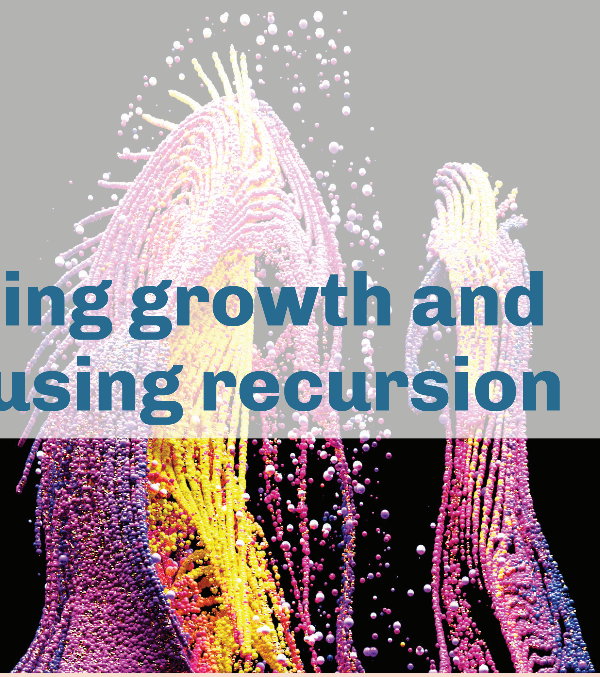
Chapter 7

Modelling growth and decay using recursion



Chapter objectives

- What is a sequence?
- How do we generate a sequence of numbers from a starting value and a rule?
- How do we identify particular terms in a sequence?
- What is recursion?
- What is linear growth and decay?
- How can recurrence relations be used to model simple interest, flat rate depreciation and unit cost depreciation on assets?
- How can recurrence relations be used to model compound interest and reducing-balance depreciation on assets?
- How can the CAS calculator be used to find the length of time or the necessary interest rate required for an investment or loan to reach a particular value?
- How can investments and loans be compared using effective interest rates?

In this chapter, the notion of a sequence and recurrence relation are introduced as well as the concepts of linear growth and decay and geometric growth and decay. Taken together, these ideas are applied to financial situations including investments, loans and the depreciation of assets to investigate how much interest must be paid on a loan, how much interest an investment earns or how much an asset depreciates, under different assumptions.

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7A Sequences and recurrence relations

Learning intentions

- To be able to generate a sequence of terms recursively.
- To be able to generate a sequence of numbers from a worded description using a calculator.
- To be able to generate a sequence from a recurrence relation.
- To be able to generate a sequence of numbers from a recurrence relation using a calculator.
- To be able to number and name terms in a sequence.

A list of numbers, written down in succession, is called a **sequence**. Each of the numbers in a sequence is called a **term**. We write the terms of a sequence as a list, separated by commas. If a sequence continues indefinitely, or if there are too many terms in the sequence to write them all, we use an *ellipsis*, '. . .'.

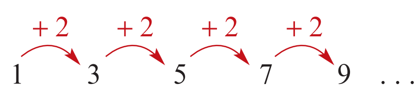
Sequences may be either generated randomly or by **recursion** using a rule. For example, this sequence

1, 3, 5, 7, 9, . . .

has a definite pattern.

The sequence of numbers has a starting value of 1. We add 2 to this number to generate the next term, 3. Then, add 2 again to generate the next term, 5, and so on.

The rule is 'add 2 to each term'.



Example 1 Generating a sequence of terms recursively (1)

Write down the first five terms of the sequence with a starting value of 6 and the rule 'add 4 to the previous term'.

Explanation	Solution
1 Write down the starting value.	6
2 Apply the rule (add 4) to generate the next term.	$6 + 4 = 10$
3 Calculate three more terms.	$10 + 4 = 14$ $14 + 4 = 18$ $18 + 4 = 22$
4 Write your answer.	The first five terms are 6, 10, 14, 18, 22.

