Chapter 9

Variation

Chapter questions

- How do we recognise relationships involving **direct variation**?
- How do we determine the **constant of variation** in cases involving direct variation?
- How do we solve problems involving direct variation?
- How do we recognise relationships involving **inverse variation**?
- How do we determine the **constant of variation** in cases involving inverse variation?
- How do we solve problems involving inverse variation?
- How do we establish the relationship that exists between two variables from given data?
- How do we use data transformations to linearise a relationship?
- How do we use and interpret **log scales**?

In Chapter 7, we looked at how we might better understand an association between two variables for which we have data from the same subject. The techniques that we developed in that chapter were based on data collected on the two variables, and to a large extent, could only be used if we could establish that a linear association existed between the two variables.

In this chapter, we again look at the relationship between two variables, but this time we take a more mathematical approach, exploring situations where an exact formula relating the two variables can be found. We use the ideas of direct and indirect variation to establish the formulae which connect the two variables. We also look in more detail at some special cases where a non-linear relationship exists and develop some new techniques which enable us to linearise such relationships using data transformation.

9A Direct variation

Learning intentions

- To be able to recognise direct variation.
- To be able to find the constant of variation for direct variation.
- To be able to solve practical problems involving direct variation.

For the variables x and y, if $y = kx$ where k is a positive constant, we say that 'y varies **directly** as x'. Sometimes we use the following phrase, which has exactly the same meaning: 'y is directly proportional to x'. The positive constant k is called the **constant of variation**.

Using symbols, we write 'y is directly proportional to x' as: $\boldsymbol{y \propto x}$.

The symbol $\propto$ means '**is proportional to**' or '**varies as**'.

We note that, as x increases, y increases.

For example, if $y = 3x$, then $y \propto x$ and 3 is the constant of variation.

An example of direct variation is with the variables *distance* and *time* when driving at a constant speed.

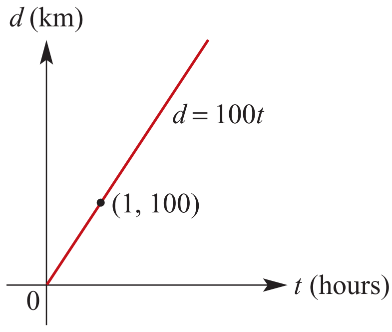
For example, Emily drives from her home in Appleton to visit her friend, Kim, who lives 600 km away in Brownsville. She drives at a constant speed of 100 km/h, and each hour, she notes how far she has travelled.

Time (t hours)	1	2	3	4	5	6
Distance (d km)	100	200	300	400	500	600

It can be seen that, as t increases, d also increases. The rule relating time and distance is: $d = 100t$.

This is an example of **direct variation**, and in this case, 100 is called the **constant of variation**.

We can say that the distance travelled **varies directly** as the time taken, or that d is **proportional** to t ($\boldsymbol{d \propto t}$). The graph of $\boldsymbol{d}$ **against** $\boldsymbol{t}$ is a straight line, passing through the origin.



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- The variable, y, is said to **vary directly** as x, if $y = kx$, for some positive constant, k.
- The constant, k, is called the **constant of variation**.
- In direct variation, $k = \dfrac{y \text{ value}}{\text{corresponding value of } x}$
- The statement 'y varies directly as x' is written symbolically as $\boldsymbol{y \propto x}$.
- If $y \propto x$, then the graph of y against x is a straight line, passing through the origin.

Determining the constant of variation

If y varies directly as x, then we can write $y = kx$, where k is the constant of variation.

The constant of variation can be found if we know just one value of x and the corresponding value of y.

Example 1 Finding the constant of variation

Use the table of values to determine the constant of variation, k, and hence complete the table:
$y \propto x$

x	3	5	7	
y	21		49	63

Explanation

1. Rewrite the variation expression as an equation, with k as the constant of variation.
2. Substitute corresponding values for x and y, and solve for k.
3. Substitute $k = 7$ in $y = kx$.
4. Substitute the value for x to find the corresponding y value.
5. Substitute the value of y to find the corresponding x value.
6. Complete the table.

Solution

$y \propto x$
$y = kx$

When $x = 3$, $y = 21$
$21 = 3k$
$\therefore \quad k = 7$
$y = 7x$

When $x = 5$
$y = 7(5)$
$\therefore \quad y = 35$

When $y = 63$
$63 = 7(x)$
$\therefore \quad x = 9$

x	3	5	7	9
y	21	35	49	63

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Now try this 1 Finding the constant of variation (Example 1)

Use the table of values to find the constant of variation and complete the table.

$y \propto x$

x	2	4	6	
y	10		30	100

Hint 1 Rewrite the variation expression as an equation with constant of variation, k.

Hint 2 Substitute known corresponding values for x and y to find k.

Hint 3 Solve for unknown y by substituting relevant x value in the equation.

Hint 4 Solve for unknown x by substituting relevant y value in the equation.

Variation involving powers

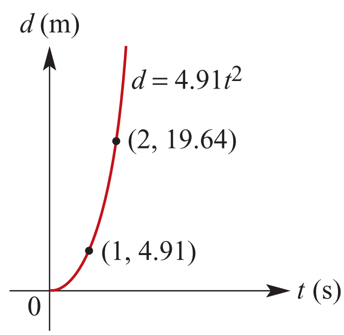
Sometimes, one of the variables may be raised to a power.
For example, a metal ball is dropped from the top of a tall building, and the distance it has fallen, in metres, is recorded each second.

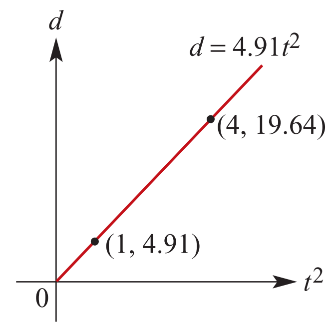
Time (t s)	0	1	2	3	4	5
Distance (d m)	0	4.91	19.64	44.19	78.56	122.75

As t increases, d also increases. The rule relating time and distance is $d = 4.91t^2$.

This is another example of direct variation. In this case, we say that the distance fallen varies directly as the *square* of the time taken, or that d is proportional to t^2. We write $d \propto t^2$.

The graph of d against t is a parabola. However, the graph of d against t^2 is a straight line passing through the origin.





- If $y \propto x^n$, then $y = kx^n$, where k is a **constant of variation**.
- If $y \propto x^n$, then the graph of y against x^n is a straight line passing through the origin.

For all examples of direct variation, where one variable increases the other will also increase. It should be noted, however, that not all increasing trends are examples of direct variation.

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Example 2 Finding the constant of variation involving powers

Given that $y \propto x^2$, use the table of values to determine the constant of variation, k, and hence complete the table.

x	2	4	6	
y	12		108	192

Explanation

1 Rewrite the variation expression as an equation, with k as the constant of variation.

2 Substitute corresponding values for x and y and solve for k.

3 Rewrite equation for y, substituting k.

4 Check with other values that the correct value for k has been found.

5 Substitute values for x to find corresponding y values.

6 Substitute the value of y to find the corresponding x value.

7 Complete the table.

Solution

$y \propto x^2$
$y = kx^2$

When $x = 2$, $y = 12$
$12 = (2^2)k$
$\therefore \quad k = 3$

$y = 3x^2$

When $x = 6$, $y = 3(6^2) = 108$

When $x = 4$, $y = 3(4^2)$
$\therefore \quad y = 48$

When $y = 192$
$192 = 3(x^2)$
$\therefore \quad x = 8$

x	2	4	6	8
y	12	48	108	192

Now try this 2 Finding the constant of variation involving powers (Example 2)

Given that $y \propto x^2$, use the table of values to find the constant of variation and then complete the table.

x	2	4	6	
y		32	72	288

Hint 1 Rewrite the variation expression as an equation, with constant of variation, k.
Hint 2 Substitute known corresponding values for x and y into the equation, remembering to square each x value.
Hint 3 Solve equation for unknown y value by multiplying x^2 value by k.
Hint 4 Solve for x by substituting known y value into equation.

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Example 3 Solving a direct variation practical problem

In an electrical wire, the resistance (R ohms) varies directly as the length (L m) of the wire.

a If a 6 m wire has a resistance of 5 ohms, what is the resistance of a 4.5 m wire?

b How long is a wire for which the resistance is 3.8 ohms?

Explanation	Solution
a	
1 Write down the variation expression.	$R \propto L$
2 Rewrite expression as an equation with k as the constant of variation.	$R = kL$
3 Find the constant of variation by substituting given values for R and L.	When $L = 6, R = 5$ $5 = k(6)$ $\therefore \quad k = \frac{5}{6}$
4 Substitute value for k and write down the equation.	$R = \frac{5L}{6}$
5 Substitute $L = 4.5$ and solve to find R.	$R = \frac{5 \times 4.5}{6}$ $= 3.75$
6 Write your answer.	A wire of length 4.5 m has a resistance of 3.75 ohms.
b	
1 Write down the equation.	$R = \frac{5L}{6}$
2 Substitute $R = 3.8$ and solve to find L.	$3.8 = \frac{5 \times L}{6}$ $\therefore \quad L = 4.56$
3 Write your answer.	A wire of resistance 3.8 ohms has a length of 4.56 m.

Now try this 3 Solving a direct variation practical problem (Example 3)

A car is travelling at a constant speed. The distance travelled (d) varies directly as the time taken (t).

a If it takes 2 hours to travel 190 km, how far has the car travelled in 3 hours?

b How long does it take to travel 500 km? Give your answer in hours and minutes, rounded to the nearest whole minute.

Hint 1 Write down the variation expression and then rewrite as an equation with constant of variation, k.

Hint 2 Substitute known values to find k.

Hint 3 Use equation to solve for unknown values.

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Section Summary

Direct variation

- The variable y **varies directly** as x if $y = kx$, for some positive constant k. We can also say that y is **proportional** to x, and we can write $y \propto x$.
- The constant, k, is called the **constant of variation** or **constant of proportionality**.
- If y is proportional to x, then the graph of y against x is a straight line through the origin. The slope of the line is the constant of variation.
- If $y \propto x$, then for any two non-zero values x_1 and x_2 and the corresponding values y_1 and y_2,

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = k$$

Skill-sheet

Exercise 9A

Building understanding

Example 1

1 Complete the following tables for the given equations.

a $y = 3x$

x	2	4	6	
y	6	12		24

b $y = 4x$

x	0	1	2	
y	0			12

2 Rewrite the following expressions of variation as equations, using k as the constant of variation.

a $y \propto x$ **b** $y \propto x^2$ **c** $y \propto x^5$
d $a \propto b$ **e** $z \propto w$ **f** $y \propto \sqrt{x}$

3 Write expressions of variation for the following statements.

a m varies directly as n
b y varies directly as the square of x
c y varies directly as the square root of x
d s varies directly as the square of t

Developing understanding

Example 2

4 For each of the following, determine the constant of variation, k, and hence complete the table of values:

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a $y \propto x$

x	3	4		12
y	21		49	84

b $y \propto x$

x	4	9	14	
y	2	4.5		10

c $y \propto x^2$

x	2	4	6	
y	8	32		128

5 If $y \propto x$ and $y = 42$ when $x = 7$, find:

a y when $x = 9$

b x when $y = 102$

6 If $M \propto n^2$, and $M = 48$ when $n = 4$, find:

a M when $n = 10$

b n when $M = 90$, correct to one decimal place.

