerian circuit, both or neither, and state why.

ii If the graph has an Eulerian trail or an Eulerian circuit, show one example.

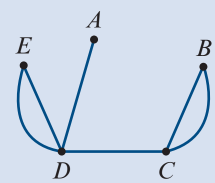
a



b



c



Applications of Eulerian trails and circuits

In everyday life, Eulerian trails and circuits relate to situations like the following:

- A postie wants to deliver mail without travelling along any street more than once.
- A visitor to a tourist park wants to minimise the distance they walk to see all of the attractions by not having to retrace their steps at any stage.
- A road inspector wants to inspect the roads linking several country towns without having to travel along each road more than once.

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Section Summary

Eulerian trail: a walk that follows every edge of a graph with no repeated edges.
An Eulerian trail will exist if the graph:

- is connected
- has exactly *zero or two* vertices that have an *odd degree*.

If there are zero odd vertices, the Eulerian trail can start at any vertex in the graph. If there are two odd vertices, the Eulerian trail will start at one of the odd vertices and finish at the other.

Eulerian circuit: an Eulerian trail that starts and finishes at the same vertex.
An Eulerian circuit will exist if the graph:

- is connected
- has vertices that *all* have an *even degree*.

An Eulerian circuit can start at *any* of the vertices.

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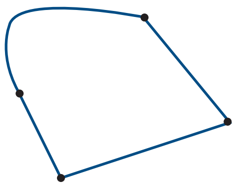
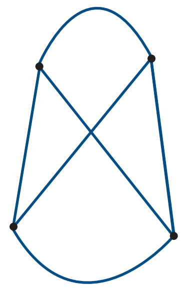
Exercise 8E

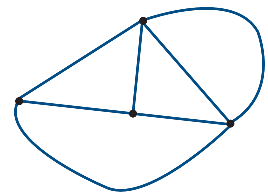
Building understanding

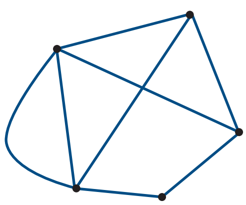
Example 10

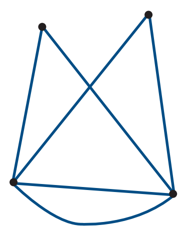
1 For each of the following graphs, determine whether the graph has an Eulerian trail, an Eulerian circuit, both or neither, and state why.

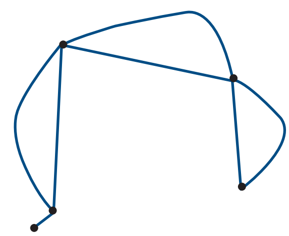
a


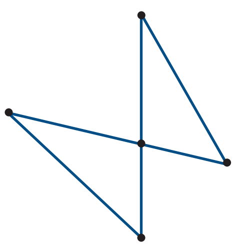
b


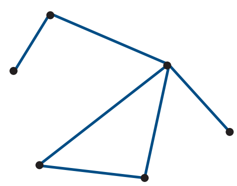
c


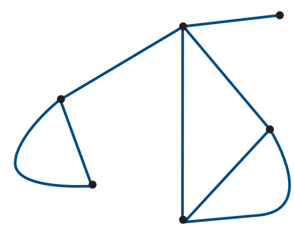
d


e


f


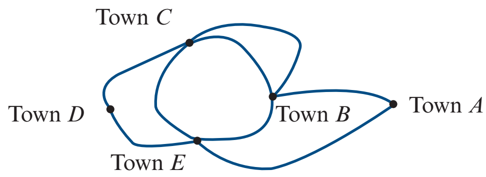
g


h


i


Developing understanding

2 A road inspector lives in Town A and is required to inspect all roads connecting the neighbouring towns, B, C, D and E. The network of roads is shown on the right.



a Is it possible for the inspector to set out from Town A, carry out their inspection by travelling over every road linking the five towns only once, and return to Town A? Explain.

b Show one possible route she can follow.

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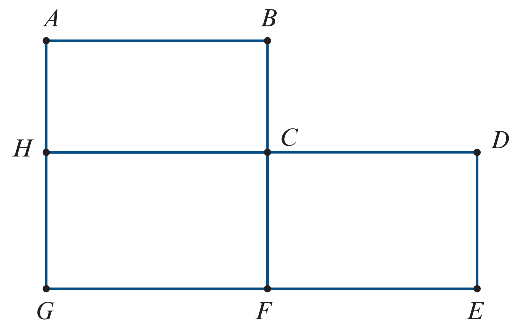
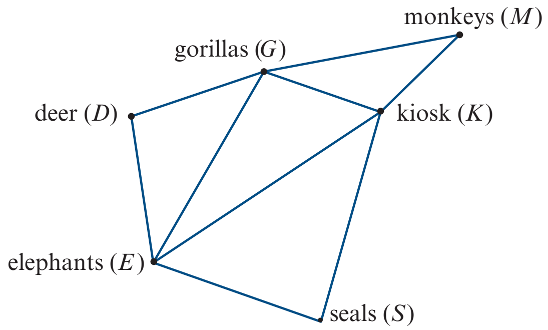
3 A postie has to deliver letters to the houses located on the graph of streets shown on the right.



a Is it possible for the postie to start and finish her deliveries at the same point in the graph without retracing her steps at some stage? If not, why not?

b It is possible for the postie to start and finish her deliveries at different points in the graph without retracing her steps at some stage. Identify one such route.

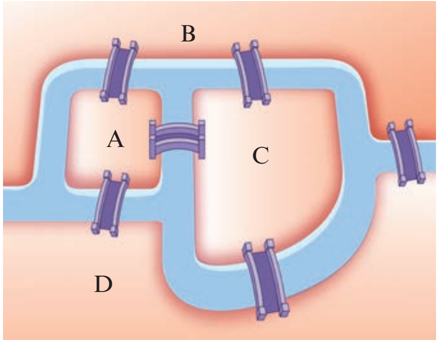
4 The graph below models the pathways linking five animal enclosures in a zoo to the kiosk and to each other.



a Is it possible for the zoo's street sweeper to follow a route that enables its operator to start and finish at the kiosk without travelling down any one pathway more than once? If so, explain why.

b If so, write down one such route.

5 Two islands are connected to the banks of a river by six bridges. See opposite.



a Draw a graph to represent this situation. Label the vertices: A, B, C and D to represent the riverbanks and the two islands. Use the edges of the graph to represent the bridges.

b It is not possible to plan a walking route that passes over each bridge once only. Why not?

c **i** Show where another bridge could be added to make such a walk possible.

ii Draw a graph to represent this situation.

iii Explain why it is now possible to find a walking route that passes over each bridge once only. Mark one such route on your graph.

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Testing understanding

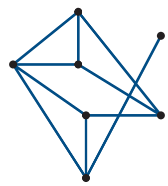
6 A graph has five vertices: A, B, C, D and E. The adjacency matrix for this graph is shown below.

a Does an Eulerian trail or Eulerian circuit exist? Give a reason for your answer.

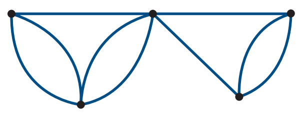
b If the element in row A, column C, changed to a zero, and the element in row C, column A, was also changed to a zero, would your answer to part **a** change? Justify your reasoning with a diagram and with reference to the degrees of the vertices.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

7 Consider the graph opposite. What is the minimum number of edges that must be added for an Eulerian circuit to exist?



8 An Eulerian trail for the graph opposite will be possible if only one edge is removed. In how many different ways could this be done?



8F Hamiltonian paths and cycles (extension)

Learning intentions

- ▶ To be able to identify a walk as a Hamiltonian path.
- ▶ To be able to identify a walk as a Hamiltonian cycle.

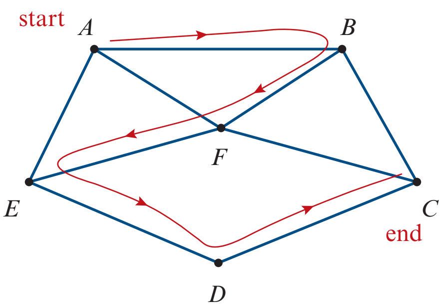
Eulerian trails and circuits focus on edges.

Hamiltonian paths and cycles focus on vertices.

Hamiltonian path

Hamiltonian path

A **Hamiltonian path** visits every vertex of a graph, with no repeated vertices.



For example, in the graph opposite, *A-B-F-E-D-C* is a Hamiltonian path. It starts at vertex A and ends at vertex C, visiting every vertex of the graph. (Follow the arrows.)

Note: A Hamiltonian path does not have to involve all edges.

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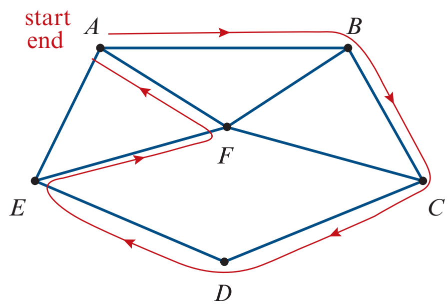
Hamiltonian cycle

Hamiltonian cycle

A **Hamiltonian cycle** visits every vertex of a graph with no repeated vertices, except for starting and finishing at the same vertex.



For example, in the second graph, *A-B-C-D-E-F-A* is a Hamiltonian cycle. It starts and finishes at vertex *A*, visiting every vertex of the graph. (Follow the arrows.)

Note: A Hamiltonian cycle does not have to involve all edges.

Unfortunately, unlike Eulerian trails and circuits, there are *no simple rules* for determining whether a network contains a Hamiltonian path or cycle. It is just a matter of 'trial and error'.

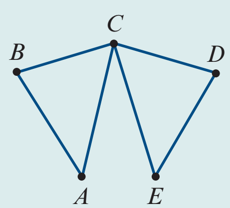
Example 11 Identifying Hamiltonian paths and cycles

For each of the following graphs:

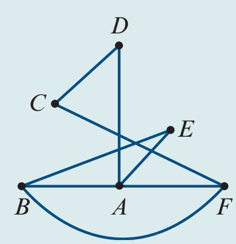
i Identify a Hamiltonian path, starting at vertex *A*.

ii State whether a Hamiltonian cycle is possible or not. If possible, identify one. If not possible, explain why not.

a



b



Solution

a **i** Hamiltonian path:

$A - B - C - D - E$

ii A Hamiltonian cycle is not possible. Whichever vertex you start at, you cannot return to the same vertex without repeating a vertex.

b **i** Hamiltonian path:

$A - E - B - F - C - D$

ii Yes, a Hamiltonian cycle is possible. One example:

$A - E - B - F - C - D - A$

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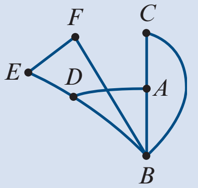
Now try this 11 **Identifying Hamiltonian paths and cycles (Example 11)**

For each of the following graphs:

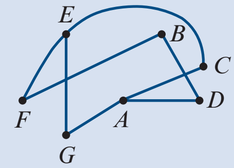
i Identify a Hamiltonian path, if possible, starting at vertex A.

ii State whether a Hamiltonian cycle is possible or not. If possible, identify one.

a



b



Applications of Hamiltonian paths and cycles

Hamiltonian paths and cycles have many practical applications. In everyday life, a Hamiltonian path would apply to situations like the following:

- You plan a trip from Melbourne to Mildura, with visits to Bendigo, Halls Gap, Horsham, Stawell and Ouyen on the way, but do not want to visit any town more than once.

Hamiltonian cycles relate to situations like the following:

- A courier leaves her depot to make a succession of deliveries to a variety of locations before returning to her depot. She does not like to go past each location more than once.
- A tourist plans to visit all of the historic sites in a city without visiting each more than once.
- You are planning a trip from Melbourne to visit Shepparton, Wodonga, Bendigo, Swan Hill, Natimuk, Warrnambool and Geelong before returning to Melbourne. You don't want to visit any town more than once.

In all these situations, there would be several suitable routes. However other factors, such as time taken or distance travelled, may need to be taken into account in order to determine the best route. This is an issue addressed in the next section: weighted graphs and networks.