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Example 2 Generating a sequence of terms recursively (2)

Write down the first five terms of the sequence with a starting value of 5 and the rule 'double the number and then subtract 3'.

Explanation	Solution
1 Write down the starting value.	5
2 Apply the rule (double **5**, then subtract 3) to generate the next term.	$5 \times 2 - 3 = 7$
3 Calculate three more terms.	$7 \times 2 - 3 = 11$ $11 \times 2 - 3 = 19$ $19 \times 2 - 3 = 35$
4 Write your answer.	The first five terms are 5, 7, 11, 19, 35.

Using a calculator to generate a sequence of numbers from a rule

All of the calculations to generate sequences from a rule are repetitive. The same calculations are performed over and over again – this is called *recursion*. A calculator can perform recursive calculations very easily because it automatically stores the answer to the last calculation it performed, as well as the method of calculation.

Example 3 Generating a sequence of numbers with a calculator

Use a calculator to generate the first five terms of the sequence with a starting value of 5 and the rule 'double and then subtract 3'.

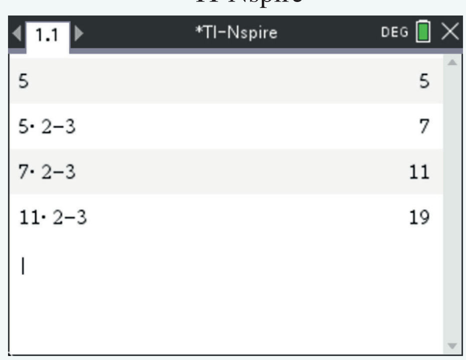
Explanation

Steps

1. Start with a blank computation screen.
2. Type 5 and press enter or EXE.
3. Next type **×2 – 3** and press enter or EXE to generate the next term in the sequence. The computation generating this value is shown as '5·2–3' on the TI-Nspire and 'ans × 2 – 3 on the ClassPad (here 'ans' represents the answer to the previous calculation).

Solution

TI-Nspire



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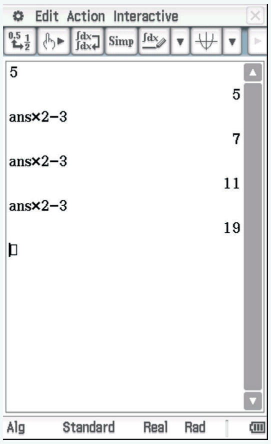
4 Pressing enter or EXE repeatedly applies the rule ×2 – 3 to the last calculated value.

ClassPad



5 State your answer.

The first five terms are 5, 7, 11, 19, 35

Recurrence relations

A **recurrence relation** is a mathematical rule that we can use to generate a sequence. It has two parts:

1 a *starting value*: the value of the first term in the sequence

2 a *rule*: that can be used to generate the next term from the current term.

For example, in words, a recurrence relation that can be used to generate the sequence:

10, 15, 20, . . .

can be written as follows:

1 Start with 10.

2 To obtain the next term, add 5 to the current term.

A more compact way of communicating this information is to translate this rule into symbolic form. We do this by defining a subscripted variable. Here we will use the variable V_n, but the V can be replaced by any letter of the alphabet.

Let V_n be the term in the sequence *after n* applications of the rule, called **iterations**.

In words	In symbols
Starting value = 10	$V_0 = 10$
Next term = current term +5	$V_{n+1} = V_n + 5$

Using this definition, we can write a formal recurrence relation where the starting value is defined, followed by the rule for generating the next term.

$$V_0 = 10, \qquad V_{n+1} = V_n + 5$$

Note: Because of the way we defined V_n, the starting value of n is 0. At the start there have been no applications of the rule.