Example 3

7 The area (A) of a triangle of fixed base length varies directly as its perpendicular height (h). If the area of the triangle is 60 cm^2 when its height is 10 cm, find:

a the area when its height is 12 cm

b the height when its area is 120 cm^2

8 The cost of potatoes varies directly with their weight. If potatotes weighing 4 kg cost $15.60, find:

a the cost of potatoes weighing 6 kg

b how many kgs of potatoes you could buy for $25, correct to one decimal place.

9 The distance (d) that a car travels at a constant speed varies directly with the time (t) taken. If the car travels 330 kms in 3 hours, find:

a the time taken to travel 500 kms

b the distance travelled after

i 5 hours

ii 90 minutes

Testing understanding

10 The weight (W) of a square sheet of lead varies directly with the square of its side length (L). If a sheet of side length 20 cm weighs 18 kg, find the weight of a sheet that has an area of 225 cm^2.

11 The distance (d) to the visible horizon varies directly with the square root of the height (h) of the observer above sea level. An observer 1.8 m tall can see 4.8 km out to sea when standing on the shoreline. How far could the person see if they climbed a 4-m tower? Give your answer to two decimal places.

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9B Inverse variation

Learning intentions

- ▶ To be able to recognise inverse variation.
- ▶ To be able to find the constant of variation for inverse variation.
- ▶ To be able to solve practical problems involving inverse variation.

For the variables x and y, if $y = \frac{k}{x}$ where k is a positive constant, we say that 'y varies **inversely** as x'. Sometimes we use the following phrase, which has exactly the same meaning: 'y is **inversely proportional** to x'. The positive constant, k, is called the **constant of variation**.

Using symbols, we write it as $y \propto \frac{1}{x}$.

We note that as x increases, y decreases.

For example, if $y = \frac{3}{x}$, then $y \propto \frac{1}{x}$ and 3 is the constant of variation.

For example, a builder employs a number of bricklayers to build a brick wall. Three bricklayers will complete the wall in 8 hours. However, if the builder employs six bricklayers, the wall will be completed in half the time. The more bricklayers employed, the shorter the time taken to complete the wall. The time taken (t) decreases as the number of bricklayers (b) increases. This is an example of **inverse variation**.

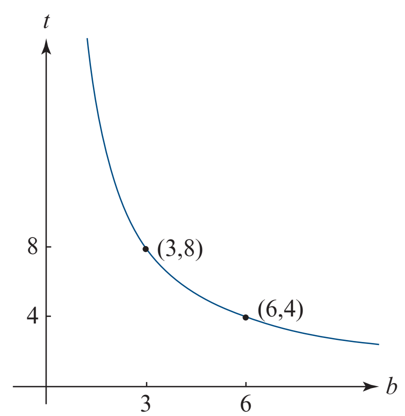
In this case we say:

t **varies inversely** as b

or t is **inversely proportional** to b

and we write $t \propto \frac{1}{b}$.

The graph of this is shown here.



If y **varies inversely** as x, then $\boldsymbol{y \propto \frac{1}{x}}$ and $y = \frac{k}{x}$, where k is the constant of variation.

In the table below, $y \propto \frac{1}{x}$ and the constant of variation is 6. We can say that $y = \frac{6}{x}$.

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2
y	18	12	6	3

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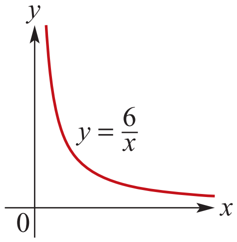
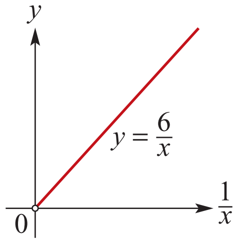
Note: $y \propto \frac{1}{x}$ is equivalent to $xy = k$, for a positive constant k. That is, the product is constant. This is often a useful way to check for inverse variation when given data in table form.

When we plot the graph of y against x, we obtain a curved shape called a **hyperbola**. However, the graph of y against $\frac{1}{x}$ is a straight line which does not have a value at the origin (as $\frac{1}{x}$ is undefined).





For all examples of inverse variation, as one variable increases, the other will decrease, and vice versa. The graph of y against x will show a downward trend. It should be noted, however, that not all decreasing trends are examples of inverse variation.

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Example 4 Determining the constant of variation for inverse variation

Use the table of values to find the constant of variation, k, and hence complete the table.

$y \propto \frac{1}{x}$

x	1	2	3	4	
y	3		1	0.75	0.6

Explanation	Solution
1 Write down the variation expression.	$y \propto \frac{1}{x}$
2 Rewrite the expression as an equation with constant of variation, k.	$y = \frac{k}{x}$
3 Substitute known values for x and y.	When $x = 1, y = 3$ $3 = \frac{k}{1}$
4 Solve for k.	$k = 3$
5 Rewrite equation, substituting k.	$y = \frac{3}{x}$
6 Check with other values that the correct value for k has been found.	When $x = 3, y = \frac{3}{3} = 1$
7 Substitute value for x to find corresponding y value.	When $x = 2$ $y = \frac{3}{2} = 1.5$
8 Substitute value for y to find corresponding x value.	When $y = 0.6$ $0.6 = \frac{3}{x}$ $x = \frac{3}{0.6}$ $\therefore \quad x = 5$
9 Complete the table.	(see table below)

x	1	2	3	4	5
y	3	1.5	1	0.75	0.6

Now try this 4 Determining the constant of variation for inverse variation (Example 4)

Use the table of values to find the constant of variation, k, and hence complete the table.

$y \propto \frac{1}{x}$

x	2	4	5	10	
y	1	0.5	0.4		0.1

Hint 1 Rewrite the variation expression as an equation with constant of variation, k.

Hint 2 Substitute known values for x and y and solve for k.

Hint 3 Substitute value for x to find corresponding unknown y value.

Hint 4 Substitute value for y to find corresponding unknown x value.

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Example 5 Solving an inverse variation practical problem

For a cylinder of fixed volume, the height (h cm) is inversely proportional to the square of the radius (r cm).
If a cylinder of height 15 cm has a base radius of 4.2 cm, how high would a cylinder of equivalent volume be if its radius was 3.5 cm?

Explanation	Solution
1 Write down the variation expression, and then rewrite it as an equation with constant of variation, k.	$h \propto \frac{1}{r^2}$ $h = \frac{k}{r^2}$
2 Substitute known values for h and r, and solve for k.	When $h = 15, r = 4.2$ $15 = \frac{k}{(4.2)^2}$ $k = 15(4.2)^2$ $\therefore\ k = 264.6$
3 Write the equation.	$h = \frac{264.6}{r^2}$
4 To find the height when the radius is 3.5 cm, substitute $r = 3.5$ into equation and solve for h.	$h = \frac{264.6}{(3.5)^2}$ $h = 21.6$
5 Write your answer with correct units.	A cylinder with radius 3.5 cm has a height of 21.6 cm.

Now try this 5 Solving an inverse variation practical problem (Example 5)

The time taken (t hours) to empty a tank is inversely proportional to the rate, r, of pumping. If it takes two hours for a pump to empty a tank at a rate of 1200 litres per minute, how long will it take to empty a tank at a rate of 2000 litres per minute?

Hint 1 Write the variation expression and then rewrite as an equation with constant of variation, k.

Hint 2 Substitute values for t and r and solve for k.

Hint 3 Write down the equation, substitute value for r and solve for t.

Section Summary

Inverse variation

- The variable y **varies inversely** as x if $y = \frac{k}{x}$, for some positive constant, k.
  We can also say that y is **inversely proportional** to x, and we can write $y \propto \frac{1}{x}$.
- If y varies inversely as x, then the graph of y against $\frac{1}{x}$ is a straight line (not defined at the origin) and the gradient is the constant of variation.
- If $y \propto \frac{1}{x}$, then $x_1y_1 = x_2y_2 = k$, for any two values x_1 and x_2 and the corresponding values y_1 and y_2.

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Skill-sheet

Exercise 9B

Building understanding

1 Complete the following tables for the given equations.

a $y = \frac{20}{x}$

x	2	4	5	
y	10	5		2

b $y = \frac{5}{x}$

x	1	2	4	
y	5			1

2 Rewrite the following expressions of variation as equations, using k as the constant of variation.

a $y \propto \frac{1}{x}$ **b** $y \propto \frac{1}{x^2}$ **c** $y \propto \frac{1}{x^3}$

d $m \propto \frac{1}{n}$ **e** $z \propto \frac{1}{w}$ **f** $y \propto \frac{1}{\sqrt{x}}$

3 Write expressions of variation for the following statements.

a A varies inversely as r

b y varies inversely as the square of x

c y varies inversely as the square root of x

d m varies inversely as the cube of n

e s varies inversely as t

Developing understanding

Example 4

4 For each of the following, determine the constant of variation, k, and hence complete the table of values:

a $y \propto \frac{1}{x}$

x	1	2	4	
y	10	5		1

b $y \propto \frac{1}{x}$

x	2	4	10	
y	1	$\frac{1}{2}$		$\frac{1}{15}$

c $y \propto \frac{1}{x}$

x	1	2	4	
y	1	$\frac{1}{2}$		$\frac{1}{5}$

d $y \propto \frac{1}{x}$

x	0.5	1		5
y	1	0.5	0.25	

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5 If $y \propto \frac{1}{x}$ and $y = 20$ when $x = 5$, find:

a y when $x = 10$ **b** x when $y = 50$

6 If $a \propto \frac{1}{b}$ and $a = 1$ when $b = 2$, find:

a a when $b = 4$ **b** b when $a = \frac{1}{8}$

7 If $a \propto \frac{1}{b}$ and $a = 5$ when $b = 2$, find:

a a when $b = 4$ **b** b when $a = 1$

Example 5

8 The current (I amperes) that flows in an electrical appliance varies inversely with the resistance (R ohms). If the current is 3 amperes when the resistance is 80 ohms, find the current when the resistance is 100 ohms.

9 The time taken (t) to check-in passengers on a flight varies inversely with the number of people (n) working at the check-in desks.

a Write a variation expression to show this.

b When 5 people are working at the check-in desks, it takes 40 minutes to check-in passengers. How long will it take if there are 8 people working?

10 The length of a string on a musical instrument varies inversely to its frequency vibration. A 32.5 cm string on a violin has a frequency vibration of 400 cycles per second. What would be the frequency vibration of a 24.2 cm string? Give your answer to the nearest whole number.

Testing understanding

11 The gas in a cylindrical canister occupies a volume of 22.5 cm^3 and exerts a pressure of 1.9 kg/cm^2. If the volume (V) varies inversely with the pressure (P), find the pressure if the volume is reduced to 15 cm^3.

9C Data transformations

Learning intentions

- To be able to use the squared transformation.
- To be able to use the reciprocal transformation.

When the data from two variables are plotted on a graph, we may see that there is a clear relationship between the variables, but that this relationship is not linear. By changing the scale of one of the variables, it is sometimes possible to change the relationship to linear form, which is easier to analyse using techniques we have already developed.

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We can change the scale of a variable by applying a mathematical function to the values of one of the variables. This is a strategy that we used in the previous sections, changing the scale from x to x^2 in Example 2.