Section Summary

- **Hamiltonian path**: a walk that involves every vertex of a graph with no repeated vertices.
- **Hamiltonian cycle**: a walk that involves every vertex of a graph with no repeated vertices, except for starting and ending at the same vertex.
- Both Hamiltonian paths and cycles can only be identified by inspection.

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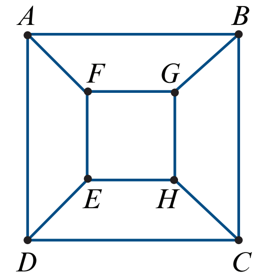
Exercise 8F

Building understanding

1 List a Hamiltonian path for the graph shown.

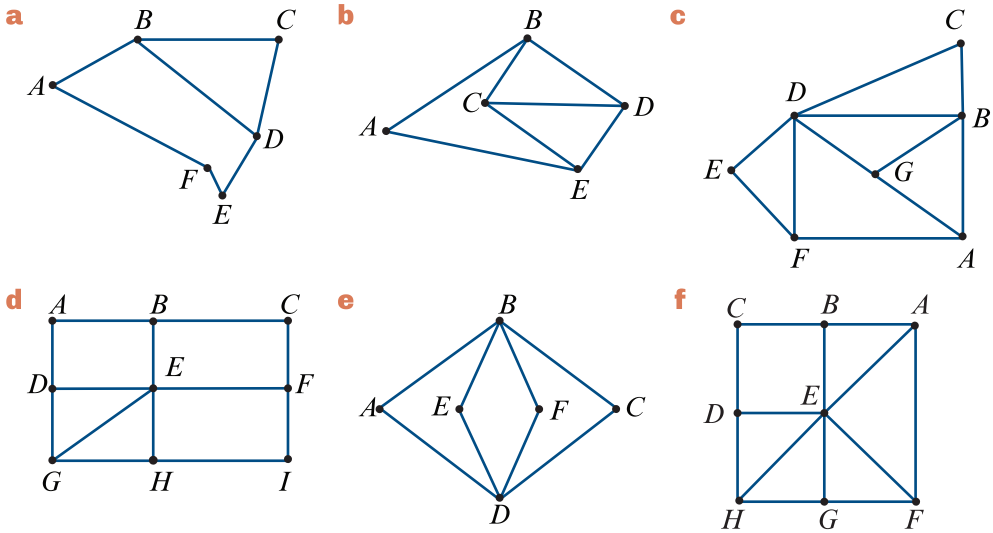
a Starting at A and finishing at D

b Starting at F and finishing at G



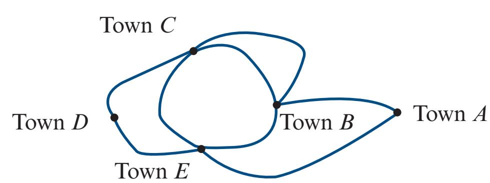
Example 11

2 Identify a Hamiltonian cycle in each of the following graphs (if possible), starting at A each time.



Developing understanding

3 A tourist wants to visit a second-hand bookshop in each of five different towns: Apsley (A), Berrigama (B), Cleverland (C), Donsley (D) and Everton (E).



The graph of roads connecting the towns is shown above.
Can a tourist start a tour that visits each town only once by starting at:

a Cleverland and finishing at Everton? If so, identify one possible route and give its mathematical name.

b Cleverland and finishing at Apsley? If so, identify one possible route and give its mathematical name.

c Everton and finishing at Everton? If so, identify one possible route and give its mathematical name.

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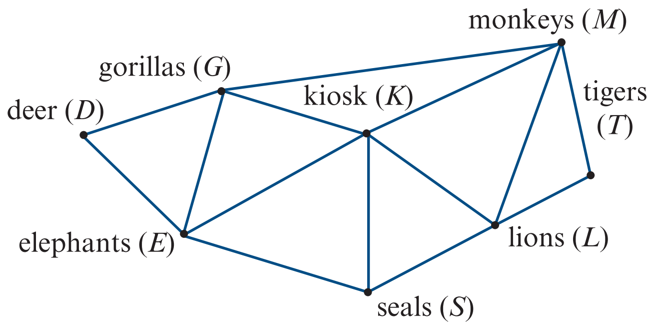
4 The graph opposite models the pathways linking seven animal enclosures in a zoo to the kiosk and to each other.



a Is it possible for a visitor to the zoo to start their visit at the kiosk and see all of the animals without visiting any one animal enclosure more than once? If so, identify a possible route, and give this route its mathematical name.

b Is it possible for a visitor to the zoo to start their visit at the deer enclosure and finish at the kiosk without visiting the kiosk or any enclosure more than once? If so, identify a possible route and give this route its mathematical name.

Testing understanding

5 The opposite adjacency matrix represents a graph.

	A	B	C	D
A	1	2	0	0
B	2	0	1	0
C	0	1	0	2
D	0	0	2	1

a How many faces does this graph have?

b State the two vertices connected by the bridge in this graph.

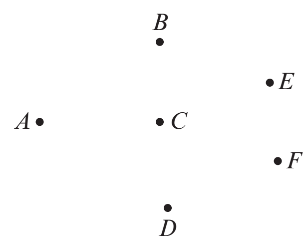
c Does this graph contain an Eulerian trail, Eulerian circuit or neither?

d Does this graph contain a Hamiltonian path, Hamiltonian cycle or neither?

6 What is the minimum number of edges that must be added to the following set of vertices for:

a a Hamiltonian path to exist

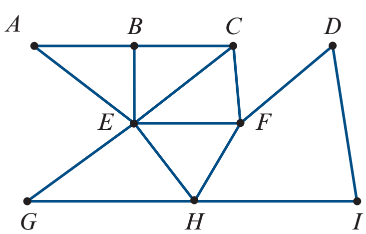
b a Hamiltonian cycle to exist.



7 What is the maximum number of edges that can be removed from the graph opposite for:

a a Hamiltonian path to still exist

b a Hamiltonian cycle to still exist.



8 **a** Do all connected graphs contain a Hamiltonian path? Explain your reasoning.

b Do all connected graphs contain a Hamiltonian cycle? Justify your reasoning with reference to the degree of vertices.

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8G Weighted graphs, networks and the shortest path problem

Learning intentions

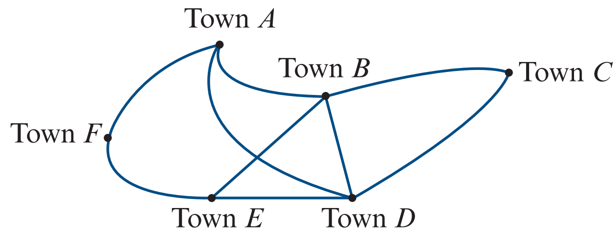
- To be able to analyse a weighted graph.
- To be able to represent a real-world situation using a network.
- To be able to find the shortest path between two vertices for a network.

Weighted graphs and networks

The graph opposite shows how six towns are connected by road.

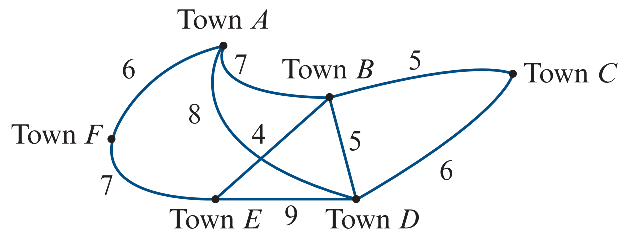
The towns are represented by the vertices of the graph.

The roads between towns are represented by the edges.



We can give more information about the situation we are representing with the graph by adding numbers to the edges.

The weighted graph opposite shows the distances between the towns (in kilometres).



Weighted graph

A **weighted graph** is a graph that has a number associated with each edge.

The weighted graph above could be called a network.

Network

A **network** is a weighted graph in which the weights are physical quantities, for example, distance, time or cost.

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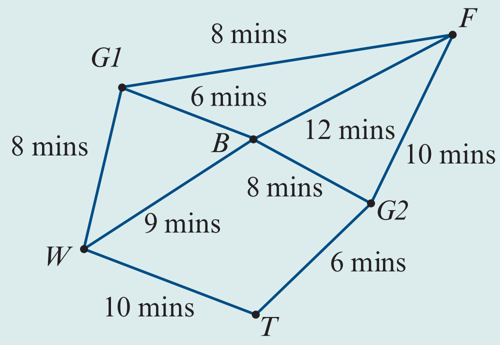
Example 12 Interpreting a network

The network opposite is used to model the tracks in a forest connecting a suspension bridge (B), a waterfall (W), a very old tree (T) and a fern gully (F).

Walkers can enter or leave the forest through either Gate 1 ($G1$) or Gate 2 ($G2$).

The numbers on the edges represent the times (in minutes) taken to walk directly between these places.



a How long does it take to walk from the bridge directly to the fern gully?

b How long does it take to walk from the old tree to the fern gully via the waterfall and the bridge?

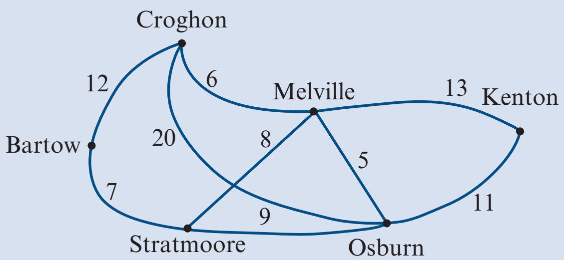
Explanation	Solution
a Identify the edge that directly links the bridge with the fern gully, and read off the time.	The edge is B-F. The time taken is 12 minutes.
b Identify the path that links the old tree to the fern gully, visiting the waterfall and the bridge on the way. Add up the times.	The path is T-W-B-F. The time taken is $10 + 9 + 12 = 31$ minutes.

Now try this 12 Interpreting a network (Example 12)

The weighted graph in the diagram below shows towns, represented by vertices, and the roads between those towns, represented by edges. The numbers represent the lengths of each road, in kilometres.

a How far is it from Stratmoore directly to Osburn?

b How far is it from Stratmoore to Kenton, via Melville then Osburn?



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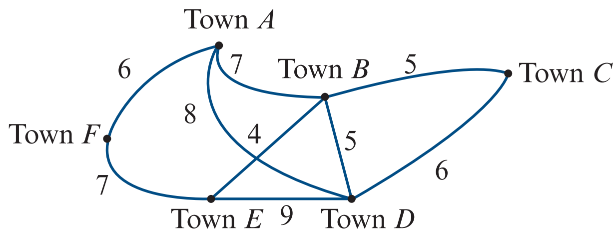
The shortest path problem

Another question we might have when presented with a road network like the one shown is, 'What is the shortest distance between certain towns?'

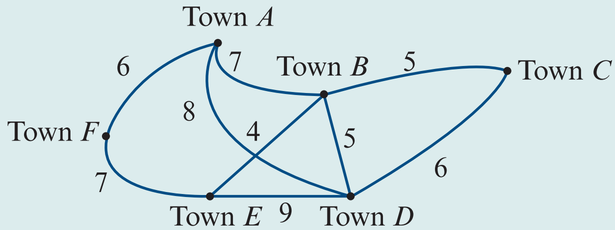


While this question is easily answered if all of the towns are directly connected by a road, for example, Town A and Town B, the answer is not so obvious if we have to travel through other towns to get there, for example, Town F and Town C.

While there are sophisticated techniques for solving the shortest path problem (which are presented in Units 3 and 4), we will identify and compare the lengths of the likely candidates for the shortest path by inspection.

Example 13 Finding the shortest path by inspection

Find the shortest route between Town C and Town F in the network shown opposite.



Explanation	Solution
1 Identify all of the likely shortest routes between Town C and Town F and calculate their lengths.	C-D-E-F: The distance is $6 + 9 + 7 = 22$ km. C-B-E-F: The distance is $5 + 4 + 7 = 16$ km. C-B-A-F: The distance is $5 + 7 + 6 = 18$ km.

Note: In theory, when using the 'by inspection' method to solve this problem, we need to list all possible routes between Town C and Town F and determine their lengths. However, we can save time by eliminating any route that passes through any town more than once or uses any road more than once. We can also eliminate any route that 'takes the long way around' rather than using the direct route, for example, when travelling from Town B to Town D, we can ignore the route that goes via Town A because it is longer.