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Example 4 Generating a sequence from a recurrence relation

Write down the first five terms of the sequence defined by the recurrence relation

$$V_0 = 29, \qquad V_{n+1} = V_n - 4$$

Explanation	Solution
1 Write down the starting value.	$V_0 = 29$
2 Use the rule to find the next term, V_1.	$V_1 = V_0 - 4$ $= 29 - 4$ $= 25$
3 Use the rule to determine three more terms.	$V_2 = V_1 - 4 = 25 - 4 = 21$ $V_3 = V_2 - 4 = 21 - 4 = 17$ $V_4 = V_3 - 4 = 17 - 4 = 13$
4 Write your answer.	The first five terms are 29, 25, 21, 17, 13

Example 5 Using a calculator to generate sequences from recurrence relations

A sequence is generated by the recurrence relation

$$V_0 = 300, \qquad V_{n+1} = 0.5V_n - 9$$

Use your calculator to generate this sequence and determine how many terms at the start of the sequence are positive.

Explanation

1 Start with a blank computation screen.

2 Type **300** and press [enter] (or [EXE]).

3 Next type ×0.5 − 9 and press [enter] (or [EXE]) to generate the next term in the sequence.

4 Continue to press [enter] (or [EXE]) until the first negative term appears.

5 Write your answer.

Solution

300	300.
$300 \cdot 0.5 - 9$	141.
$141 \cdot 0.5 - 9$	61.5
$61.5 \cdot 0.5 - 9$	21.75
$21.75 \cdot 0.5 - 9$	1.875
$1.875 \cdot 0.5 - 9$	−8.625

The first five terms of the sequence are positive.

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Example 6 Naming terms in a sequence

Consider the recurrence relation

$$V_0 = 3, \qquad V_{n+1} = V_n + 6$$

State the values of:

a V_1 **b** V_4 **c** V_5

Explanation	Solution
1 Write the name for each term under its value in the sequence.	3, 9, 15, 21, 27, 33 V_0 V_1 V_2 V_3 V_4 V_5
2 Read the value of each required term.	$V_1 = 9$, $V_4 = 27$ $V_5 = 33$

Exercise 7A

Generating a sequence recursively

Example 1

1 Use the following starting values and rules to generate the first five terms of the following sequences recursively.

a Starting value: 2
Rule: add 6

b Starting value: 5
Rule: subtract 3

c Starting value: 1
Rule: multiply by 4

d Starting value: 64
Rule: divide by 2

Example 2

2 Use the following starting values and rules to generate the first five terms of the following sequences recursively.

a Starting value: 6
Rule: multiply by 2 then add 2

b Starting value: 24
Rule: multiply by 0.5 then add 4

c Starting value: 1
Rule: multiply by 3 then subtract 1

d Starting value: 124
Rule: multiply by 0.5 then subtract 2

Example 3

3 Use the following starting values and rules to generate the first five terms of the following sequences recursively using a calculator.

a Starting value: 4
Rule: add 2

b Starting value: 24
Rule: subtract 4

c Starting value: 2
Rule: multiply by 3

d Starting value: 50
Rule: divide by 5

e Starting value: 5
Rule: multiply by 2 then add 3

f Starting value: 18
Rule: multiply by 0.8 then add 2

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Generating sequences using recurrence relations

Example 4

4 Write down the first five terms of the sequences generated by each of the recurrence relations below.

a $W_0 = 2$, $W_{n+1} = W_n + 3$
b $D_0 = 50$, $D_{n+1} = D_n - 5$
c $M_0 = 1$, $M_{n+1} = 3M_n$
d $L_0 = 3$, $L_{n+1} = -2L_n$
e $K_0 = 5$, $K_{n+1} = 2K_n - 1$
f $F_0 = 2$, $F_{n+1} = 2F_n + 3$
g $S_0 = -2$, $S_{n+1} = 3S_n + 5$
h $V_0 = -10$, $V_{n+1} = -3V_n + 5$

Example 5

5 Using your calculator, write down the first five terms of the sequence generated by each of the recurrence relations below.