Changing the scale of a variable is called transforming the data.

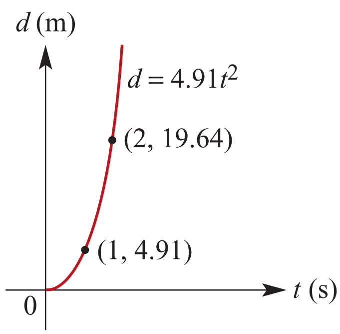
The process of changing the scale so that the relationship is linear is called **linearisation.**

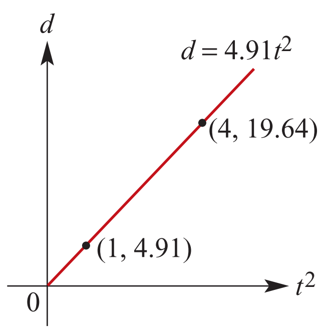
Knowing which transformation to apply to linearise a particular relationship is complex and will be studied in further detail in General Mathematics Units 3 and 4. In this section, we will look at two very useful transformations: the squared transformation and the reciprocal transformation.

The squared transformation: x^2

In this transformation, we change the scale on the horizontal axis from $\boldsymbol{x}$ to $\boldsymbol{x^2}$.

Referring back to the example on direct variation involving powers (page 555) where a metal ball was dropped from a tall building and the distance and time were recorded, we noticed that the graph of $\boldsymbol{d}$ against $\boldsymbol{t}$ was a parabola (curved graph).
However, when we plotted $\boldsymbol{d}$ against $\boldsymbol{t^2}$ we obtained a straight-line graph, passing through the origin.





This is an example of using the squared transformation, and it has the effect of linearising the graph (changing the data to a straight line), which is easier to analyse. The slope of this line is the constant of variation.

Example 6 Transforming data using the x^2 transformation

Plot the points for the given table of values, and then use an x^2 transformation to check if this gives a straight line.

x	0	1	2	3	4
y	1	2	5	10	17

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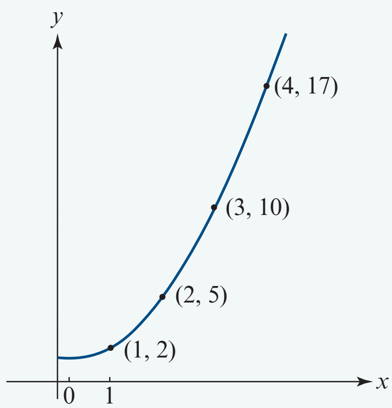
Explanation

1 Plot corresponding x and y values on a graph. This is clearly non-linear.

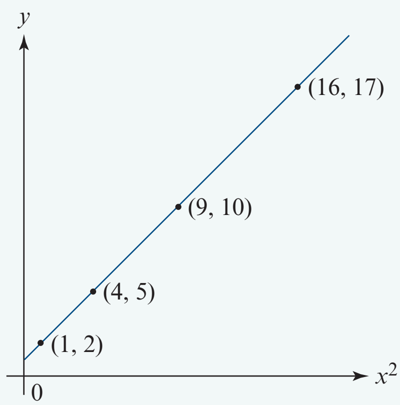
Solution



2 Square all x values.

x	0	1	2	3	4
x^2	0	1	4	9	16
y	1	2	5	10	17

3 Plot corresponding x^2 and y values on a graph. **Remember** to label the horizontal axis as x^2.



4 Check to see if the graph has become linear.

Graph is a straight line, so the graph has been linearised with the x^2 transformation.

Now try this 6 **Transforming data using the x^2 transformation (Example 6)**

Plot the points for the given table of values and then use an x^2 transformation to check if this gives a straight line.

x	0	1	2	3	4
y	3	4	7	12	19

Hint 1 Square each x value.

Hint 2 Plot each corresponding x^2 value and y value, and check to see if this gives a straight line.

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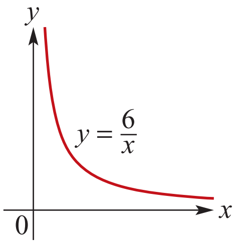
The reciprocal transformation: $\frac{1}{x}$

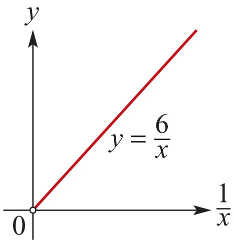
In this transformation we change the scale on the horizontal axis from $\boldsymbol{x}$ to $\boldsymbol{\frac{1}{x}}$.

Using our earlier example (page 561) of $y \propto \frac{1}{x}$ with a constant of variation of 6, the graph of y against x gives a curved line, called a hyperbola. However, the graph of y against $\frac{1}{x}$ gives a straight line with a slope of 6. The graph will not have a point at the origin (indicated on the graph by an open circle) because $\frac{1}{0}$ is undefined.

The table of values has been rewritten here with the values of $\frac{1}{x}$ included. Next to the table are the graphs showing $\boldsymbol{y}$ plotted against $\boldsymbol{x}$, and then $\boldsymbol{y}$ plotted against $\boldsymbol{\frac{1}{x}}$.

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2
$\frac{1}{x}$	3	2	1	$\frac{1}{2}$
y	18	12	6	3





We can see that by changing the horizontal axis from x to $\frac{1}{x}$ we obtain a linear graph.

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Example 7 Transforming data using the $\frac{1}{x}$ transformation

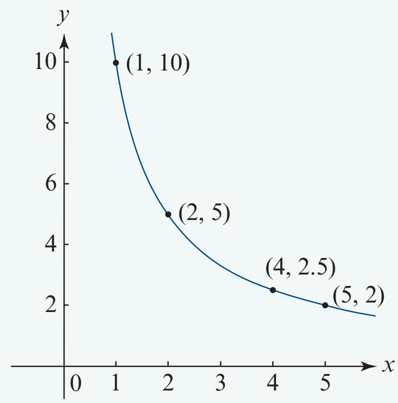
Plot the points for the given table of values, and then use the $\frac{1}{x}$ transformation to check if this gives a straight line.

x	1	2	4	5
y	10	5	2.5	2

Explanation

1 Plot corresponding x and y values on a graph.

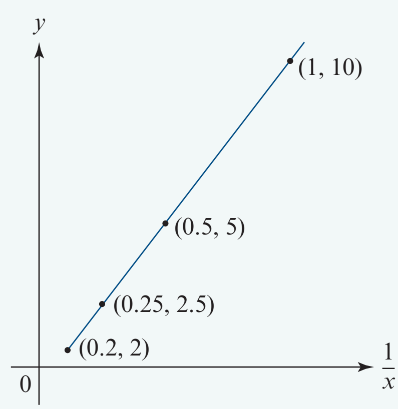
Solution



2 Find the reciprocal $(\frac{1}{x})$ of all x values.

x	1	2	4	5
$\frac{1}{x}$	1	0.5	0.25	0.2
y	10	5	2.5	2

3 Plot corresponding $\frac{1}{x}$ and y values on a graph. **Remember** to label horizontal axis as $\frac{1}{x}$.



4 Check to see if the graph has become linear.

Graph is a straight line, so the graph has been linearised with the $\frac{1}{x}$ transformation.

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Now try this 7 Transforming data using the $\frac{1}{x}$ transformation (Example 7)

Plot the points for the given table of values, and then use a $\frac{1}{x}$ transformation to check if this gives a straight line.

x	1	2	4	8
y	4	2	1	0.5

Hint 1 Divide 1 by each x value to find $\frac{1}{x}$.

Hint 2 Plot each corresponding $\frac{1}{x}$ value and y value, and check to see if this gives a straight line.

The CAS calculator can be used to perform the x^2 and the $\frac{1}{x}$ transformations.

Example 8 Using a CAS calculator to perform the x^2 transformation

For the given table of values, use the x^2 transformation and plot the graph of y against x^2 to check if this transformation gives a linear graph.

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

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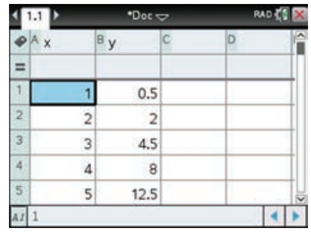
Using the TI-Nspire CAS to perform a squared transformation

Steps

1 Start a new document by pressing (ctrl) + (N).

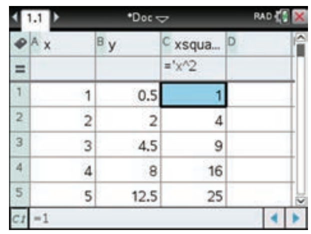
2 Select **Add Lists & Spreadsheet**.

Enter the data into lists named *x* and *y*, as shown.



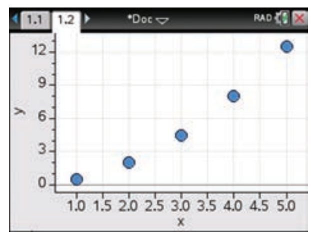
3 Name column C as *xsquared*.

4 Move the cursor to the grey cell below *xsquared*. Enter the expression = xˆ2 by pressing (=), then typing xˆ2. Pressing (enter) calculates and displays the values of x^2.



5 Press (ctrl)+(I) and select **Add Data & Statistics**.

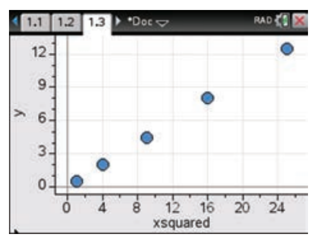
Construct a scatterplot of *y* against *x*. The plot is clearly non-linear.



6 Press (ctrl)+(I) and select **Add Data & Statistics**.

Construct a scatterplot of *y* against x^2.

The plot is now linear.



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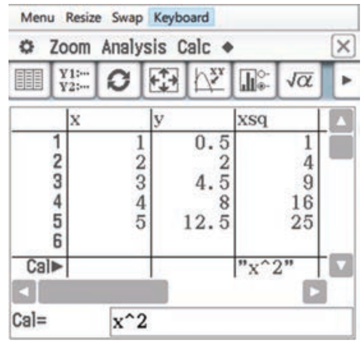
Using the CAS Classpad to perform a squared transformation

Steps

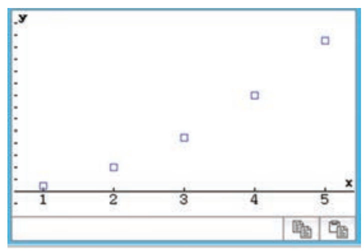
1 In the Statistics application, enter the data into lists named *x* and *y*. Name the third list *xsq* (for x^2).

2 Place the cursor in the calculation cell at the bottom of the third column and type x^2. This will calculate the values of x^2.



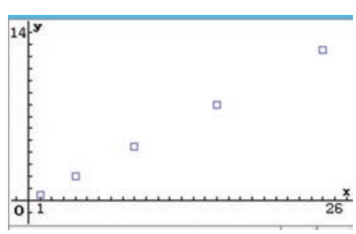
3 Construct a scatterplot of *y* against *x*.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The scatterplot is clearly non-linear.



4 Construct a scatterplot of *y* against x^2.

- Tap and complete the **Set StatGraphs** dialog box as shown.
- Tap to view the scatterplot. The plot is now clearly linear.