Explanation	Solution
2 Compare the different path lengths to identify the shortest path and write your answer. Note: You should compare the lengths of all likely paths because there can be more than one shortest path in a network.	The shortest path is C-B-E-F.

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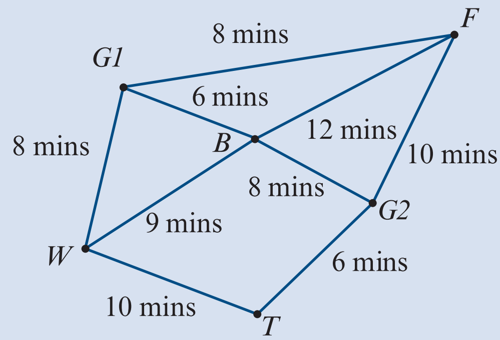
Now try this 13 Finding the shortest path by inspection (Example 13)

Find the shortest route between the waterfall (W) and the fern gully (F) in the network shown on the right.



Section Summary

- A **weighted graph** is a graph where a number is associated with each edge. These numbers are called weights.
- A **network** is a weighted graph in which the weights are physical quantities, for example, distance, time and cost.
- Determining the shortest distance or time or the least cost to move around a network is called the **shortest path problem**.

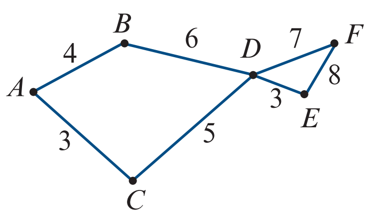
Skill-sheet

Exercise 8G

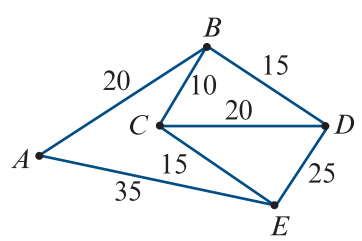
Building understanding

Example 13

1 Find the shortest path from vertex A to vertex E in this network. The numbers represent time, in hours.



2 Find the shortest path from vertex A to vertex D in this network. The numbers represent lengths, in metres.



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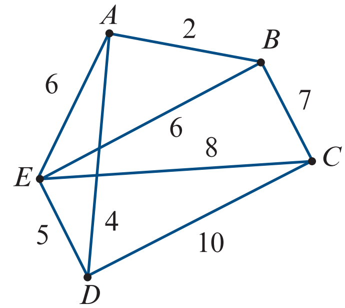
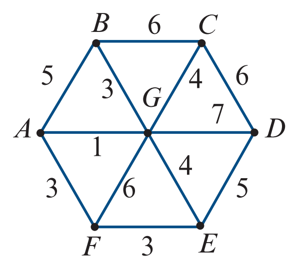
3 Find the shortest path from vertex *B* to vertex *D* in this network. The numbers represent cost, in dollars.



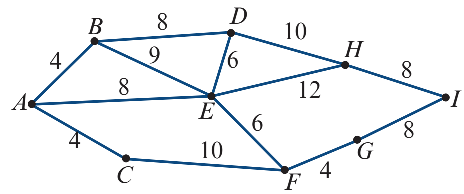
4 Find the shortest path from vertex *B* to vertex *F* in this network. The numbers represent time, in minutes.



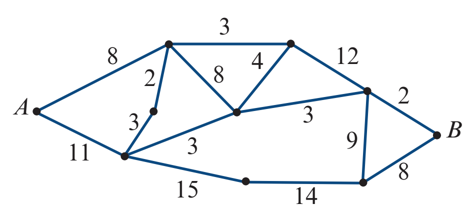
Developing understanding

Example 12

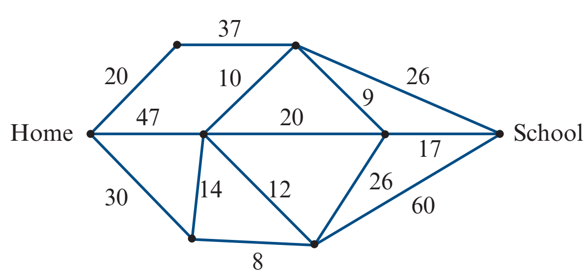
5 The network opposite shows the distance, in kilometres, along walkways that connect the landmarks *A, B, C, D, E, F, G, H* and *I* in a national park. Find the shortest path from *A* to *I*.



6 In the network opposite, the vertices represent small towns and the edges represent roads. The weights on the edges indicate the distance, in km, between towns. Determine the length of the shortest path between towns *A* and *B*.



7 Victoria rides her bike to school each day. The edges of the network opposite represent the roads that Victoria can use to ride to school. The numbers on the edges give the time taken, in minutes, to travel along each road. What is the shortest time that Victoria can ride between home and school?



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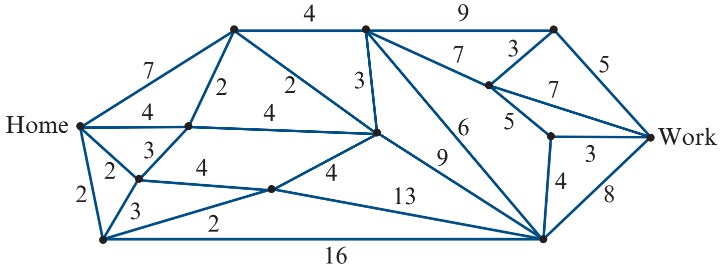
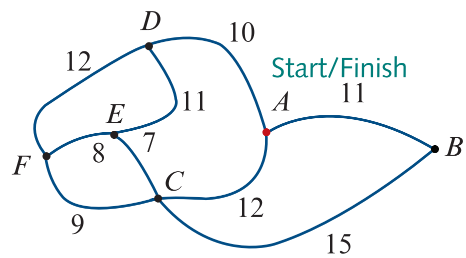
8 Renee drives to work each day. The edges of the network below represent the roads that Renee can use to drive to work. The numbers on the edges give the time, in minutes, to travel along each road. What is the shortest time that Renee can drive between home and work?



9 The graph opposite shows a mountain bike rally course. Competitors must pass through each of the checkpoints, *A*, *B*, *C*, *D*, *E* and *F*. The average times, in minutes, taken to ride between the checkpoints are shown on the edges of the graph.

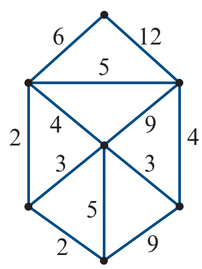


Competitors must start and finish at checkpoint *A* but can pass through the other checkpoints in any order they wish.
Which route would have the shortest average completion time?

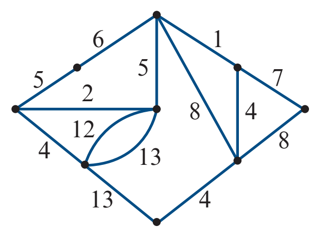
Testing understanding

10 For each of these graphs, determine the length of the shortest Hamiltonian path.

a



b



11 Is an Eulerian trail possible for each of the graphs above? Why is it unnecessary to determine the length of the shortest Eulerian trail?

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8H Minimum spanning trees and greedy algorithms

Learning intentions

- To be able to identify a tree.
- To be able to find a spanning tree for a graph.
- To be able to find the minimum spanning tree for a weighted graph using greedy algorithms such as Prim's or Kruskal's.
- To be able to apply a greedy algorithm to find the shortest path in a network from one vertex to another.

In the previous applications of networks, the weights on the edges of the graph were used to determine a minimum weight pathway through the graph from one vertex to another. In other applications, it is more important to minimise the number and weights of the edges in order to keep all vertices connected to the graph. For example, a number of towns might need to be connected to a water supply. The cost of connecting the towns can be minimised by connecting each town into a network of water pipes only once, rather than connecting each town to every other town. Problems of this type can be solved with the use of a **tree**.

Trees

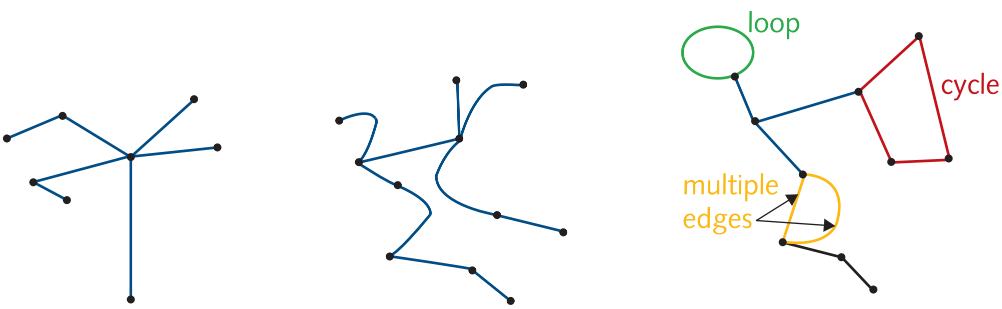
Tree

A **tree** is a connected graph that contains no cycles, multiple edges or loops.

A tree may be part of a larger graph.

If a tree has n vertices, it will have $n - 1$ edges.

For example, Graphs 1 and 2 below are examples of trees. Graph 3 is *not* a tree.



Graph 1: a tree Graph 2: a tree Graph 3: not a tree

Graphs 1 and 2 are trees: they are connected and have no cycles, multiple edges or loops.

Graph 3 is *not* a tree because it has several cycles (loops and multiple edges count as cycles).

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For trees, there is a relationship between the number of vertices and the number of edges.

- In Graph 1, the tree has 8 vertices and 7 edges.
- In Graph 2, the tree has 11 vertices and 10 edges.

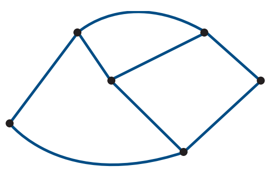
An inspection of other trees would show that, in general, the number of edges in a tree is one less than the number of vertices. A tree with n vertices has $n - 1$ edges.

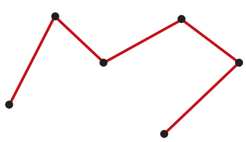
Spanning trees

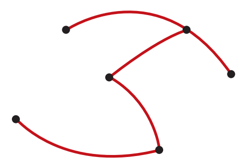
Spanning tree

A **spanning tree** is a tree that connects *all* of the vertices in a connected graph.

For example, Graphs 2 and 3 below are different spanning trees of Graph 1. Note that the spanning trees connect all of the vertices from Graph 1; each spanning tree has 6 vertices and 5 edges.


Graph 1


Graph 2


Graph 3

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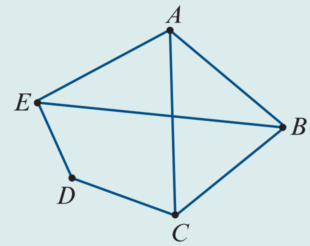
Example 14 Finding a spanning tree in a graph

Find two spanning trees for the graph shown opposite.



Explanation

1. The graph has five vertices and seven edges. A spanning tree will have five (n) vertices and four ($n - 1$) edges.

2. To form a spanning tree, remove any *three* edges, provided that:
   - all the vertices remain connected
   - there are no multiple edges or loops.

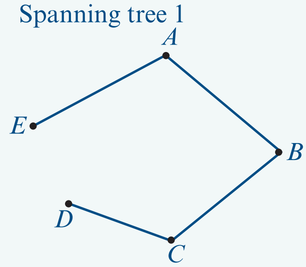
   Spanning tree 1 is formed by removing edges *EB*, *ED* and *CA*.

   Spanning tree 2 is formed by removing edges *EA*, *AC* and *CD*.

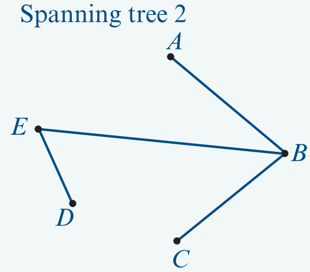
   Note: Several other possibilities exist.

Solution

Spanning tree 1



Spanning tree 2

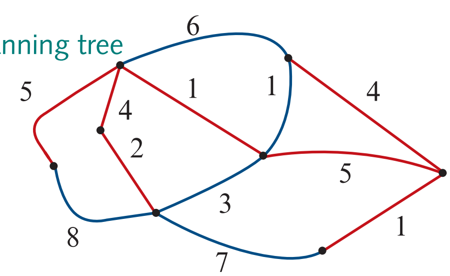


Minimum spanning trees

For weighted graphs or networks, it is possible to determine the 'length' of each spanning tree by adding up the weights of the edges in the tree.