a $A_0 = 12$, $A_{n+1} = 6A_n - 15$
b $Y_0 = 20$, $Y_{n+1} = 3Y_n + 25$
c $V_0 = 2$, $V_{n+1} = 4V_n + 3$
d $H_0 = 64$, $H_{n+1} = 0.25H_n - 1$
e $G_0 = 48\,000$, $G_{n+1} = G_n - 3000$
f $C_0 = 25\,000$, $C_{n+1} = 0.9C_n - 550$

Example 6

6 Consider the following recurrence relations. Find the required term for each.

a $A_0 = 2$, $A_{n+1} = A_n + 2$. Find A_2.
b $B_0 = 11$, $B_{n+1} = B_n - 3$. Find B_4
c $C_0 = 1$, $C_{n+1} = 3C_n$. Find C_3
d $D_0 = 3$, $D_{n+1} = 2D_n + 1$. Find D_5

7 Write a recurrence relation for each of the following worded descriptions.

a Starting value: 4
Rule: add 2

b Starting value: 24
Rule: subtract 4

c Starting value: 2
Rule: multiply by 3

8 State a recurrence relation that could be used to generate each of the following sequences.

a $5, 10, 15, 20, 25, \ldots$
b $13, 9, 5, 1, -3, \ldots$
c $1, 4, 16, 64, 256, \ldots$
d $64, 32, 16, 8, 4, \ldots$

Exploring sequences with a calculator

9 How many terms of the sequence formed from the recurrence relation below are positive?

$$F_0 = 150, \; F_{n+1} = 0.6F_n - 5$$

10 How many terms of the sequence formed from the recurrence relation below are negative?

$$Y_0 = 30, \; Y_{n+1} = 1.2Y_n + 2$$

Exam 1 style questions

11 A sequence of numbers is generated by the recurrence relation shown below

$$A_0 = 3, \quad A_{n+1} = 4A_n + 1$$

The value of A_4 is

A 3 **B** 4 **C** 13 **D** 213 **E** 853

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12 The following recurrence relation can generate a sequence of numbers

$$A_0 = 15, \qquad A_{n+1} = A_n + 4$$

The number 51 appears in this sequence as

A A_1 **B** A_7 **C** A_8 **D** A_9 **E** A_{10}

13 The first five terms of a sequence are

$$3, 7, 15, 31, 63$$

The recurrence relation that generates this sequence could be

A $B_0 = 3, \quad B_{n+1} = B_n + 4$

B $B_0 = 3, \quad B_{n+1} = B_n + 8$

C $B_0 = 3, \quad B_{n+1} = 3B_n - 1$

D $B_0 = 3, \quad B_{n+1} = 4B_n - 5$

E $B_0 = 3, \quad B_{n+1} = 2B_n + 1$

7B Modelling linear growth and decay

Learning intentions

- To be able to graph the terms of a linear growth/decay sequence.
- To be able to model simple interest loans and investments using recurrence relations.
- To be able to use a recurrence relation to analyse a simple interest investment.
- To be able to model and analyse flat rate depreciation using a recurrence relation.
- To be able to model and analyse unit cost depreciation using a recurrence relation.

Linear growth means a value is increasing by the same amount in each unit of time. For example, if you have \$300 in your bank account and you add \$20 each week, then your savings will have linear growth. Similarly, **linear decay** is characterised as decreasing by the same amount in each unit of time. For example, the depreciation of a new car by a constant amount each year.

A recurrence model for linear growth and decay

The recurrence relations

$$P_0 = 20, \qquad P_{n+1} = P_n + 2$$
$$Q_0 = 20, \qquad Q_{n+1} = Q_n - 2$$

both have rules that generate sequences with linear patterns, as can be seen from the table below. The first generates a sequence whose successive terms have a linear pattern of growth, and the second a linear pattern of decay.