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Now try this 8 Using a CAS calculator to perform the x^2 transformation (Example 8)

For this table of values, use the x^2 transformation and plot the graph of y against x^2 to check if this transformation gives a linear graph.

x	1	2	3	4
y	3	12	27	48

Example 9 Using a CAS calculator to perform the $\frac{1}{x}$ transformation

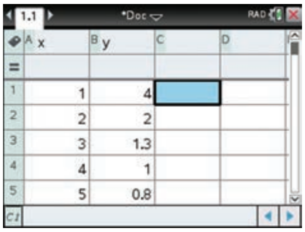
For the given table of values, use the $\frac{1}{x}$ transformation, and plot the graph of y against $\frac{1}{x}$ to check if this transformation gives a linear graph.

x	1	2	3	4	5
y	4	2	1.3	1	0.8

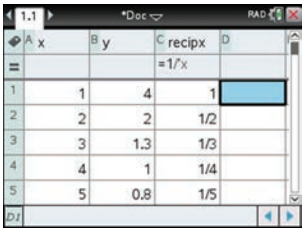
Using the TI-Nspire CAS to perform a $\frac{1}{x}$ transformation

Steps

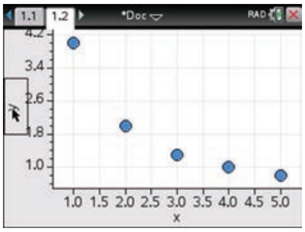
1. Start a new document by pressing ctrl + N.
2. Select **Add Lists & Spreadsheet**.
   Enter the data into lists named x and y, as shown.



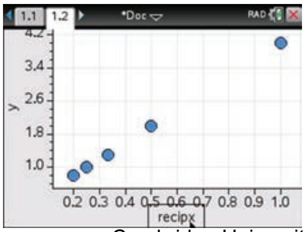
3. Name column C as *recipx*.
4. Move the cursor to the grey cell below *recipx*. Enter the expression $= \frac{1}{x}$ by pressing =, then typing **1/x**.
   Pressing enter calculates and displays the values of $\frac{1}{x}$.



5. Press ctrl + I and select **Add Data & Statistics**.
   Construct a scatterplot of y against x. The plot is clearly non-linear.



6. Press ctrl + I and select **Add Data & Statistics**.
   Construct a scatterplot of y against $\frac{1}{x}$.
   The plot is now linear.



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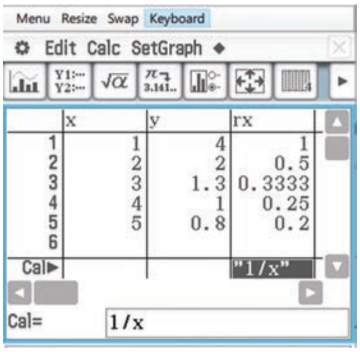
Using the CAS Classpad to perform a $\frac{1}{x}$ transformation

Steps

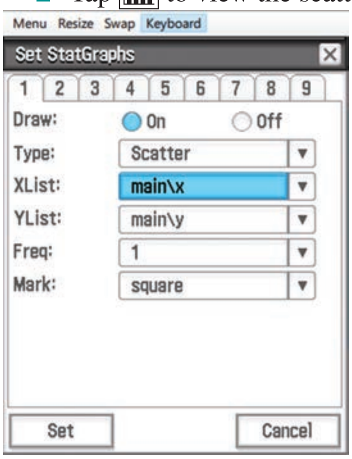
1 In the Statistics application, enter the data into lists named x and y. Name the third list rx (for $\frac{1}{x}$, the reciprocal of x).

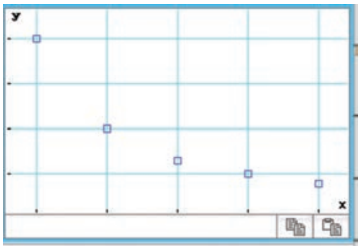
2 Place the cursor in the calculation cell at the bottom of the third column and type $\frac{1}{x}$. This will calculate the values of $\frac{1}{x}$.



3 Construct a scatterplot of y against x.

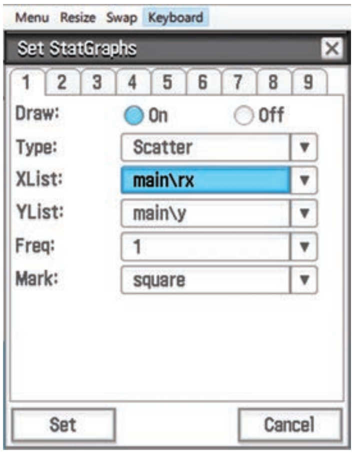
- Tap [icon] and complete the **Set StatGraphs** dialog box as shown.
- Tap [icon] to view the scatterplot. The scatterplot is clearly non-linear.





4 Construct a scatterplot of y against $\frac{1}{x}$.

- Tap [icon] and complete the **Set StatGraphs** dialog box as shown.
- Tap [icon] to view the scatterplot. The plot is now clearly linear.



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Now try this 9 Using a CAS calculator to perform the $\frac{1}{x}$ transformation (Example 9)

For the given table of values, use the $\frac{1}{x}$ transformation and plot the graph of y against $\frac{1}{x}$ to check if this transformation gives a linear graph.

x	1	2	4	5
y	0.5	0.25	0.125	0.1

Hint 1 Follow instructions for the relevant CAS calculator above.

Section Summary

- A graph or a table of values may be used to help decide between direct and inverse variation.
- A set of data can be transformed using a squared (x^2) or a reciprocal ($\frac{1}{x}$) transformation.
- Direct variation If $y \propto x^2$, then the graph of y against x^2 will be a straight line through the origin. The slope of this line will be the constant of variation, k.
- Inverse variation If $y \propto \frac{1}{x}$, then the graph of y against $\frac{1}{x}$ will be a straight line, not defined at the origin. The slope of this line will be the constant of variation, k.

Exercise 9C

Building understanding

1 Copy and complete the following:

a In direct variation, if the values of x increase, the values of y ☐.

b Direct variation graphs are ☐ lines that pass through the ☐.

c In inverse variation, as the values of x increase, the values of y ☐.

2 Do the following tables show direct or inverse variation?

a

x	3	6	9	12
y	4	2	$\frac{4}{3}$	1

b

x	3	6	9	12
y	18	72	162	288

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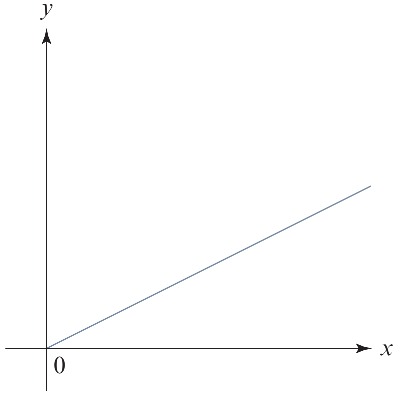
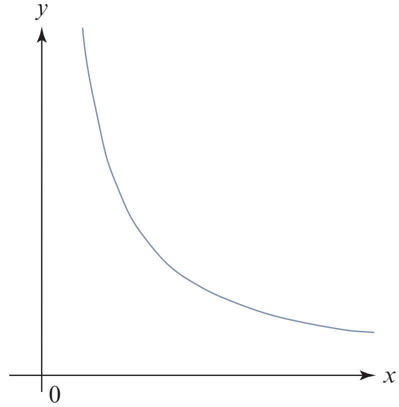
3 Do the following graphs show direct or inverse variation?

a 

b 

4 Complete the following tables to give x^2 and $\frac{1}{x}$ values.

a

x	2	4	6	8
x^2	4			64
$\frac{1}{x}$	$\frac{1}{2}$		$\frac{1}{6}$	

b

x	10	20	30	40
x^2			900	
$\frac{1}{x}$		$\frac{1}{20}$		

Developing understanding

5 The following tables show different types of variation. Which one shows:

a direct, $y \propto x$ **b** inverse, $y \propto \frac{1}{x}$ **c** direct, $y \propto x^2$

i

x	1	2	3	4	5
y	4	16	36	64	100

ii

x	0	3	6	9	12
y	0	2	4	6	8

iii

x	1	5	10	15	20
y	5	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

Example 6

6 Use the x^2 transformation to check if plotting the values of x^2 and y is linear.

x	1	2	3	4	5
y	7.5	9	11.5	15	19.5

7 Use the x^2 transformation to check if plotting the values of x^2 and y is linear.

x	2	2.5	3	3.5	4
y	6.3	15	21.6	27	30.5

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Example 7 **8** Use the $\frac{1}{x}$ transformation to check if plotting the values of $\frac{1}{x}$ and y is linear.

x	0.2	0.4	0.5	1
y	50	20	14	2

Example 8 **9** Use a CAS calculator to check if y plotted against x^2 is linear.

x	2	4	6	8	10
y	9	21	41	69	105

Example 9 **10** Use a CAS calculator to check if y plotted against $\frac{1}{x}$ is linear.

x	2	3	4	8	12
y	12	8	6	3	2

Testing understanding

11 Given the following data:

x	1	2	3	4
y	8	19	34	55

a would you use the x^2 or the $\frac{1}{x}$ transformation to obtain a straight-line graph?

b Perform the transformation that you selected in part **a**, and check to see that this gives a straight-line graph.

12 The time taken to mow a lawn is inversely proportional to the number of mowers. It takes 2 hours to mow a public area of grass with one ride-on mower. If we have 2 ride-on mowers, it will take half the time. This information is represented in the table below, where the time taken, t, is in minutes, and the number of ride-on mowers is represented by n.

n	1	2	3	4	5	6
t	120	60	40			

a Complete the table.

b What transformation would you use to linearise this data?

c Plot the resulting graph.

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9D Logarithms

Learning intentions

- To be able to understand orders of magnitude.
- To be able to use the log transformation ($\log_{10} x$).

What are logarithms?

Possibly the most commonly used data transformation when working with real data is the logarithmic transformation. Before we see how useful this transformation can be, we need to develop our understanding of logarithms. Consider the numbers:

1, 10, 100, 1000, 10 000, 100 000, 1 000 000

Such numbers can also be written as:

$10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6$

In fact, if we make it clear we are only talking about powers of 10, we can simply write down the powers:

0, 1, 2, 3, 4, 5, 6

These powers are called the **logarithms** of the numbers, or **logs** for short. When we use logarithms to write numbers as powers of 10, we say we are working with logarithms to the base 10.

We write $\log_{10} 100$ to mean log to the base 10 of 100. Often we leave the base 10 out and simply write log 100. (We only do this if the base is 10).

Knowing the powers of 10 is important when using logarithms to the base 10.

Example 10 Evaluating a logarithm

Write the number 100 as a power of 10, and then write down its logarithm.

Explanation	Solution
1 Write 100 as a power of 10.	$100 = 10^2$
2 Write down the logarithm.	$\log(100) = \log(10^2)$ $= 2$

Now try this 10 Evaluating a logarithm (Example 10)

Write the number 10 000 as a power of 10, and then write down its logarithm.

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Example 11 Using a CAS calculator to find logarithms

Find the log of 45 to one decimal place.