For the spanning tree (highlighted in red) opposite:

$$\text{Length} = 5 + 4 + 2 + 1 + 5 + 4 + 1$$
$$= 22 \text{ units}$$



Minimum spanning tree

A **minimum spanning tree** is a spanning tree of minimum length. There may be more than one.

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A minimum spanning tree may represent the minimum distance, minimum time, minimum cost, etc. There may be more than one minimum spanning tree in a weighted graph.

Minimum spanning trees have many real-world applications, such as planning the layout of a computer network or a water supply system for a new housing estate. In these situations, we might want to minimise the amount of cable or water pipe needed for the job. Alternatively, we might want to minimise the time needed to complete the job or its cost.

Algorithmic methods for determining minimum spanning trees

To date, we have used inspection to identify minimum spanning trees in a weighted graph. While this is practical for simple weighted graphs, it is less appropriate when solving practical problems which are likely to involve more complex weighted graphs. A more systematic or algorithmic approach is needed.

An algorithm is a set of instructions or rules to follow for solving a problem or performing a complex task. In the everyday world, a recipe for baking a cake is an algorithm. In arithmetic, it could be the series of steps we can follow to reliably perform a long division between two multi-digit numbers by hand.

In the world of graphs and networks, an algorithm is a series of steps you can follow to enable you to evaluate some property of a graph or network. In this topic, we are interested in algorithms that will enable us to identify a minimum spanning tree in a network and determine its length in a systematic and routine manner.

There are several algorithms that have been developed for this purpose. We will consider two: Prim's algorithm and Kruskal's algorithm, which are the most used algorithms for this task.

Prim's algorithm

Prim's algorithm for finding a minimum spanning tree

Prim's algorithm is a set of rules to determine a minimum spanning tree for a graph.

1. Choose a starting vertex (any will do).
2. Inspect the edges, starting from this vertex, and choose the one with the lowest weight. (If there are two edges that have the same weight, it does not matter which one you choose.) The starting vertex, the edge and the vertex it connects to form the beginning of the minimum spanning tree.
3. Next, inspect all the edges, starting from both of the vertices you have in the tree so far. Choose the edge with the lowest weight, ignoring edges that would connect the tree back to itself. The vertices and edges you already have, plus the extra edge and vertex it connects to, form the minimum spanning tree so far.
4. Repeat the process until all the vertices are connected.

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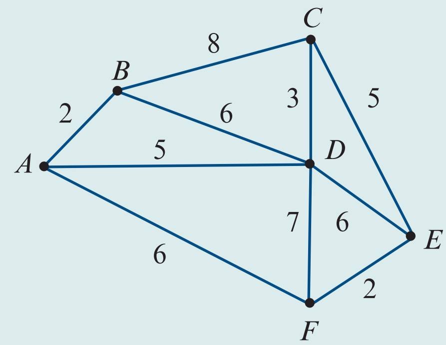
Example 15 Finding the minimum spanning tree by applying Prim's algorithm

Apply Prim's algorithm to find the minimum spanning tree for the graph shown on the right. Write down the total weight of the minimum spanning tree.



Explanation

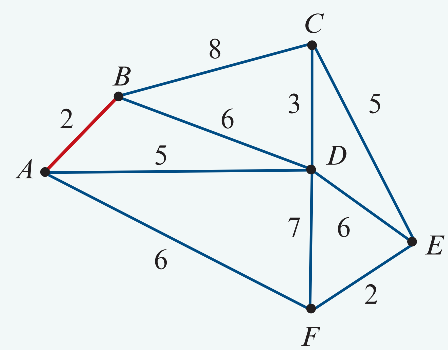
1 Start with vertex A.

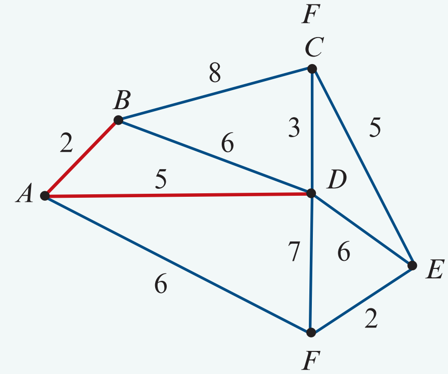
The smallest weighted edge from vertex *A* is to *B*, with weight 2.

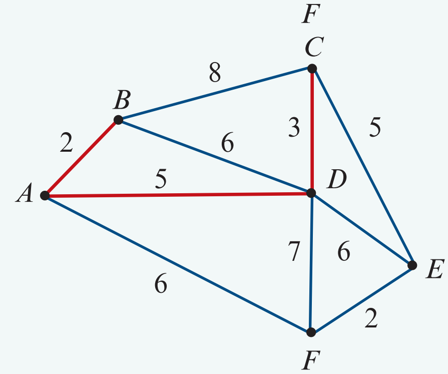
2 Look at vertices *A* and *B*. The smallest weighted edge from either vertex *A* or vertex *B* is from *A* to *D*, with weight 5.

3 Look at vertices *A*, *B* and *D*. The smallest weighted edge from vertix *A*, *B* or *D* is from *D* to *C*, with weight 3.

Solution







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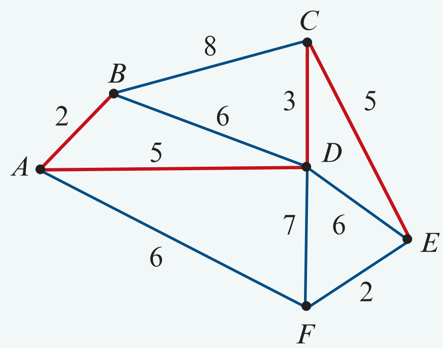
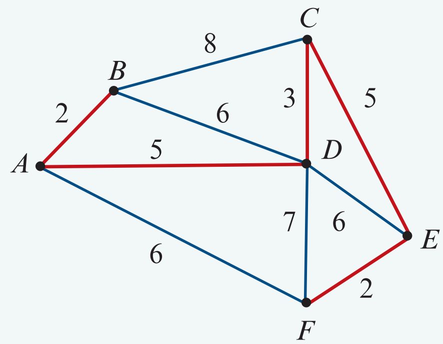
4 Look at vertices A, B, D and C. The smallest weighted edge from vertex A, B, D or C is from C to E, with weight 5.



5 Look at vertices A, B, D, C and E. The smallest weighted edge from either vertex A, B, D, C or E is from E to F, with weight 2.
All vertices have been included in the graph. This is the minimum spanning tree.

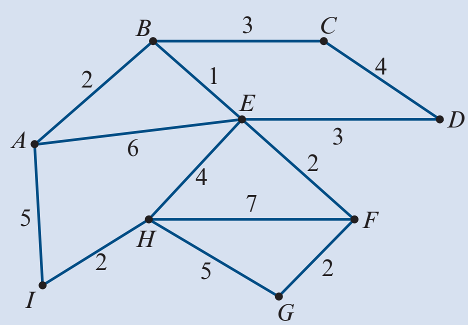


6 Add the weights to find the total weight of the minimum spanning tree.

The total weight of the minimum spanning tree = 2 + 5 + 3 + 5 + 2 = 17.

Now try this 15 **Finding the minimum spanning tree by applying Prim's algorithm (Examples 14 and 15)**

Apply Prim's algorithm to obtain a minimum spanning tree for the graph shown, and calculate its length.



Kruskal's algorithm

Kruskal's algorithm is another algorithm that can be used to determine a minimum spanning tree for a network.

Prim's algorithm starts from a vertex and builds up the spanning tree step-by-step from one vertex to an adjacent vertex, to eventually form a minimum spanning tree for the network.

By contrast, Kruskal's algorithm sorts all the edges from lowest to highest weight and keeps adding the edges of the same or next lowest weight (that do not form cycles) until all the vertices have been covered.

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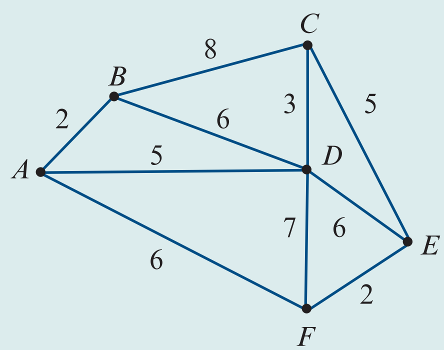
Kruskal's algorithm for finding a minimum spanning tree

1. Choose the edge with the least weight as the starting edge. If there is more than one least-weight edge, any will do.
2. Next, from the remaining edges, choose an edge of least weight which does not form a cycle. If there is more than one least-weight edge, any will do.
3. Repeat the process until all vertices are connected. The result is a minimum spanning tree.
4. Determine the length of the spanning tree by summing the weights of the chosen vertices.

Example 16 Kruskal's algorithm for determining a minimum spanning tree

Apply Kruskal's algorithm to obtain a minimum spanning tree for the graph shown, and calculate its length.

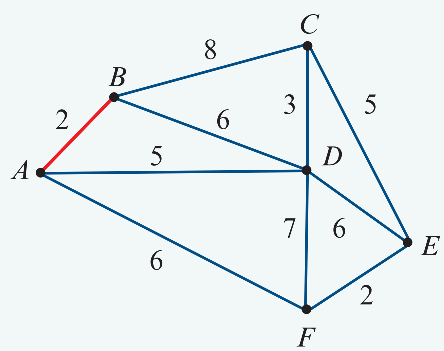


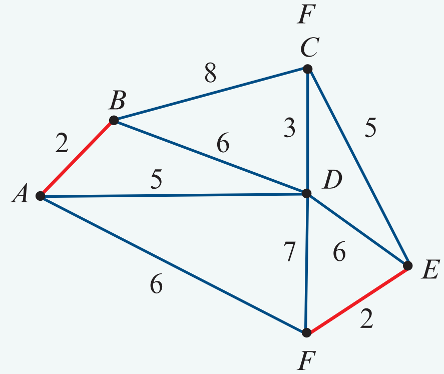
Explanation

1. Choose the edge with the least weight. If there is more than one, it doesn't matter which you choose. There are two edges of weight 2. We will choose edge *AB*. Draw it in as shown in red, opposite.

2. From the remaining edges, choose the edge with the least weight. There is one, edge *FE*, with weight 2. Draw *FE* in, as shown opposite.

Solution





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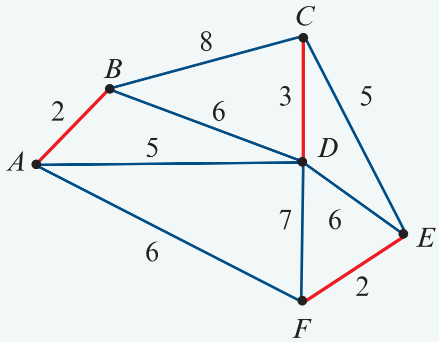
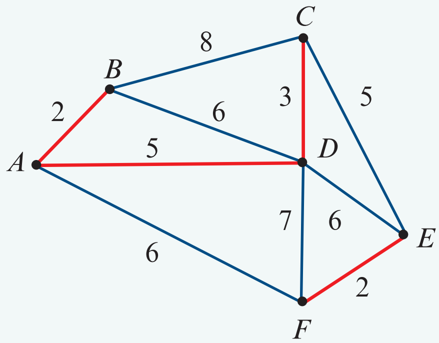
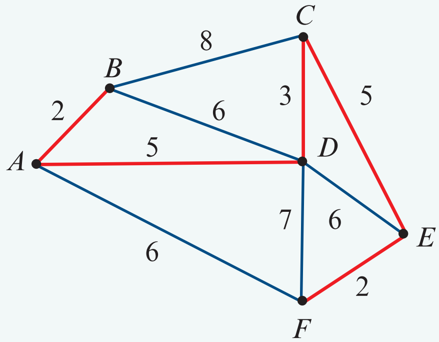
3 Continuing the process, the next edge we choose is *CD*, with a weight of 3. Draw *CD* in, as shown opposite.