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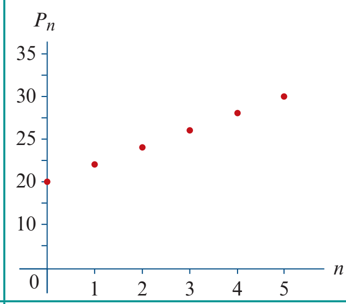
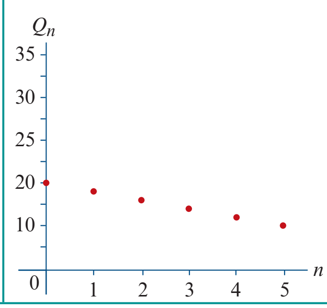
Recurrence relation	*Rule*	*Sequence*	*Graph*
$P_0 = 20, P_{n+1} = P_n + 2$	'add 2'	20, 22, 24, ...	
$Q_0 = 20, Q_{n+1} = Q_n - 2$	'subtract 2'	20, 18, 16, ...	

As a general rule, if D is a positive constant, a recurrence relation rule of the form:

- $V_{n+1} = V_n + D$ can be used to model **linear growth**.
- $V_{n+1} = V_n - D$ can be used to model **linear decay**.

We refer to D as the **common difference** and can graph the sequence to obtain a straight line graph of dots (do not join the dots). An upward slope indicates growth and a downward slope reveals decay.

Example 7 Graphing the terms of linear growth/decay sequence

For each of the following recurrence relations, list the first four terms and graph the corresponding points.

a $V_0 = 2, \quad V_{n+1} = V_n + 5$

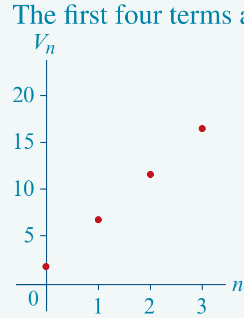
b $W_0 = 20, \quad W_{n+1} = W_n - 3$

Explanation

a From the rule, the starting value is 2.
The rule is 'add 5'.
The corresponding points can then be graphed.

Solution

The first four terms are 2, 7, 12, 17.



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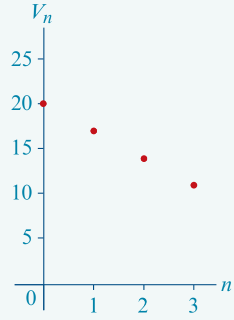
b From the rule, the starting value is 20.
The rule is 'subtract 3'.
The corresponding points can then be graphed.

The first four terms are 20, 17, 14, 11.



Simple interest loans and investments

Simple interest is an example of linear growth in which the starting value is the amount borrowed or invested. The amount borrowed or invested is called the **principal**. The amount added at each step is the interest and is usually a percentage of this principal, found by multiplying the annual interest rate $r\%$ by the principal for each year of the loan.

Recurrence model for simple interest

Let V_n be the value of the loan or investment after n years and r be the annual percentage **interest rate**.

The recurrence relation for the value (or **balance**) of the loan or investment after n years is

$$V_0 = \text{principal}, \qquad V_{n+1} = V_n + D$$

where $D = \frac{r}{100} \times V_0$.

Example 8 Modelling simple interest investments with a recurrence relation

Cheryl invests \$5000 in an investment account that pays 4.8% per annum simple interest.

Model this simple investment using a recurrence relation of the form:

$$V_0 = \text{the principal}, \qquad V_{n+1} = V_n + D, \qquad \text{where } D = \frac{r}{100}V_0.$$

Let V_n be the value of the investment after n years.

Explanation	Solution
1 Write down the value of V_0.	$V_0 = 5000$
2 Write down the interest rate r and use it to determine the value of $D = \frac{r}{100}V_0$.	$r = 4.8$ $D = \frac{4.8}{100} \times 5000 = 240$
3 Use the values of V_0 and D to write down the recurrence relation.	$V_0 = 5000, \qquad V_{n+1} = V_n + 240.$

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Once we have a recurrence relation, we can use it to determine the value of an investment after a given number of years.