Explanation	Solution
1 Open a calculator screen, enter log(45) from the keyboard, and press ENTER (TI-Nspire) or EXE (Casio).	$\log_{10}(45)$ 1.65321
2 Write the answer to one decimal place.	$\log(45) = 1.65\ldots$ $= 1.7$ to one decimal place

Now try this 11 Using a CAS calculator to find logarithms (Example 11)

Find the logarithm of 245 to one decimal place.

Hint 1 Use your calculator.

Hint 2 Correct to one decimal place means that there will be only one number after the decimal point.

Hint 3 Round up if the digit after the first decimal place is 5 or more.

Example 12 Using a CAS calculator to evaluate a number if logarithm is known

Find the number whose logarithm is 3.1876 to one decimal place.

Explanation	Solution
1 If the logarithm of a number is 3.1876, then the number is $10^{3.1876}$.	
2 Enter the expression and press ENTER (TI-Nspire) or EXE (Casio).	$10^{3.1876} = 1540.281\ldots$
3 Write the answer to one decimal place.	$= 1540.3$ to one decimal place

Now try this 12 Using a CAS calculator to evaluate a number if logarithm is known (Example 12)

Find the number whose log is 2.8517 to one decimal place.

Hint 1 Use your calculator to evaluate $10^{2.8517}$.

Hint 2 Correct to one decimal place means that there will be only one number after the decimal point.

Hint 3 Round up if the digit after the first decimal place is 5 or more.

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Order of magnitude

Increasing an object by an order of magnitude of 1 means that the object is ten times larger.

An increase by order of magnitude	Increase in size
1	10^1 = 10 times larger
2	10^2 = 100 times larger
3	10^3 = 1000 times larger
6	10^6 = 1 000 000 times larger

Decreasing an object by an order of magnitude of 1 means that the object is ten times smaller.

An increase by order of magnitude

In general, an increase by n orders of magnitude is the equivalent of multiplying a quantity by 10^n.

A decrease by order of magnitude

In general, a decrease by n orders of magnitude is the equivalent of dividing a quantity by 10^n.

It is easy to see the order of magnitude of various numbers when they are written in standard form (e.g. 500 in standard form is $5 \times 100 = 5 \times 10^2$, so the order of magnitude of 500 is 2).

Example 13 Finding the order of magnitude of a number written in standard form

What is the order of magnitude of 1200?

Explanation

1 Write 1200 in standard form.

2 Look at the power of 10 to find the order of magnitude. Write your answer.

Note: The order of magnitude of 1.2 is 0.

Solution

$1200 = 1.2 \times 10^3$

The power of 10 is 3, so the order of magnitude of 1200 is 3.

Now try this 13 Finding the order of magnitude of a number written in standard form (Example 13)

What is the order of magnitude of 35 500?

Hint 1 Write 35 500 in standard form.

Hint 2 Look at the power of 10 to find the order of magnitude.

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Using the logarithmic transformation: $\log_{10}(x)$

As well as using the squared and the reciprocal transformation, we can also use a log transformation to linearise data. In this case, we change the scale on the horizontal axis from x to $\log_{10}(x)$.

Example 14 Transforming data using the $\log_{10}(x)$ transformation

For the given table of values, perform a $\log_{10}(x)$ transformation to check if this gives a straight line, as the graph of y against x is not a straight line.

x	5	10	150	500
y	3.1	4.0	7.5	9.1

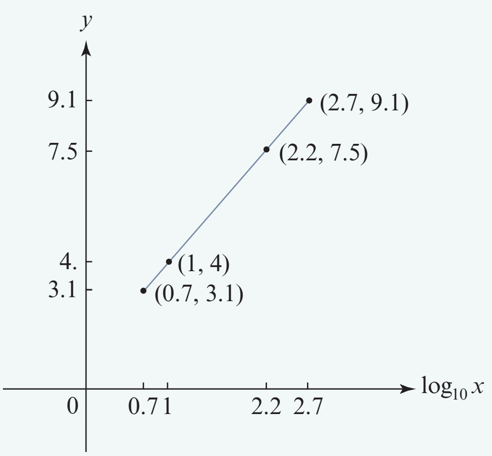
Explanation

1 Find $\log_{10}(x)$ for all x values.

Solution

x	5	10	150	500
$\log_{10}(x)$	0.7	1	2.2	2.7
y	3.1	4.0	7.5	9.1

2 Plot corresponding $\log_{10}(x)$ and y values on a graph. **Remember** to label horizontal axis as $\log_{10}(x)$.



3 Check to see if the graph has become linear.

Graph has been linearised with the $\log_{10}(x)$ transformation.

Now try this 14 Transforming data using the $\log_{10}(x)$ transformation (Example 14)

Plot the points for the given table of values, and then use a $\log_{10}(x)$ transformation to check if this gives a straight line.

x	2	4	6	8
y	1.6	2.2	2.6	2.8

Hint 1 Find $\log_{10}(x)$ for each x value.

Hint 2 Plot each corresponding $\log_{10}(x)$ value and y value, and check to see if this gives a straight line.

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A CAS calculator can also be used to perform the $\log_{10}(x)$ transformation.

Example 15 **Using a CAS calculator to perform the $\log_{10}(x)$ transformation**

For the given table of values, use the $\log_{10}(x)$ transformation to check if this transformation gives a linear graph.

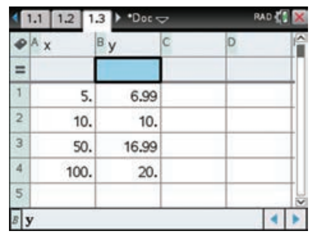
x	5	10	50	100
y	6.99	10	16.99	20

Using the TI-Nspire CAS to perform a $\log_{10}(x)$ transformation

Steps

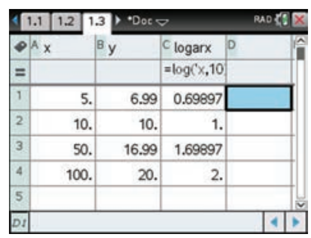
1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named x and y, as shown.

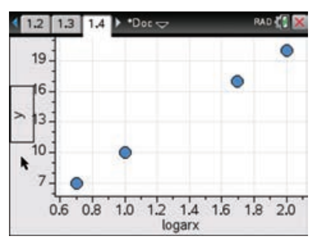


3 Name column C as *logarx* .

4 Move the cursor to the grey cell below *logarx*. Enter the expression = $\log_{10} x$ by pressing =, then entering $\log_{10} x$. Pressing enter calculates and displays the values of $\log_{10} x$.



5 Press ctrl+I and select **Add Data & Statistics**.
Construct a scatterplot of y against $\log_{10} x$.

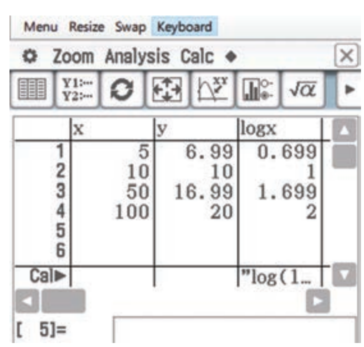


Using the CASIO ClassPad to perform a $\log_{10}(x)$ transformation

Steps

1 In the Statistics application, enter the data into lists named x and y. Name the third list: $\log x$ (for $\log_{10}(x)$) .

2 Place the cursor in the calculation cell at the bottom of the third column, and type $\log_{10}(x)$. This will calculate the values of $\log_{10}(x)$.



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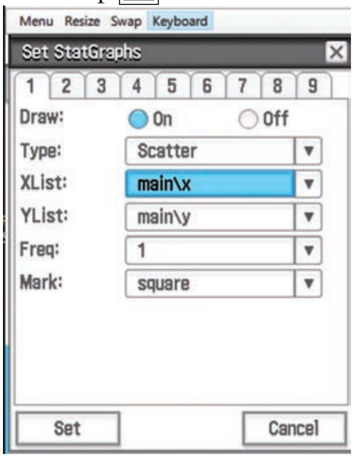
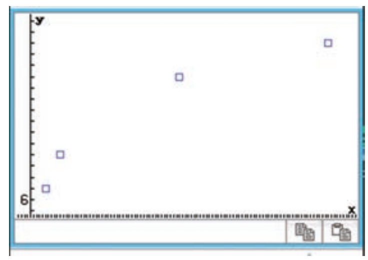
3 Construct a scatterplot of y against x.

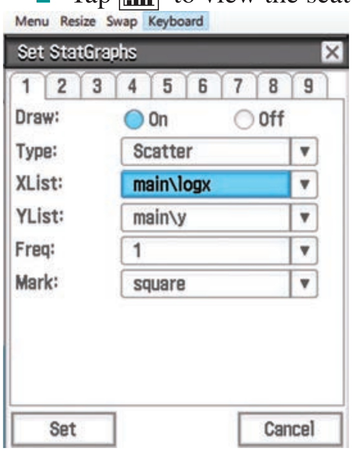
- Tap [icon] and complete the **Set StatGraphs** dialog box as shown.
- Tap [icon] to view the scatterplot. The scatterplot is clearly non-linear.





4 Construct a scatterplot of y against $\log_{10}(x)$.

- Tap [icon] and complete the **Set StatGraphs** dialog box as shown.
- Tap [icon] to view the scatterplot. The plot is now clearly linear.



Now try this 15 **Using a CAS calculator to perform the $\log_{10}(x)$ transformation (Example 15)**

For the given table of values, use the $\log_{10}(x)$ transformation to check if this transformation gives a linear graph.

x	10	100	500	1000
y	5	8	10.1	11

Hint 1 Follow instructions for the relevant CAS calculator above.

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Example 16 Applying the logarithmic transformation

Plot the heartbeat/minute of mammals against the logarithm of their body weight.

Mammal	Body weight (g)	Heartbeat/minute
Shrew	2.5	1400
Chick	50	400
Rabbit	1000	205
Monkey	5000	190
Tree kangaroo	8000	192
Giraffe	900 000	65
Elephant	5 000 000	30
Blue whale	170 000 000	16

Note 1: If we plot the heartbeat/minute of mammals against their body weight, we will be starting from a very small weight value of 2.5 grams for a shrew to 170 tonne = 170 000 kilograms = 170 000 000 grams for a blue whale.

Plotting the actual body weight values on a horizontal axis is difficult because of the large range of values for the body weight of mammals.

However, if the body weight values are written more compactly as logarithms (powers) of 10, then these logarithms can be placed on a logarithmic scale graph.

We have seen that $\log_{10} 100 = 2$. This can also be expressed as $\log(100) = 2$.

Note 2: $\log_{10} x$ is often written as $\log(x)$.

Explanation	Solution
1 Convert each mammal's body weight to logarithms. Weight of shrew is 2.5 grams. Find logarithm (log) of 2.5.	$\log(2.5) = 0.40$ (to two decimal places)
Weight of tree kangaroo is 8000 grams. Use a calculator to find log(8000).	$\log(8000) = 3.90$ (to two decimal places)
Weight of giraffe is 900 000 grams. Use a calculator to find log(900 000).	$\log(900\,000) = 5.95$ (to two decimal places)
Weight of blue whale is 170 000 000 grams. Use a calculator to find log(170 000 000).	$\log(170\,000\,000) = 8.23$ (to two decimal places)

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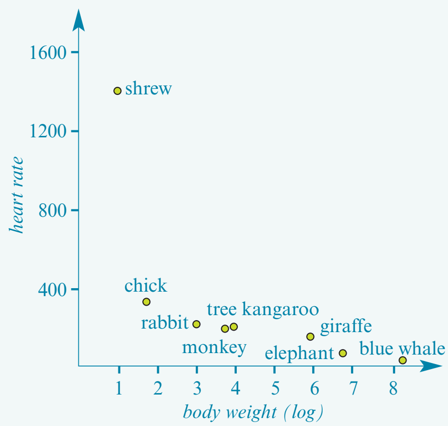
2 Use a calculator to find the logarithms (logs) of the body weight of the other mammals.
Record your results.