4 Continuing the process, there are two edges of weight 5. Neither form a cycle so it does not matter which of these we choose. We will choose edge *AD*. Draw *AD* in, as shown opposite.



5 We can now add in the remaining edge of weight 5, *CE*. Draw *CE* in, as shown opposite.
All vertices have now been joined and a minimum spanning tree has been determined.

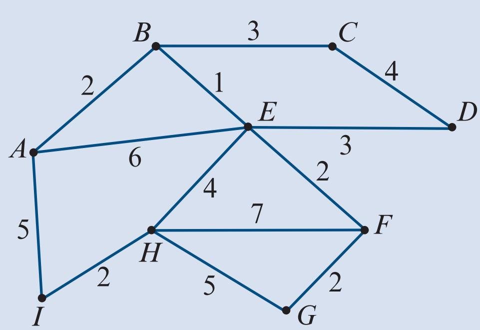


6 Find the length of the minimum spanning tree by adding the weights of the chosen edges.

Minimum spanning tree
Length = 2 + 5 + 3 + 5 + 2 = 17

Now try this 16 **Kruskal's algorithm for determining a minimum spanning tree (Example 16)**

Apply Kruskal's algorithm to obtain a minimum spanning tree for the graph shown, and calculate its length



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Greedy algorithms

Prim's and Kruskal's algorithms are examples of greedy algorithms.

What is a greedy algorithm?

A **greedy algorithm** is a simple, intuitive set of rules that can be used to solve optimisation problems. It breaks up the solution of the optimisation problem into a series of simple steps that find the optimum solution at each step in the process. The expectation is that finding the optimum solution at each step in the solution will lead to the optimum solution for the entire problem.

Because a greedy algorithm must consider all the directly available information at each step in the process, it can be very time consuming compared to some alternative but less intuitive methods that do not have this requirement. This is particularly true when dealing with large and complex networks, but these situations are beyond the scope of this course.

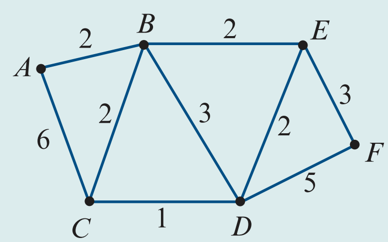
Another problem with using greedy algorithms to solve optimisation problems is that the methodology does not always lead to the optimal solution for all optimisation problems. However, this is not the case for Prim's and Kruskal's algorithms. A greedy algorithm can be used to identify the shortest path from one vertex to another.

Dijkstra's greedy algorithm for finding the shortest path (Extension)

Next year, you will learn to use Dijkstra's algorithm to find the shortest path between two points in a network. It is also a greedy algorithm.

Example 17 **Dijkstra's algorithm for determining the shortest path**

Find the shortest path from A to F in the weighted graph shown on the right.

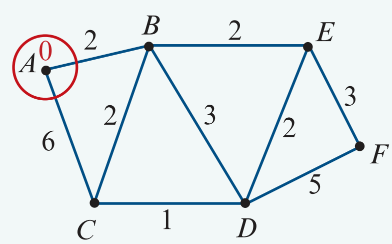


Explanation

1 Assign the starting vertex a zero, and circle the vertex and its new value of zero.

Solution

A is the starting vertex; it is assigned zero and it is circled.



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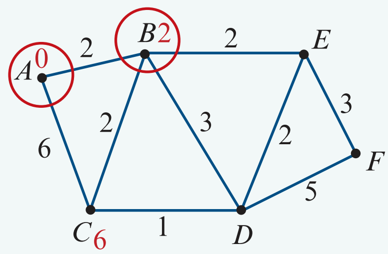
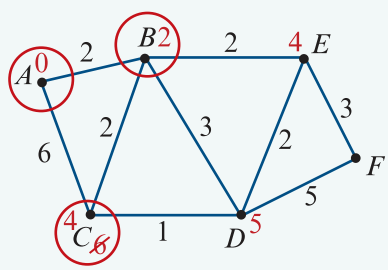
2 Assign a value to each vertex connected to the starting vertex. The value assigned is the length of the edge connecting it to the starting vertex. Circle the vertex with the lowest assigned value.

The starting vertex, A, is connected to vertices B and C. The vertex B is assigned 2 and the vertex C is assigned 6. Vertex B is circled because it has the lowest value.



3 From the newly circled vertex, assign a value to each vertex connected to it by *adding* the value of each connecting edge to the newly circled vertex's value. If a connecting vertex already has a value assigned to it, and the new value is less than it, replace it with the new value. If a vertex is circled, it cannot have its value changed. Consider all uncircled vertices and circle the one with the lowest value.

The newly circled vertex, B, is connected to three vertices; C, D and E. Starting with vertex B's value of 2, E is assigned 4 (adding 2 from the connecting edge) and D is assigned 5 (adding 3 from the connecting edge). The vertex C will be re-assigned 4 (adding 2 from the connecting edge) because it is lower than 6. Now there are two uncircled vertices with the lowest assigned value of 4, vertices C and E; it **does not** matter which one you circle.



Continued

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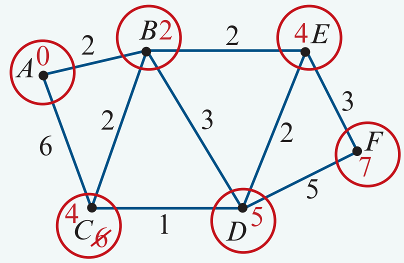
4 Repeat Step 3 until the destination vertex and its assigned value are circled. The length of the shortest path will be the assigned value of the destination vertex. The shortest path is found by backtracking. Starting at the destination vertex, move to the circled vertex whose value is equal to the destination vertex's assigned value minus the connecting edge value. Continue to minus the connecting edge value from one circled vertex to the next until you reach the starting vertex.

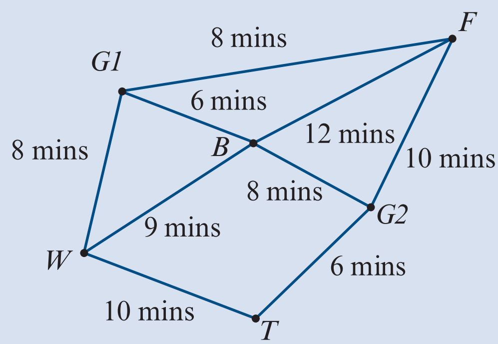
Note: Once a vertex is assigned a value, it cannot be assigned a larger value, even if it has not been circled yet. You do not need to circle all vertices. Stop when the destination vertex is circled.

Vertex F is the destination vertex, assigned a value of 7. Therefore the shortest path from A to F has a length of 7. To find the shortest path, start at F and consider the two connecting edges to it. The edge of length 3 is correct because 7 minus 3 equals 4, the value of vertex E. Likewise, minus the connecting edge of 2 to vertex B to equal 2, then minus the 2 to the connecting edge with A to equal zero. Therefore, the shortest path from A to F is: A-B-E-F, with a length of 7.



Now try this 17 Dijkstra's algorithm for determining the shortest path (Example 17)

Find the shortest route between the waterfall *(W)* and the fern gully *(F)* in the network shown on the right.



Section Summary

- A **tree** is a connected graph that contains no cycles, multiple edges or loops. A tree may be part of a larger graph.
- A tree with n vertices has $n - 1$ edges.
- A **spanning tree** is a tree that connects *all* of the vertices in a connected graph.
- A **minimum spanning tree** is the spanning tree of *minimum* length.
- **Prim's and Kruskal's algorithms** are different types of **greedy algorithms.** They give a set of rules to determine a minimum spanning tree for a graph.

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Skill-sheet

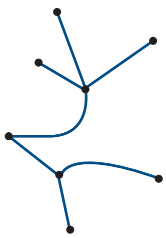
Exercise 8H

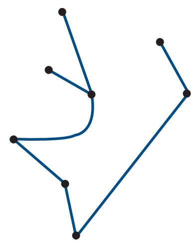
Building understanding

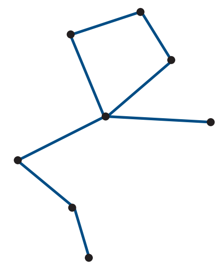
1 **a** How many edges are there in a tree with 15 vertices?
b How many vertices are there in a tree with 5 edges?
c Draw two different trees with four vertices.
d Draw three different trees with five vertices.

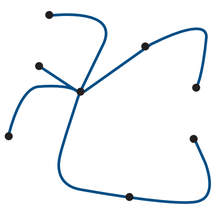
2 A connected graph has eight vertices and ten edges. Its spanning tree has ☐ vertices and ☐ edges.

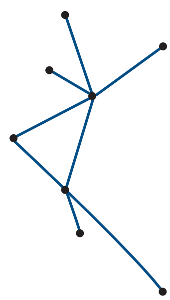
3 Which of the following graphs are trees?

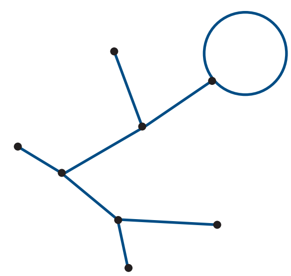
A


B


C


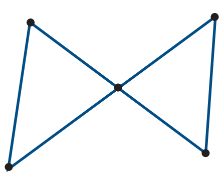
D


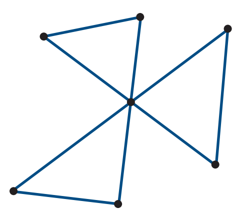
E


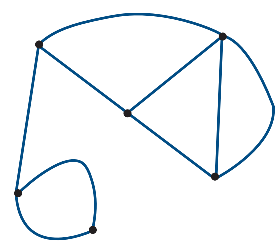
F


Example 14

4 For each of the following graphs, draw three different spanning trees.

a


b


c


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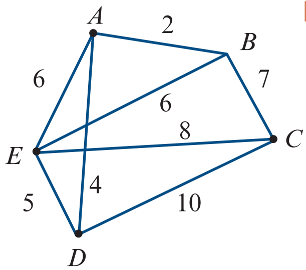
Developing understanding

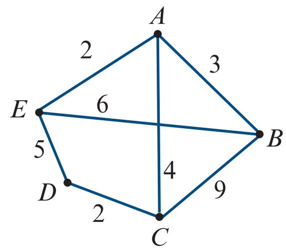
Example 15

5 For each of the following connected graphs, use Prim's algorithm to determine the minimum spanning tree and its length.

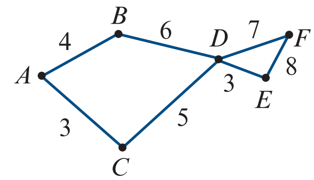
a



b



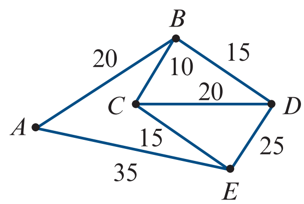
c



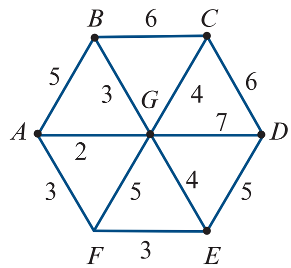
Example 16

6 For each of the following connected graphs, use Kruskal's algorithm to determine the minimum spanning tree and its length.