Example 9 **Using a recurrence relation to analyse a simple interest investment**

Cheryl's simple interest investment is modelled by

$$V_0 = 5000, \qquad V_{n+1} = V_n + 240$$

where V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of Cheryl's investment after 3 years is \$5720.

b When will Cheryl's investment first exceed \$6000, and what will its value be then?

Explanation

a Calculate V_0, V_1, V_2 and V_3.

Solution

$V_0 = 5000$

$V_1 = 5000 + 240 = 5240$

$V_2 = 5240 + 240 = 5480$

$V_3 = 5480 + 240 = 5720$

Thus, after three years, the value of Cheryl's investment is \$5720.

Explanation

b **i** On a blank calculation screen, type **5000** and press enter (or EXE).

ii Type **+240** and press enter (or EXE) until the value of the investment first exceeds \$6000.

iii Count the number of times that 240 was added. Write your answer.

Solution

5000	5000.
5000. + 240	5240.
5240. + 240	5480.
5480. + 240	5720.
5720. + 240	5960.
5960. + 240	6200.

After 5 years; \$6200.

Depreciation

For some large items, their value decreases over time. This is called **depreciation**. Businesses take into account the impact of depreciation by tracking the likely value of an asset at a point in time, called the **future value**. At some point in time or at a particular value, called the **scrap value**, the item will be sold or disposed of as it is no longer useful to the business.

There are a number of techniques for estimating the future value of an asset. Two of them, **flat rate depreciation** and **unit cost depreciation**, can be modelled using linear decay recurrence relations.

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Flat rate depreciation

Flat rate depreciation is an example of linear decay where a constant amount is subtracted from the value of the asset each time period. This constant amount is called the depreciation amount and is often given as a percentage of the initial purchase price of the asset. The **scrap value** is the value at which the item is no longer of use to the business.

Recurrence model for flat-rate depreciation

Let V_n be the value of the asset after n years and r be the percentage depreciation rate.

The recurrence relation for the value of the asset after n years is

$$V_0 = \text{initial value of the asset}, \qquad V_{n+1} = V_n - D$$

where $D = \frac{r}{100} \times V_0$.

Example 10 Modelling flat rate depreciation with a recurrence relation

A new car was purchased for \$24 000 in 2014. The car depreciates by 20% of its purchase price each year. Model the depreciating value of this car using a recurrence relation of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n - D, \qquad \text{where } D = \frac{r}{100}V_0$$

Let V_n be the value of the car after n years depreciation.

Explanation	Solution
1 Write down the value of V_0. Here, V_0 is the value of the car when new.	$V_0 = 24\,000$
2 Write down the annual rate of depreciation, r, and use it to determine the value of $D = \frac{r}{100}V_0$.	$r = 20$ $D = \frac{20}{100} \times 24\,000 = 4800$
3 Use the values of V_0 and D to write down the recurrence relation.	$V_0 = 24\,000, \qquad V_{n+1} = V_n - 4800$

Once we have a recurrence relation, we can use it to determine things such as the value of an asset after a given number of years of flat rate depreciation.

Example 11 Using a recurrence relation to analyse flat rate depreciation

The flat rate depreciation of a car is modelled by

$$V_0 = 24\,000, \qquad V_{n+1} = V_n - 4800$$

where V_n is the value of the car after n years.

a Use the model to determine the value of the car after 2 years.

b If the car was purchased in 2023, in what year will the car's value depreciate to zero?

c What was the percentage depreciation rate?

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Explanation

a **i** Write down the recurrence relation.

ii On a blank calculation screen, type **24 000** and press enter (or EXE).

iii Type **–4800** and press enter (or EXE) twice to obtain the value of the car after 2 years' depreciation. Write your answer.

b **i** Continue pressing enter (or EXE) until the car has no value.

ii Write your answer.

c Use the amount of depreciation and initial value.