Mammal	Body weight (g)	log(*weight*)
Shrew	2.5	0.40
Chick	50	1.70
Rabbit	1000	3.00
Monkey	5000	3.70
Tree kangaroo	8000	3.90
Giraffe	900 000	5.95
Elephant	5 000 000	6.70
Blue whale	170 000 000	8.23

3 Plot the logarithms of the animals' body weights on the horizontal axis of the graph, and the heart rate on the vertical axis.



Now try this 16 **Applying the logarithmic transformation (Example 16)**

Plot the heart beat/minute of the animals below against the logarithm of their life span.

Animal	Life span (days)	Heartbeat/minute
Mayfly	1	1260
Fly	28	250
Mouse	365	600
Rabbit	3 650	205
Horse	10 950	40
Elephant	31 390	30
Galapagos Tortoise	69 350	6

Hint 1 Use a calculator to find the logarithm of each animal's life span.
Hint 2 Place the logarithms of the animal's life span on the horizontal axis.
Hint 3 The vertical axis will represent the heart rate.
Hint 4 Label axes and points clearly.

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On a log scale:

- In moving from 1 to 2 we are increasing by a factor of 10.
- In moving from 2 to 3 we are increasing by a factor of 10.
- In moving from 2 to 5 we are increasing by a factor of 1000 ($10^3 = 10 \times 10 \times 10$).

Example 17 Applying logarithms

Using the logarithmic values from the table in Example 16, how many times heavier than a rabbit is a giraffe? Give your answer to the nearest hundred.

Explanation	Solution
1 Subtract the logarithm of the weight of a rabbit from the logarithm of the weight of a giraffe.	$5.95 - 3.00$ $= 2.95$
2 The difference between these logarithms means that a giraffe is $10^{2.95}$ times heavier than a rabbit.	
3 Evaluate $10^{2.95}$	$10^{2.95} = 891.25$
4 Write your answer to the nearest hundred.	A giraffe is 900 times heavier than a rabbit.

Now try this 17 Applying logarithms (Example 17)

Using the logarithmic values from the table in Example 16, how many times heavier than a tree kangaroo is an elephant? Give your answer to the nearest whole number.

Hint 1 Subtract the logarithm of a tree kangaroo from the logarithm of an elephant.

Hint 2 Evaluate 10 to the power of this logarithm.

Section Summary

- A logarithm is the power to which a number must be raised in order to get another number.
- The order of magnitude of a physical quantity is its magnitude in powers of 10.
- A logarithmic transformation, $\log_{10}(x)$, can be used to linearise data.

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Exercise 9D

Building understanding

1 Write these numbers as powers of 10.

a 100 **b** 1000 **c** 10 **d** 1 **e** 10 000

2 Write down the logarithm (log) of the following:

a 10^5 **b** 10^8 **c** 10^0 **d** 10^9 **e** $10^{4.5}$

3 Use your calculator to evaluate to 2 decimal places:

a $10^{1.5}$ **b** $10^{2.2}$ **c** $10^{3.8}$ **d** $10^{0.7}$ **e** $10^{6.9}$

4 What is the order of magnitude of:

a 10^4 **b** 10^{33} **c** 10^{100}

Developing understanding

Example 10

5 Write the number as a power of 10, and then write down its logarithm.

a 1000 **b** 1 000 000 **c** 10 000 000 **d** 1 **e** 10

Example 11

6 Use your calculator to evaluate to three decimal places.

a log(300) **b** log(5946) **c** log(10 390) **d** log(7.25)

Example 12

7 Find the numbers, to two decimal places, with logarithms of:

a 2.5 **b** 1.5 **c** 0.5 **d** 0

Example 13

8 What is the order of magnitude of the following numbers?

a 46 000 **b** 559 **c** 3 000 000 000
d 4.21×10^{12} **e** 600 000 000 000

9 A city has two TAFE colleges with 4000 students each. What is the order of magnitude of the total number of TAFE students in the city?

10 At the football stadium, 35 000 people attend a football match each week. What is the order of magnitude of the number of people who attend 8 weeks of games?

11 A builder buys 9 boxes, each containing 1000 screws, to build a deck.

a What is the order of magnitude of the total number of screws?

Once the deck is completed, the number of screws left is 90.

b What is the order of magnitude of the number of screws that are left?

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Example 14

12 Use the $\log_{10}(x)$ transformation to check if plotting the values of $\log_{10}(x)$ and y is linear.

x	20	125	250	500	1000
y	8.6	10.2	10.8	11.4	12

Example 15

13 Use a CAS calculator and the $\log_{10}(x)$ transformation to check if plotting the values of $\log_{10}(x)$ and y is linear.

x	5	50	500	5000
y	3.7	4.7	5.7	6.7

Example 16
Example 17

14 Use the logarithmic values for the animals' weights in Example 16 to find how much heavier a tree kangaroo is than a shrew, to the nearest thousand.

Testing understanding

15 The radius of the planet Jupiter is 69 910 km. What is the order of magnitude of Jupiter's diameter?

16 If the logarithm of a gorilla's body weight (in kg) is 2.3, what is the gorilla's actual weight? Give your answer to two decimal places.

9E Further modelling of non-linear data

Learning intentions

- To be able to model non-linear data with $kx^2 + c$.
- To be able to model non-linear data with $\frac{k}{x} + c$.
- To be able to model non-linear data with $k\log(x) + c$.

Modelling non-linear data

Once we are able to transform data so that we have a straight line, we can then use our knowledge of straight-line graphs to model the non-linear data. We will investigate three different ways of modelling our data. They are, using:

- $y = kx^2 + c$
- $y = \dfrac{k}{x} + c$ where $k > 0$
- $y = k\log(x) + c$ where $k > 0$.

Instead of plotting y against x, we will plot y against x^2, $\dfrac{1}{x}$ or $\log_{10}(x)$. For each of these, k is the slope and c is the y-intercept.

In Chapter 5, we used the straight-line equation: $y = a + bx$, where a is the y-intercept and b is the slope.

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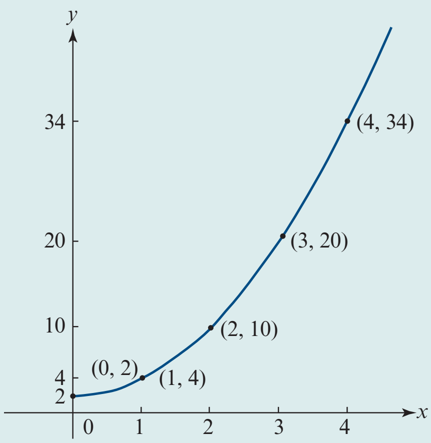
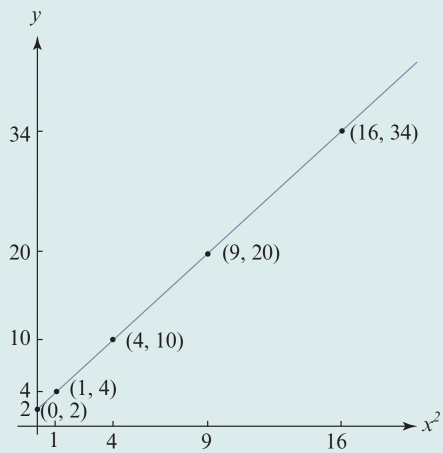
Example 18 Using the model $y = kx^2 + c$

The data in the given table has undergone an x^2 transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	0	1	2	3	4
x^2	0	1	4	9	16
y	2	4	10	20	34





Explanation

Since the graph of y against x^2 gives a straight line, we can find the equation of this straight line.

1 Find k by finding the slope of the straight-line graph. Select any two points on the line to find the slope.

2 Find the y-intercept (c). This is where the horizontal axis is 0 ($x^2 = 0$).

Alternatively, we can substitute k and a known value for x and y into the equation $y = kx^2 + c$ and solve for c.

3 Substitute k and c values into the equation $y = kx^2 + c$.

Solution

$k = \text{slope} = \dfrac{\text{rise}}{\text{run}}$

Use (4, 10) and (9, 20)

$k = \dfrac{20 - 10}{9 - 4} = \dfrac{10}{5} = 2$

The graph crosses the y-axis at 2.

So $c = 2$.

or

$y = kx^2 + c$ Use (2, 10)

$10 = 2(2^2) + c$

$c = 2$

$y = 2x^2 + 2$

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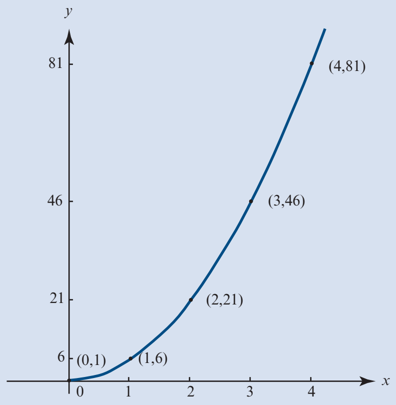
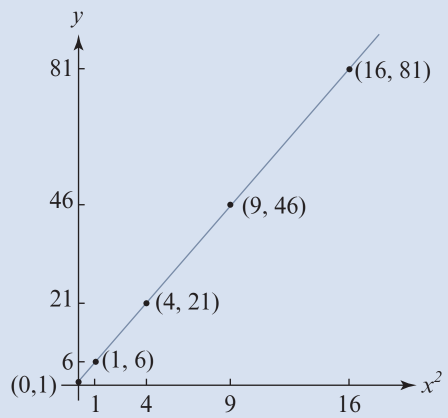
Now try this 18 **Using the model $y = kx^2 + c$ (Example 18)**

The data in the given table has undergone an x^2 transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	0	1	2	3	4
x^2	0	1	4	9	16
y	1	6	21	46	81





Hint 1 Find k, the slope of the straight-line graph.

Hint 2 Find c, the y-intercept.

Hint 3 Substitute k and c in the equation $y = kx^2 + c$.