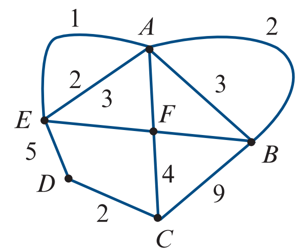
a



b



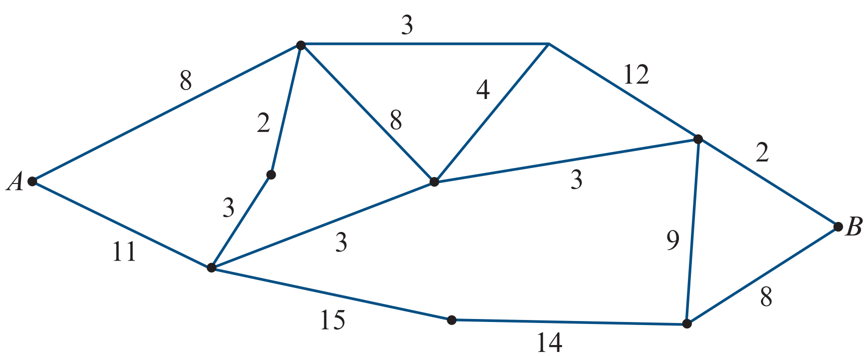
c



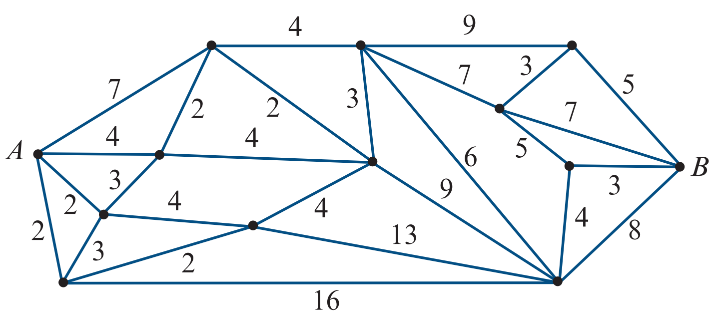
Example 17

7 Using a greedy algorithm, find the shortest path from A to B in each of the following.

a



b



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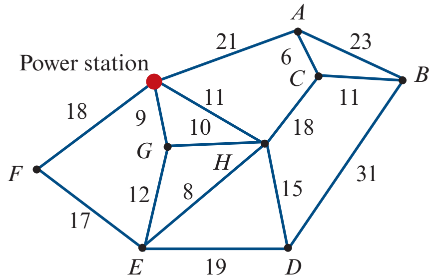
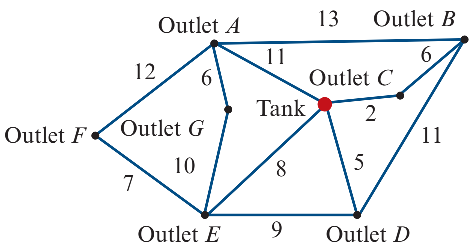
8 Power is to be connected by cable from a power station to eight substations (*A* to *H*). The distances (in kilometres) of the substations from the power station and from each other are shown in the network opposite. Determine the minimum length of cable needed.



9 Water is to be piped from a water tank to seven outlets on a property. The distances (in metres) of the outlets from the tank and from each other are shown in the network below.



a Starting at the tank, determine the minimum length of pipe needed.

b Due to high demand, the pipe connecting outlet *A* to outlet *F* and the pipe connecting outlet *A* to outlet *B* must always be in use. Given these new conditions, how much longer will the new minimum length of pipe be?

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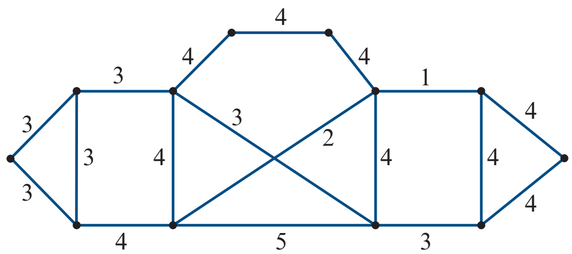
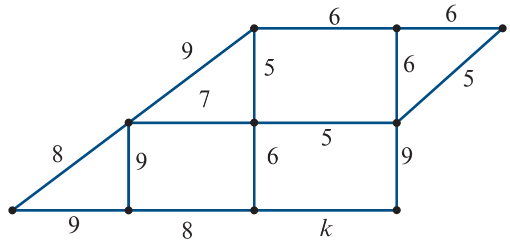
Testing understanding

10 A minimum spanning tree is to be drawn for the weighted graph below.



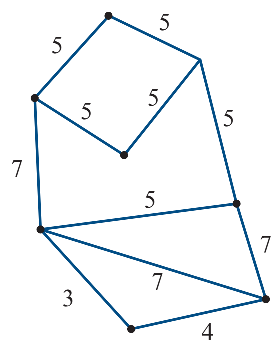
How many edges with weight 4 will **not** be included in the minimum spanning tree?

11 The minimum spanning tree for the graph below includes the edge with weight labelled *k*.



The total weight of all edges for the minimum spanning tree is 58. Find the value of *k*.

12 Consider the weighted graph below. How many different minimum spanning trees are possible?



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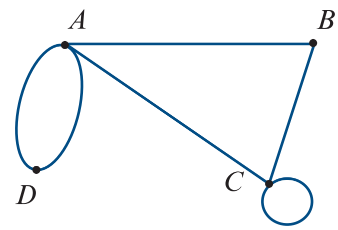
Key ideas and chapter summary

Assignment

Graph or network

A **graph** is a diagram that consists of a set of points called **vertices** and a set of lines called **edges**. *C* has one edge which links *C* to itself. This edge is called a **loop**.



Vertices and edges

In the graph above, *A, B, C,* and *D* are **vertices** and the lines *AB, AD, AC,* and *BC* are **edges**.

Degree of a vertex

The **degree of vertex** *A*, written deg(*A*), is the *number of edges attached to the vertex*. A loop will contribute to the degree of a vertex twice.

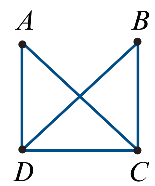
For example, in the graph above: deg(*B*) = 2 and deg(*C*) = 4.

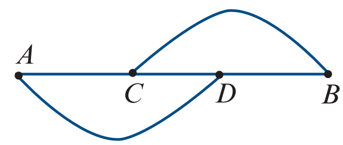
Isomorphic graphs

Two graphs are said to be **isomorphic** (equivalent) if:

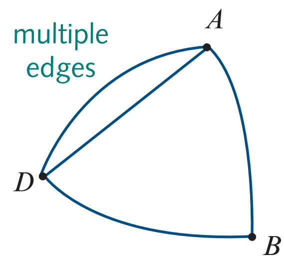
- they both have the same number of edges and vertices
- corresponding vertices have the same degree and the edges connect to the same corresponding vertices.

For example, the two graphs below are isomorphic (or equivalent).





Multiple edges



The graph above is said to have **multiple edges**, as there are two edges joining *A* and *D*.

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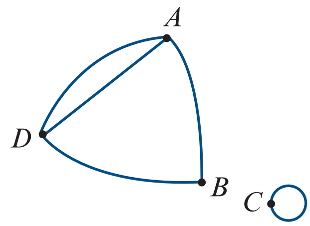
Review

Adjacency matrix

An **adjacency matrix** is a square matrix that uses a zero or positive integer to record the number of edges connecting each pair of vertices in the graph.

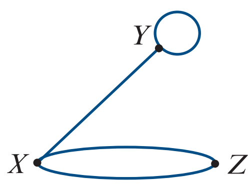
An example of a graph and its adjacency matrix is shown below.



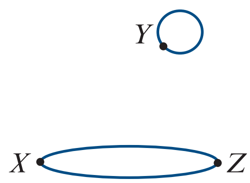
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Connected graph and bridges

A **graph** is **connected** if there is a path between each pair of vertices.
A **bridge** is a single edge in a connected graph that, if removed, leaves the graph disconnected. A graph can have more than one bridge.
Graph 1 is a connected graph.
Graph 2 is not a connected graph.
Edge XY in Graph 1 is a bridge because removing it leaves Graph 1 disconnected.



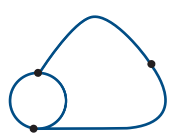
Graph 1

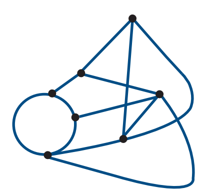


Graph 2

Planar graph

A graph that can be drawn in such a way that no two edges intersect, except at the vertices, is called a **planar graph**.


planar graph


non-planar graph

Euler's formula

For any **connected planar** graph, **Euler's formula** states:

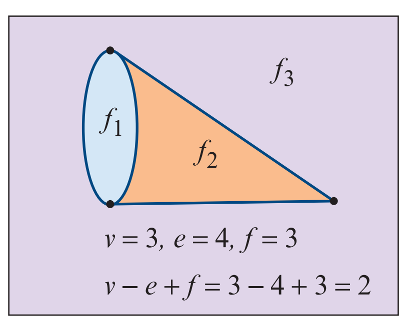
$v + f = e + 2$

v = the number of vertices

e = the number of edges

f = the number of faces.

and also, $v - e + f = 2$



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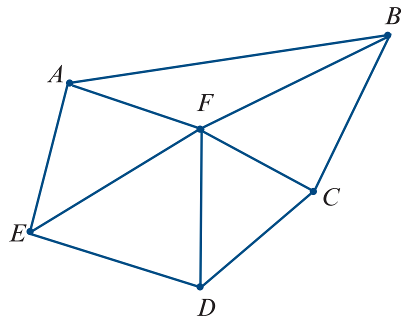
Walk, trail, path, circuit and cycle



A **walk** is a sequence of edges linking successive vertices in a graph.
In the graph above, *E-A-F-D-C-F-E-A* is a walk.
A **trail** is a walk with no repeated edges.
In the graph, *A-F-D-E-F-C* is a trail.
A **circuit** is a walk that has no repeated edges that starts and ends at the same vertex.
In the graph, *A-F-D-E-F-B-A* is a circuit.
A **path** is a walk with no repeated vertices.
In the graph, *F-A-B-C-D* is a path.
A **cycle** is a walk with no repeated vertices that starts and ends at the same vertex.
In the graph, *B-F-C-B* is a cycle.

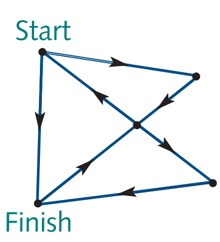
Eulerian trail

A trail that includes every edge just once (but does not start and finish at the same vertex) is called an **Eulerian trail**.

Condition for an Eulerian trail

To have an Eulerian trail (but not an Eulerian circuit), a graph must be connected and have exactly zero or two odd vertices, with the remaining vertices being even.

For example, the graph opposite is connected.
It has two odd vertices and three even vertices.
It has an Eulerian trail that starts at one of the odd vertices and finishes at the other.



Eulerian circuit

An Eulerian trail that starts and finishes at the same vertex is called an **Eulerian circuit**.

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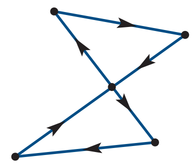
Review

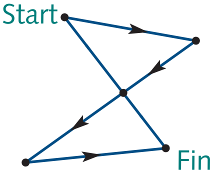
Condition for an Eulerian circuit

To have an **Eulerian circuit**, a graph must be connected, and all vertices must be even.
In the network shown, all vertices are even. It has an Eulerian circuit.
The circuit starts and finishes at the same vertex.



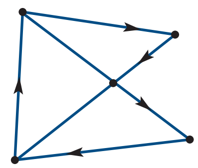
Hamiltonian path

A **Hamiltonian path** is a path through a graph that passes through each vertex exactly once but does not necessarily start and finish at the same vertex.



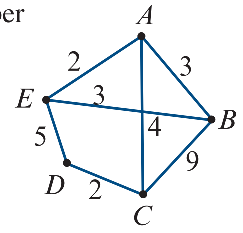
Hamiltonian cycle

A **Hamiltonian cycle** is a Hamiltonian path that starts and finishes at the same vertex.



Weighted graphs and networks

A **weighted graph** is one where a number is associated with each edge. These numbers are called weights. When the weights are physical quantities, for example, distance, time or cost, a weighted graph is often called a **network**.

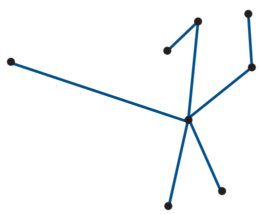


The shortest path problem

Determining the shortest distance or time or the least cost to move around a network is called the **shortest path problem**.

Tree

A **tree** is a connected graph that contains no cycles, multiple edges or loops.
A tree with n vertices has $n - 1$ edges.
The tree (right) has 8 vertices and 7 edges.



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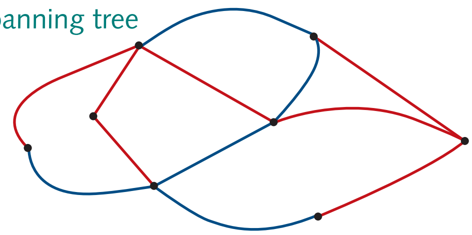
Spanning tree

A **spanning tree** of a graph is a subgraph that contains all the vertices of a connected graph, without multiple edges, cycles or loops.