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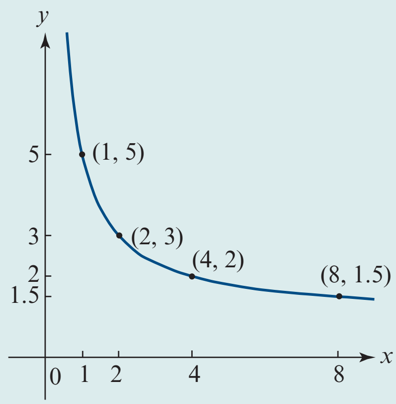
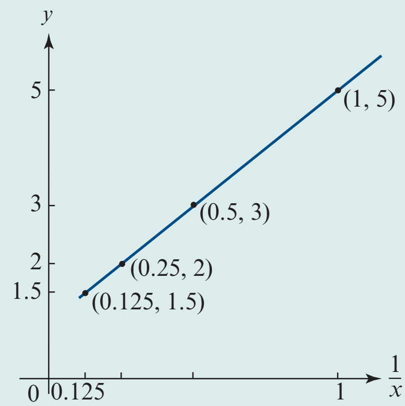
Example 19 Using the model $y = \frac{k}{x} + c$

The data in the given table has undergone a $\frac{1}{x}$ transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	1	2	4	8
$\frac{1}{x}$	1	0.5	0.25	0.125
y	5	3	2	1.5





Explanation	Solution
Since the graph of y against $\frac{1}{x}$ gives a straight line, we can find the equation of this straight line.	
1 The k value is given by the value of the slope. Find k by selecting any two points on the line. e.g. $(0.5, 3)$ and $(1, 5)$.	$k = \text{slope} = \frac{\text{rise}}{\text{run}}$ $k = \frac{5-3}{1-0.5} = \frac{2}{0.5} = 4$
2 Substitute k and a known value for x and y into the equation $y = \frac{k}{x} + c$ and solve for c.	$y = \frac{4}{x} + c$ Use (2, 3) $3 = \frac{4}{2} + c$ $\therefore c = 1$
3 Substitute values for k and c in the equation $y = \frac{k}{x} + c$.	$y = \frac{4}{x} + 1$

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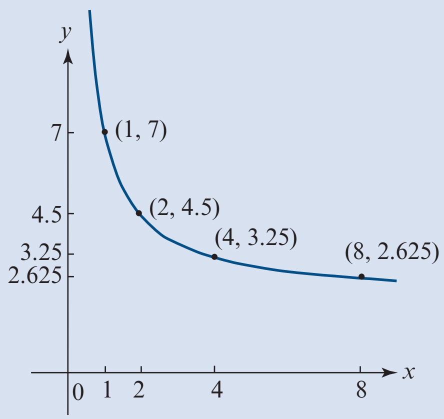
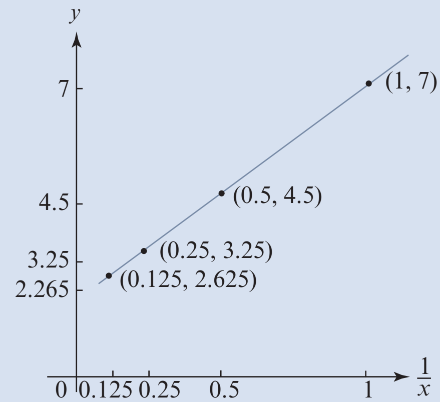
Now try this 19

Using the model $y = \frac{k}{x} + c$ (Example 19)

The data in the given table has undergone a $\frac{1}{x}$ transformation, and the graph of both the original data and the transformed data is shown. Find a rule connecting the variables x and y.

x	1	2	4	8
$\frac{1}{x}$	1	0.5	0.25	0.125
y	7	4.5	3.25	2.625

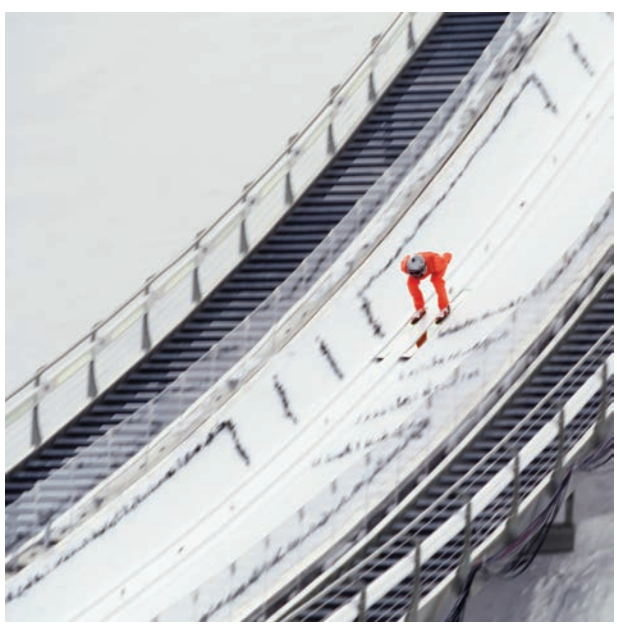




Hint 1 Find k, the slope of the straight-line graph.

Hint 2 Find c, the y-intercept.

Hint 3 Substitute k and c in the equation $y = \frac{k}{x} + c$.



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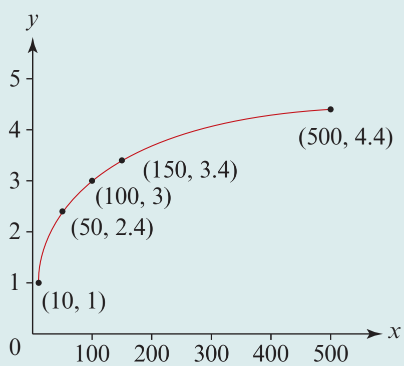
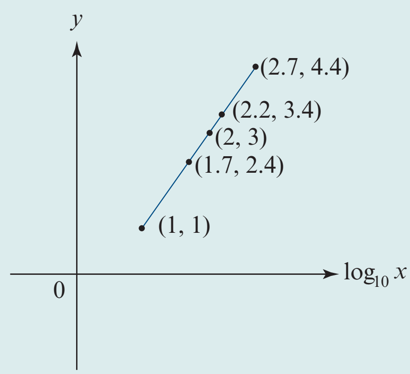
If the data can be linearised with a $\log_{10}(x)$ transformation, we can use the model $y = k\log_{10}(x) + c$.

Example 20 Using the model $y = k\log_{10}(x) + c$

The data in the given table has undergone a $\log_{10}(x)$ transformation, and the graph of the transformed data is shown. Find a rule connecting the variables x and y.

x	10	50	100	150	500
$\log_{10}(x)$	1	1.7	2	2.2	2.7
y	1	2.4	3	3.4	4.4





Explanation

Since the graph of y against $\log_{10}(x)$ gives a straight line, we can find the equation of this straight line.

1 The k value is given by the value of the slope. Find k by selecting any two points on the line, e.g. (1.7, 2.4) and (2, 3).

2 Find the y-intercept (c). Substitute k and a known value for x and y into the equation $y = k\log_{10}(x) + c$ and solve for c.

3 Substitute k and c values into the equation $y = k\log_{10}(x) + c$.

Solution

$k = \text{slope} = \dfrac{\text{rise}}{\text{run}}$

$k = \dfrac{3 - 2.4}{2 - 1.7} = \dfrac{0.6}{0.3} = 2$

$y = k\log_{10}(x) + c$ Use (10, 1)

$1 = 2\log_{10}(10) + c$

$\therefore c = -1$

$y = 2\log_{10}(x) - 1$

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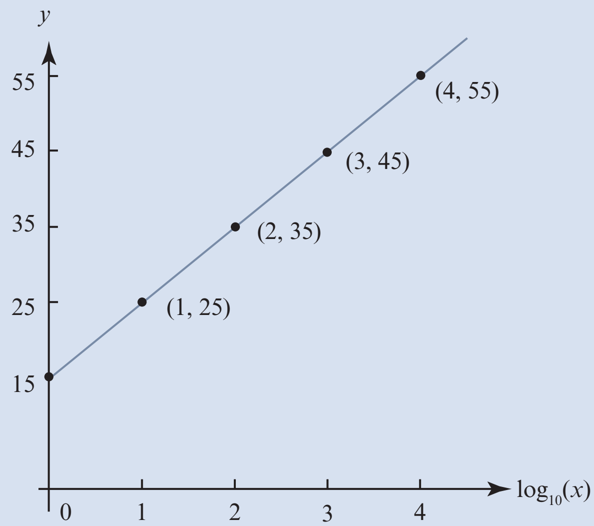
Now try this 20 Using the model $y = k\log_{10}(x) + c$ (Example 20)

The data in the given table has undergone a $\log_{10}(x)$ transformation, and the graph of the transformed data is shown. Find a rule connecting the variables x and y.

x	1	10	100	1000	10 000
$\log_{10}(x)$	0	1	2	3	4
y	15	25	35	45	55



Hint 1 Find the slope of the straight-line graph to give k.

Hint 2 c is the y-intercept.

Hint 3 Substitute k and c into the equation $y = k\log_{10}(x) + c$.

Section Summary

- Non-linear data can be modelled using:

1 $kx^2 + c$ **2** $\frac{k}{x} + c$ **3** $k\log_{10}(x) + c$

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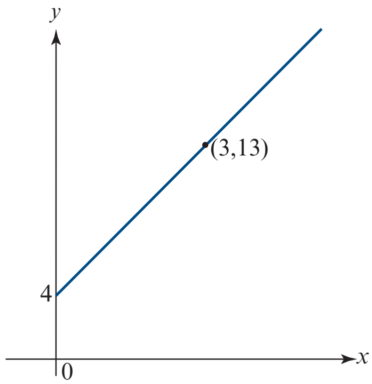
Skill-sheet

Exercise 9E

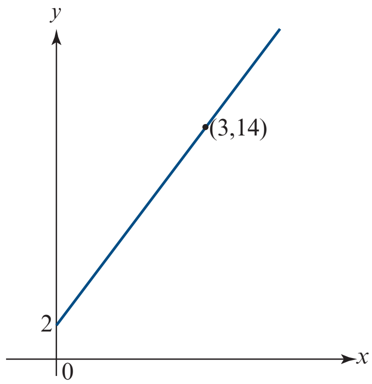
Building understanding

1 For the graphs shown below, give the slope and the y-intercept.

a



b



2 If y is plotted against x^2 for the equation $y = 3x^2 + 7$, what is the value of:

a the slope **b** the y-intercept?

3 If y is plotted against $\frac{1}{x}$ for the equation $y = \frac{2}{x} - 5$, what is the value of:

a the slope **b** the y-intercept?

4 If y is plotted against $\log_{10}(x)$ for the equation $y = 4\log_{10}(x) + 3$, what is the value of:

a the slope **b** the y-intercept?

Developing understanding

Example 18

5 The following data has undergone an x^2 transformation. Complete the table, sketch the graph of y against x^2, and find a rule connecting variables x and y.

x	1	2	3	4
x^2		4		16
y	5	11	21	35

Example 19

6 The following data has undergone a $\frac{1}{x}$ transformation. Complete the table, sketch the graph of y against $\frac{1}{x}$, and find a rule connecting variables x and y.

x	0.1	0.2	0.5	1
$\frac{1}{x}$	10		2	
y	102	52	22	12

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Example 20

7 The following data has undergone a $\log_{10}(x)$ transformation. Complete the table, sketch the graph of y against $\log_{10}(x)$, and find the relationship between x and y.

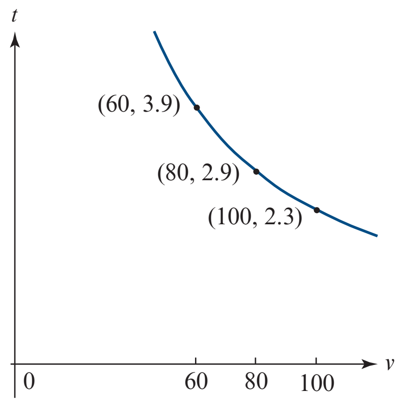
x	10	100	1000	10 000
$\log_{10}(x)$	1			4
y	11	17	23	29

Testing understanding

8 During a mice plague, the population, P, of mice increases as shown in the table, where t is the time in days. Perform a squared transformation on the variable time, t, and find a rule which allows population, P, to be predicted from time, t.

t	1	5	10	20
P	15	135	510	2010

9 The time taken to travel from Santiago to Valparaiso, in Chile, and the average speed travelled by a car is shown in the graph below.



a Find a rule describing the relationship between the time taken to travel from Santiago to Valparaiso, t hours, and the average speed, v km/hr.
Hint: Use $\frac{1}{v}$ transformation.

b If it takes Pablo two hours and six minutes to travel from Santiago to Valparaiso, what was his average speed? Give your answer to the nearest kilometre/hour.

10 The table below gives the number of people, N, infected by a virus after t days. Apply a $\log_{10}(t)$ transformation to find the relationship between N and t.

t	1	10	100	500	1000
N	15	115	215	285	315

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Key ideas and chapter summary

Assign-ment

Direct variation

y **varies directly** as x is written as $\boldsymbol{y \propto x}$

As x increases, y will also increase.

If $\boldsymbol{y \propto x}$, then $y = kx$, where k is the constant of proportionality.

The graph of y against x is a straight line through the origin.

Inverse variation

y **varies inversely** as x is written as $\boldsymbol{y \propto \frac{1}{x}}$

As x increases, y will decrease.

If $\boldsymbol{y \propto \frac{1}{x}}$ then $y = \frac{k}{x}$, where k is the constant of proportionality.

Order of magnitude

The order of magnitude of a physical quantity is its magnitude in powers of 10.

Transformations to linearity

Relationships between variables can be established by transforming data to linearity. This process is called **linearisation**.

The **squared** transformation involves changing the $\boldsymbol{x}$ values to $\boldsymbol{x^2}$.

The **reciprocal** transformation involves changing the $\boldsymbol{x}$ values to $\boldsymbol{\frac{1}{x}}$.

The **logarithmic** transformation involves changing the $\boldsymbol{x}$ values to $\boldsymbol{\log_{10}(x)}$.

Modelling non-linear data

We can model non-linear data using either:

$y = kx^2 + c$,

$y = \frac{k}{x}$ or

$y = k\log_{10}(x) + c$ where $k > 0$.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

9A **1** I can solve for the constant of variation, k, in direct variation. ☐

e.g. The distance (d) travelled by a vehicle is directly proportional to the time (t) taken. Using the information in the following table, find the constant of variation.

Time (t)	1	2	3
Distance (d)	95	190	285

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Review

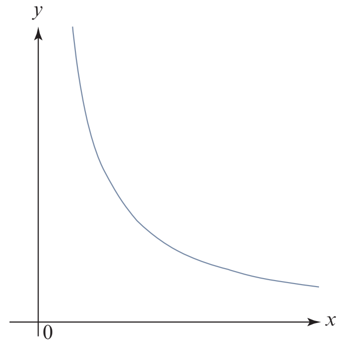
9A **2** I can solve for the constant of variation, k, in inverse variation. ☐

e.g. If F is inversely proportional to d, use the following table to find the constant of variation.

d	10	20	50
F	10	5	2

9B **3** I can determine direct or inverse variation from a graph. ☐

e.g. Does the following graph show direct or inverse variation?



9C **4** I can use the x^2 transformation (the squared transformation). ☐

e.g. Complete the following table giving an x^2 transformation, and check that this linearises the data.

x	1	2	5
x^2			
y	12	27	132

9C **5** I can use the $\frac{1}{x}$ transformation (the reciprocal transformation). ☐

e.g. Complete the following table giving a $\frac{1}{x}$ transformation, and check that this linearises the data.

x	0.5	0.2	0.1
$\frac{1}{x}$			
y	5	35	85

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9D **6** I can evaluate logarithms. ☐

e.g. Find the log of 592 to two decimal places.

9D **7** I can state the order of magnitude of a number. ☐

e.g. What is the order of magnitude of 478 000?

9D **8** I can use and interpret log scales when used to represent quantities that range over multiple orders of magnitude. ☐

e.g. Using the logarithmic values for the mammals' weights in Example 16, find how much heavier than a monkey is an elephant.

9D **9** I can use a calculator to evaluate a number if the logarithm is known. ☐

e.g. Find the number whose logarithm is 2.1567 to one decimal place.

9D **10** I can use the $\log_{10}(x)$ transformation (log transformation). ☐

e.g. Complete the following table giving a $\log_{10}(x)$ transformation, and check that this linearises the data.

x	10	100	1000
$\log_{10}(x)$			
y	3	6	9

9E **11** I can model non-linear data using either: ☐

$$y = kx^2 + c, \quad y = \frac{k}{x} + c \quad \text{or} \quad y = k\log_{10}(x) + c.$$

e.g. Find a rule connecting the variables x and y if the data in the given table has undergone an x^2 transformation.

x	1	5	10	15
x^2	1	25	100	225
y	23	95	320	695

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Review

Multiple-choice questions

1 For the values in the table shown, it is known that $y \propto x$. The value of k, the constant of variation, is equal to:

x	0	3	9	18
y	0	12	36	72

A 1 **B** 3 **C** 4 **D** 9 **E** 12

2 For the values in the table shown, it is known that $y \propto x$. The value of k, the constant of variation, is equal to:

x	2	8	16
y	20	80	160

A 2 **B** 4 **C** 10 **D** 20 **E** 60

3 For the values in the table shown, it is known that $y \propto \frac{1}{x}$. The value of k, the constant of variation, is equal to:

x	1	2	3	4
y	3	1.5	1	0.75

A $\frac{1}{3}$ **B** $\frac{1}{2}$ **C** 1 **D** 2 **E** 3

4 For the values in the table shown, it is known that $y \propto x^2$. The value of k, the constant of variation, is equal to:

x	2	3	6
y	$\frac{4}{3}$	3	12

A 3 **B** 9 **C** $\frac{1}{3}$ **D** 2 **E** $\frac{4}{3}$

5 For the values in the table shown, it is known that $y \propto \frac{1}{x}$. The value of k, the constant of variation, is equal to:

x	2	4	8
y	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

A $\frac{1}{2}$ **B** 1 **C** 4 **D** 2 **E** $\frac{1}{4}$

6 Assume that $a \propto b$, and that $a = 18$ when $b = 3$. If $b = 7$, then $a =$

A 4 **B** 6 **C** 15 **D** 22 **E** 42

7 Assume that $a \propto b^2$ and that $a = 32$ when $b = 2$. If $b = 4$, then $a =$

A 4 **B** 16 **C** 32 **D** 64 **E** 128

8 Assume that $p \propto \frac{1}{q}$ and that $p = \frac{1}{3}$ when $q = 3$. If $p = 1$, then $q =$

A -3 **B** $\frac{1}{3}$ **C** $\sqrt{3}$ **D** 1 **E** -3

9 The given table of values follows the rule: $y = kx^2 + c$.

x	1	2	3	4
y	4	22	52	94

The values of k and c respectively are:

A 6 and 2 **B** 6 and -2 **C** 2 and -1 **D** -2 and 6 **E** 2 and 1

ISBN 978-1-009-11034-1

10 The given table of values follows the rule: $y = \frac{k}{x} + c$.

x	1	2	4	5
y	6	3.5	2.25	2

The values of k and c respectively are:

A 5 and 1 **B** 2 and 3 **C** 2 and 4 **D** 4 and 2 **E** 4 and 1

11 The following data can be modelled by $y = k\log_{10}(x) + c$.

x	1	10	100	1000
y	50	350	650	950

The values of k and c respectively are:

A 1 and 50 **B** 1 and 300 **C** 50 and 0 **D** 50 and −150 **E** 300 and 50

Short-answer questions

1 If $a \propto b$ and $a = 8$ when $b = 2$,

a find a when $b = 50$

b find b when $a = 60$

2 If $y \propto x^2$ and $y = 108$ when $x = 6$,

a find y when $x = 4$

b find x when $y = 90$. Give your answer to two decimal places.

3 If $y \propto \frac{1}{x}$ and $y = 3$ when $x = 2$,

a find y when $x = \frac{1}{2}$

b find x when $y = \frac{2}{3}$

4 The distance, d metres, which an object falls, varies directly as the square of the time, t seconds, for which it has been falling. If an object falls 78.56 m in 4 s, find (to two decimal places):

a the constant of variation

b the formula connecting d and t

c the distance fallen in 10 s

d the time taken to fall 19.64 m (to the nearest second).

ISBN 978-1-009-11034-1

5 The time taken for a journey is inversely proportional to the average speed of travel. If it takes 4 hours travelling at 30 km/h, how long will it take travelling at 50 km/h?

6 For a constant resistance, the voltage (v volts) of an electrical circuit varies directly as the current (I amps). If the voltage is 24 volts when the current is 6 amps, find the current when the voltage is 72 volts.

7 The number of square tiles needed to surface the floor of a hall varies inversely as the square of the side length of the tile used. If 2016 tiles of side length 0.4 m would be needed to surface the floor of a certain hall, how many tiles of side length 0.3 m would be required?

8 The time taken to fill a tank with water varies inversely as the volume of water poured in per minute. It takes 45 minutes to fill a tank using a pipe with a flow rate of 22 litres per minute. What flow rate is being used if it takes 30 minutes to fill the tank?

9 How many times stronger is a magnitude 6 earthquake compared to a magnitude 3 earthquake?

Written-response questions

1 A certain type of hollow sphere is designed in such a way that the mass varies directly as the square of the diameter. Three spheres of this type are made. The first has mass 5 kg and diameter 10 cm, the second has diameter 14 cm and the third has mass 6 kg. Find, to two decimal places:

- **a** the mass of the second sphere
- **b** the diameter of the third sphere.

2 The time taken, t, to paint a building varies inversely as the number, n, of painters working. It takes 6 painters 20 days to paint a building.

- **a** How long will it take 4 painters?
- **b** The building needs to be painted in 15 days. How many painters should be employed?

ISBN 978-1-009-11034-1

Cambridge University Press

3 **a** The air in a tube occupies 43.5 cm^3 and the pressure is 2.8 kg/cm^2. If the volume (V) varies inversely as the pressure (P), find the formula connecting V and P.

b Calculate the pressure when the volume is decreased to 12.7 cm^3 (to two decimal places).

4 The weight (w kg) which a beam supported at each end will carry without breaking varies inversely as the distance (d m) between supports. A beam which measures 6 m between supports will just carry a load of 500 kg.

a Find the formula connecting w and d.

b What weight could the beam carry if the distance between the supports were 5 m?

c What weight could the beam carry if the distance between the supports were 9 m? Give your answer to two decimal places.

5 The table shows the relationship between the pressure and the volume of a fixed mass of gas when the temperature is constant.

Pressure (p)	12	16	18
Volume (v)	12	9	8

a What is a possible equation relating p and v?

b Using this equation, find:

i the volume when the pressure is 72 units

ii the pressure when the volume is 3 units.

c Sketch the graph relating v and $\frac{1}{p}$.

ISBN 978-1-009-11034-1 Cambridge University Press