Minimum spanning tree

A **minimum spanning tree** is a spanning tree for which the sum of the weights of the edges is as small as possible.

Greedy algorithm

A **greedy algorithm** is a simple, intuitive set of rules that can be used to solve optimisation problems. The expectation is that finding the optimum solution at each step in the solution will lead to the optimum solution for the entire problem.

Prim's algorithm

Prim's algorithm is a systematic method for determining a minimum spanning tree in a connected graph. It can start at any vertex. Prim's algorithm is a greedy algorithm.

Kruskal's algorithm

Kruskal's algorithm is a systematic method for determining a minimum spanning tree in a connected graph. It starts at the edge with the least weight. Kruskal's algorithm is a greedy algorithm.

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Review

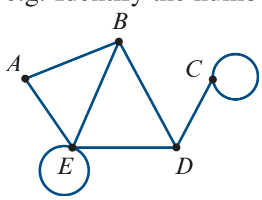
Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

8A **1** I can identify the number of vertices, edges and loops of a graph. ☐

e.g. Identify the number of vertices, edges and loops in the graph below.



8A **2** I can determine the degree of a vertex and the sum of degrees of a graph. ☐

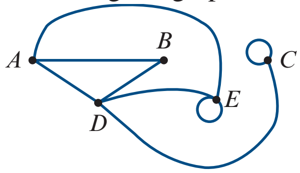
e.g. Determine the degree of each vertex in the graph above and the sum of degrees of the graph.

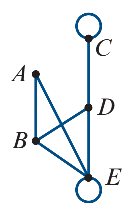
8B **3** I can identify a connected graph and a bridge. ☐

e.g. Give a reason why the graph above would be considered connected. Are there any bridges present? If yes, identify the two vertices the bridge(s) exist between.

8B **4** I can identify isomorphic graphs. ☐

e.g. Identify whether both, one or none of the following graphs are isomorphic to the original graph above.





8B **5** I can use an adjacency matrix to represent a graph. ☐

e.g. Complete the following matrix to represent the original graph above.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \end{array}$$

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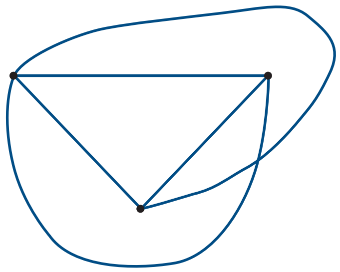
8C **6** I can classify a graph as planar and use isomorphic graphs to help identify them. ☐

e.g. Is the following graph planar? Justify your reasoning.



8C **7** I can identify the number of faces of a graph. ☐

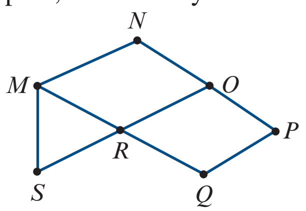
e.g. Determine the number of faces for the graph above.

8C **8** I can verify Euler's formula. ☐

e.g. Verify Euler's formula for the graph above.

8D **9** I can classify a walk as either a trail, path, circuit or cycle. ☐

e.g. For the graph below, is the walk S-R-Q-P-O-N-M-S considered a trail, path, circuit or cycle?



8E **10** I can identify an Eulerian trail. ☐

e.g. For the graph in Question **9**, find an Eulerian trail.

8E **11** I can identify an Eulerian circuit. ☐

e.g. For the graph in Question **9**, an Eulerian circuit does not exist. Give a reason why this graph does not contain one, referring to the degrees of vertices.

8F **12** I can identify a Hamiltonian path. ☐

e.g. For the graph in Question **9**, find a Hamiltonian path.

8F **13** I can identify a Hamiltonian cycle. ☐

e.g. For the graph in Question **9**, find a Hamiltonian cycle.

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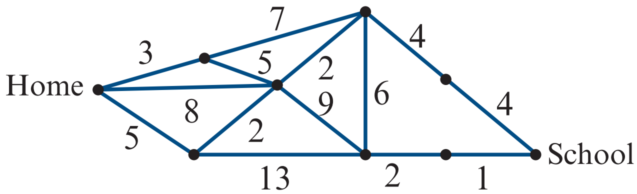
Review

8G **14** I can find the shortest path of a weighted network. ☐

e.g. For the graph below, find the length of the shortest path from Home to School.

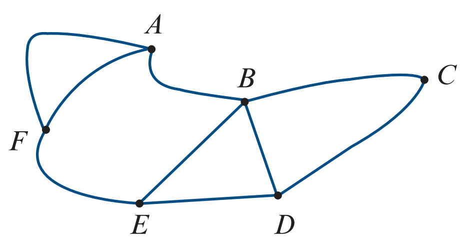


8H **15** I can apply a greedy algorithm such as Prim's or Kruskal's algorithm to find a minimum spanning tree in a network. ☐

e.g. For the graph above, find the minimum spanning tree and state its length.

Multiple-choice questions

The following graph relates to Questions 1 to 4.



1 The number of vertices in the graph above is:

A 3 **B** 5 **C** 6 **D** 7 **E** 9

2 The number of edges in the graph above is:

A 3 **B** 5 **C** 6 **D** 7 **E** 9

3 The degree of vertex B in the graph above is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

4 The number of even vertices in the graph above is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

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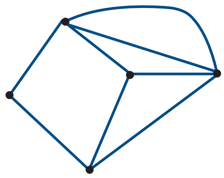
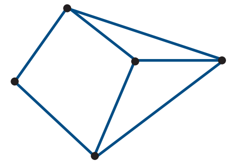
5 For which graph below is the sum of the degrees of the vertices equal to 14?

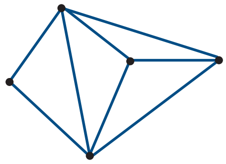
A



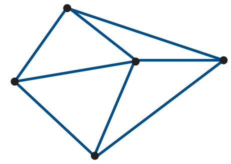
B



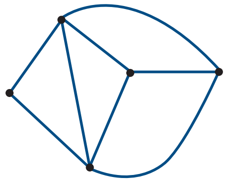
C



D



E

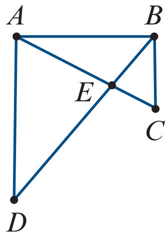


6 The graph that matches the matrix

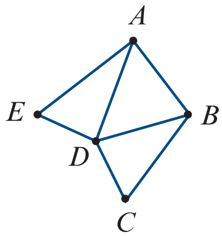
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

is:

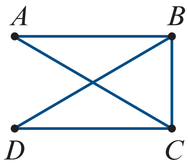
A



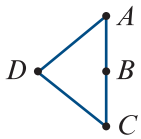
B



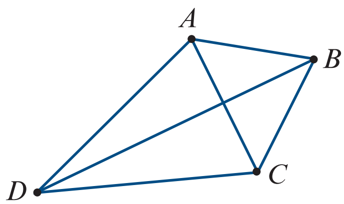
C



D



E



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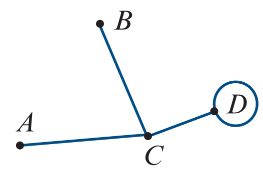
7 The adjacency matrix that matches the graph shown is:



A
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

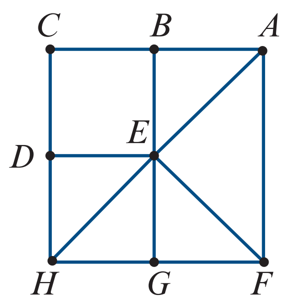
B
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

C
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{array}$$

D
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

E
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

The graph below is to be used when answering Questions 8 to 11.



8 The sequence of vertices *C-B-E-A-E-G* represents:

A a walk only
B a trail
C a path
D a circuit
E a cycle

9 The sequence of vertices *D-E-H-G-E-A-B-C-D* represents:

A a walk only
B a trail but not a circuit
C a path but not a cycle
D a circuit
E a cycle

10 The sequence of vertices *C-B-E-A-F-E-G-H* represents:

A a walk only
B a trail but not a circuit
C a path but not a cycle
D a circuit
E a cycle

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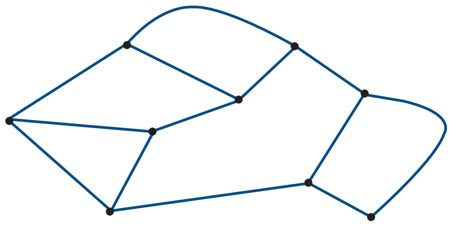
11 The sequence of vertices *D-E-A-F-G-H-D* represents:

A a walk only
B a trail but not a circuit
C a path but not a cycle
D a cycle
E none of these

The graph below is to be used when answering Questions 12 and 13.



12 The sum of the degrees of the vertices is:

A 22 **B** 23 **C** 24 **D** 25 **E** 26

13 For this graph:

A $v = 9, e = 13, f = 5$
B $v = 9, e = 13, f = 6$
C $v = 10, e = 13, f = 5$
D $v = 9, e = 14, f = 6$
E $v = 11, e = 13, f = 5$

14 Euler's formula for a planar graph is:

A $v - e = f + 2$
B $v - e + f = 2$
C $v + e + f = 2$
D $e - v + f = 2$
E $v - e = f - 2$

15 A connected graph with 10 vertices divides the plane into five faces. The number of edges connecting the vertices in this graph will be:

A 5
B 7
C 10
D 13
E 15

16 For a connected graph to have an Eulerian trail but not an Eulerian circuit:

A all vertices must be odd
B all vertices must be even
C there must be exactly two odd vertices and the rest even
D there must be exactly two even vertices and the rest odd
E an odd vertex must be followed by an even vertex

17 For a connected graph to have an Eulerian circuit:

A all vertices must be odd
B all vertices must be even
C there must be exactly two odd vertices and the rest even
D there must be exactly two even vertices and the rest odd
E an odd vertex must be followed by an even vertex

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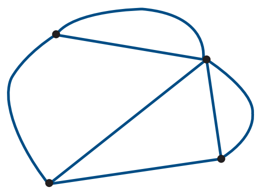
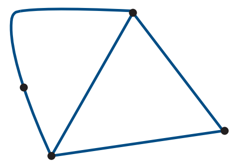
The graphs below are to be used when answering Questions 18 and 19.

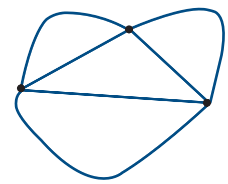
A



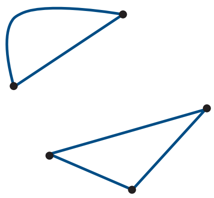
B



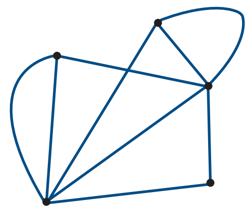
C



D



E

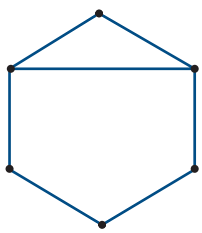


18 Which one of the graphs above has an Eulerian trail but not an Eulerian circuit?

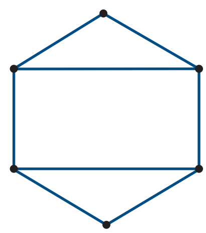
19 Which one of the graphs above has an Eulerian circuit?

20 Which one of the following graphs has an Eulerian circuit?

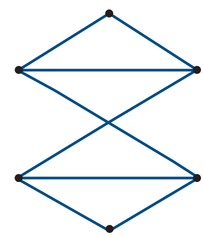
A



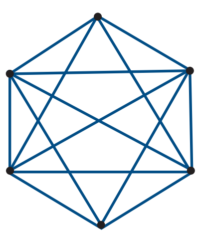
B



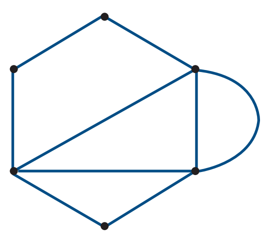
C



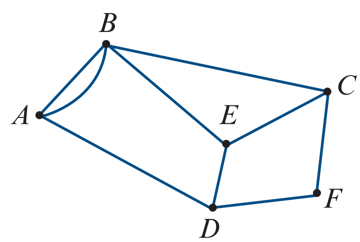
D



E



21 For the graph shown, which additional edge could be added to the graph so that the graph formed would contain an Eulerian trail?

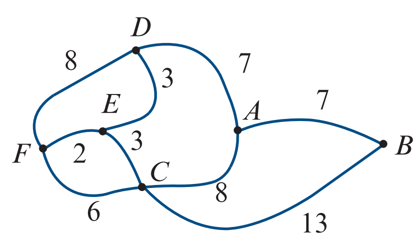


A *AF* **B** *AD*

C *AB* **D** *CF*

E *BF*

22 The length of the shortest path from *F* to *B* in the graph shown is:



A 17 **B** 18

C 19 **D** 20

E 21

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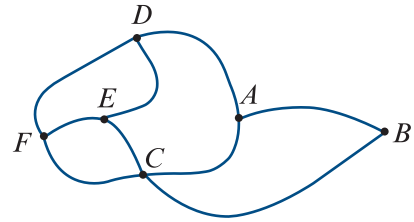
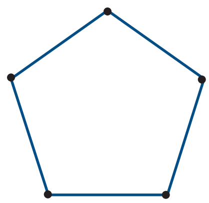
23 Which one of the following paths is a Hamiltonian cycle for the graph shown?



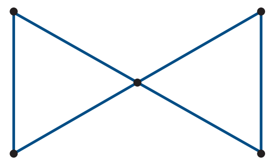
A *F-E-D-F*
B *F-E-D-A-B-C-E-F*
C *F-E-D-A-B-C-F*
D *F-C-A-B-D-E-F*
E *F-D-E-C-A-B-C-F*

24 Of the following, which graph has both an Eulerian circuit and Hamiltonian cycle?

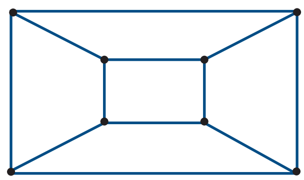
A



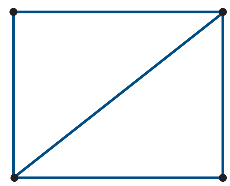
B



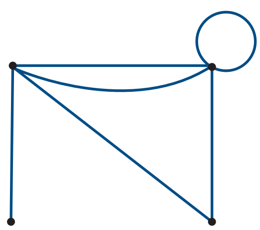
C



D

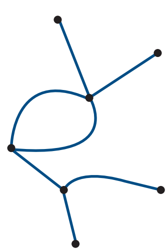


E

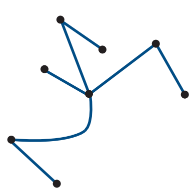


25 Which one of the following graphs is a tree?

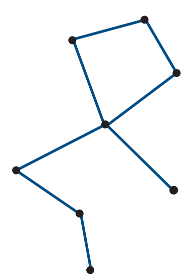
A



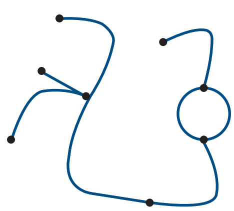
B



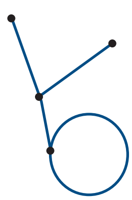
C



D



E



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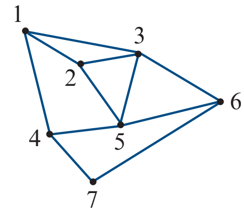
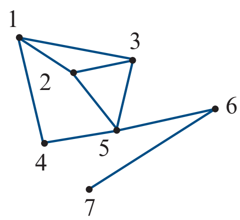
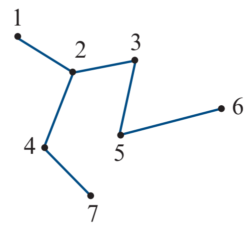
26 Which one of the following graphs is a spanning tree for the graph shown?



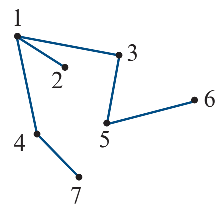
A



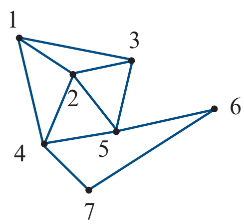
B



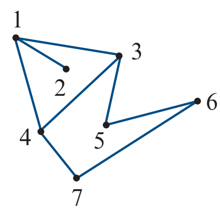
C



D



E



The campsite information below is to be used when answering Questions 27 to 29.

27 A park ranger wants to check all of the campsites in a national park, starting at and returning to the park office. The campsites are all interconnected with walking tracks. She would like to check the campsites without having to visit each campsite more than once. If possible, the route she should follow is:

A an Eulerian trail but not a circuit
B an Eulerian circuit
C a Hamiltonian path but not a cycle
D a Hamiltonian cycle
E a minimum spanning tree

28 The park authorities plan to pipe water to each of the campsites from a spring located in the park. They want to use the least amount of water pipe possible. If possible, the water pipes should follow:

A an Eulerian trail but not a circuit
B an Eulerian circuit
C a Hamiltonian path but not a cycle
D a Hamiltonian cycle
E a minimum spanning tree

29 Each week, a garbage collection route starts at the tip and collects the rubbish left at each of the campsites before returning to the tip to dump the rubbish collected. The plan is to visit each campsite only once. If possible, the garbage collection route should follow:

A an Eulerian trail but not a circuit
B an Eulerian circuit
C a Hamiltonian path but not a cycle
D a Hamiltonian cycle
E a minimum spanning tree

ISBN 978-1-009-11034-1

Cambridge University Press

Short-answer questions

1 Draw a connected graph with:

a four vertices, four edges and two faces

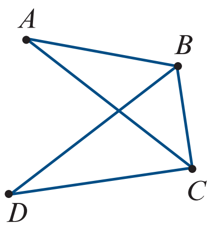
b four vertices, five edges and three faces

c five vertices, eight edges and five faces

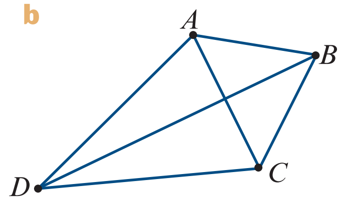
d four vertices, five edges, three faces and two bridges

2 Redraw each of the following graphs in planar form.

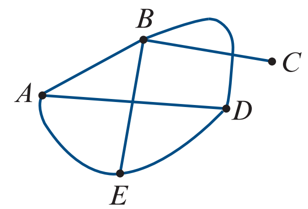
a



b



c

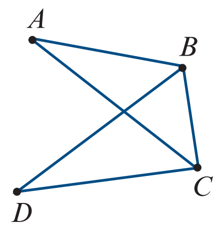


3 For the graph shown, write down:

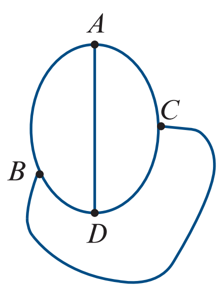
a the degree of vertex C

b the numbers of odd and even vertices

c the route followed by an Eulerian trail, starting at vertex B.



4 Construct an adjacency matrix for the graph below.

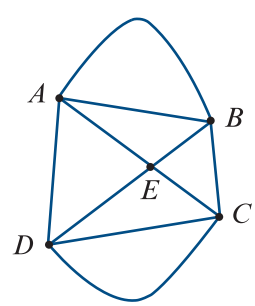


5 For the graph shown, write down:

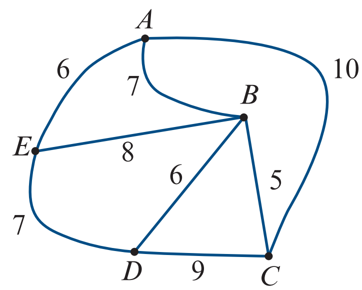
a the degree of vertex C

b the number of odd and even vertices

c the route followed by an Eulerian circuit, starting at vertex A.



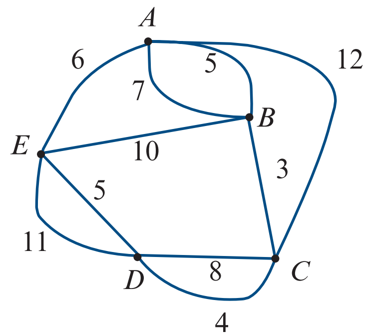
6 For the weighted graph shown, determine the length of the minimum spanning tree.



7 In the network opposite, the numbers on the edges represent distances in kilometres.
Determine the length of:

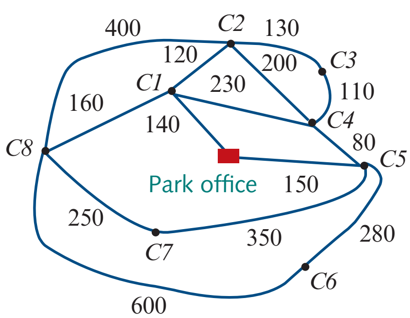
a the shortest path between vertex A and vertex D

b the length of the minimum spanning tree.



Written-response questions

1 The diagram opposite shows the network of walking tracks in a small national park. These tracks connect the campsites to each other and to the park office. The lengths of the tracks (in metres) are also shown.



a The network of tracks is planar. Explain what this means.

b Verify Euler's formula for this network.

c A ranger at campsite *C8* plans to visit campsites *C1, C2, C3, C4* and *C5* on her way back to the park office. What is the shortest distance she will have to travel?

d How many even and how many odd vertices are there in the network?

e Each day, the ranger on duty has to inspect each of the tracks to make sure that they are all passable.

i Is it possible for her to do this, starting and finishing at the park office and travelling only once on each track? Explain why.

ii Identify one route that she could take.

f Following a track inspection after wet weather, the Head Ranger decides that it is necessary to put gravel on some walking tracks to make them weatherproof. What is the minimum length of track that will need to be gravelled to ensure that all campsites and the park office are accessible along a gravelled track?

ISBN 978-1-009-11034-1 Cambridge University Press

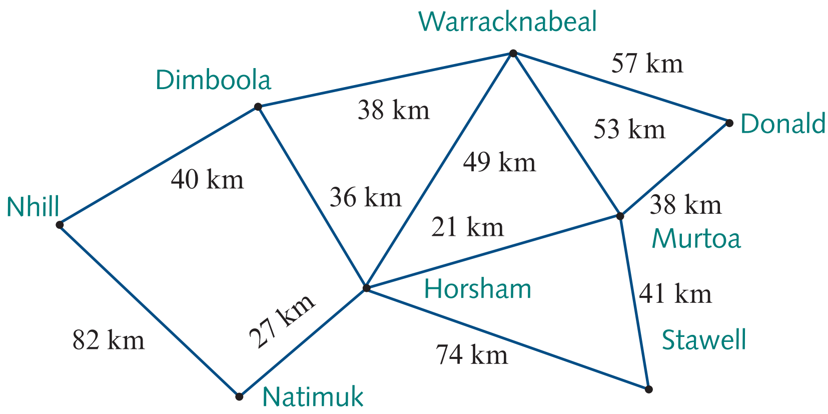
g A ranger wants to inspect each of the campsites but not pass through any campsite more than once on her inspection tour. She wants to start and finish her inspection tour at the park office.

i What is the technical name for the route she wants to take?

ii With the present layout of tracks, she cannot inspect all the tracks without passing through at least one campsite twice. Suggest where an additional track could be added to solve this problem.

iii With this new track, write down a route she could follow.

2 The network below shows the major roads connecting eight towns in Victoria and the distances between them, in kilometres.



a Find the shortest distance between Nhill and Donald using these roads.

b Verify Euler's formula for this network.

c An engineer based in Horsham needs to inspect each road in the network without travelling along any of the roads more than once. He would also like to finish his inspection at Horsham.

i Explain why this cannot be done.

ii The engineer can inspect each road in the network without travelling along any of the roads more than once if he starts at Horsham but finishes at a different town. Which town is that? How far will he have to travel in total?

iii Identify one route, starting at Horsham, that the engineer can take to complete his inspection without travelling along any of the roads more than once.

d A telecommunications company wants to connect all of the towns to a central computer system located in Horsham. What is the minimum length of cable that they will need to complete this task?

e The engineer can complete his inspection in Horsham by only travelling along one of the roads twice. Which road is that?

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