Solution

$V_0 = 24\,000, \ V_{n+1} = V_n - 4800$

24000	24000.
24000. – 4800	19200.
19200. – 4800	14400.
14400. – 4800	9600.
9600. – 4800	4800.
4800. – 4800	0.

a \$14 400

b In 2028

c $\frac{4800}{24000} \times 100\% = 20\%$

The percentage depreciation rate is 20%

Unit cost depreciation

Some items lose value because of how often they are used. A photocopier that is 2 years old but has never been used could still be considered to be in 'brand new' condition and therefore worth the same as it was 2 years ago. But if that photocopier was 2 years old and had printed many thousands of pages over those 2 years, it would be worth much less than its original value.

When the future value of an item is based upon usage, we use a **unit cost** depreciation method. Unit cost depreciation can be modelled using a linear decay recurrence relation.

Recurrence model for unit-cost depreciation

Let V_n be the value of the asset after n units of use and D be the cost per unit of use.

The recurrence relation for the value of the asset after n units of use is:

$$V_0 = \text{initial value of the asset}, \qquad V_{n+1} = V_n - D$$

Example 12 Modelling unit cost depreciation with a recurrence relation

A professional gardener purchased a lawn mower for \$270. The mower depreciates in value by \$3.50 each time it is used.

a Model the depreciating value of this mower using a recurrence relation of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n - D$$

where D is the depreciation in value per use and V_n is the value of the mower after being used to mow n lawns.

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b Use the model to determine the value of the mower after it has been used three times.

c How many times can the mower be used until its depreciated value is first less than $250?

Explanation	Solution
a **1** Write down the value of V_0. Here, V_0 is the value of the mower when new.	$V_0 = 270$
2 Write down the unit cost rate of depreciation, D.	$D = 3.50$
3 Write your answer.	$V_0 = 270, \quad V_{n+1} = V_n - 3.50$
b **1** Write down the recurrence relation.	$V_0 = 270, \quad V_{n+1} = V_n - 3.50$
2 On a blank calculation screen, type **270** and press [enter] (or [EXE]). Type **–3.50** and press [enter] (or [EXE]) three times to obtain the value of the mower after three mows.	270 → 270. 270. – 3.5 → 266.5 266.5. – 3.5 → 263. 263. – 3.5 → 259.5 259.5 – 3.5 → 256. 256. – 3.5 → 252.5 252.5 – 3.5 → 249.
3 Write your answer.	$259.50
c **1** Continue pressing [enter] (or [EXE]) until the value of the lawn mower is first less than $250.	
2 Write your answer.	After six mows

Skill-sheet

Exercise 7B

Modelling linear growth and decay using recurrence relations

Example 7

1 For each of the following recurrence relations, write down the first four terms and graph the corresponding points.

a $V_0 = 3, \quad V_{n+1} = V_n + 2$

b $V_0 = 38, \quad V_{n+1} = V_n - 5$

Modelling and analysing simple interest with recurrence relations

Example 8

2 Ashwin invests $8000 in an account that pays 4% per annum simple interest.

a Let V_n be the value of Ashwin's investment after n years. State the starting value, V_0, given by the principal.

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b Calculate the value of D using the interest rate and the rule $D = \frac{r}{100}V_0$ to find the amount of interest paid each year.

c Model this simple investment using a recurrence relation of the form

$$V_0 = \text{starting value}, \qquad V_{n+1} = V_n + D$$

3 Huang invests \$41 000 in an account that pays 6.2% per annum simple interest.

a Let H_n be the value of Huang's investment after n years. State the value of H_0.

b Find the amount, in dollars, that Huang will receive each year from the investment.

c Complete the recurrence relation, in terms of H_0, H_{n+1} and H_n, that would model the investment over time. Write your answers in the boxes below.

$$H_0 = \boxed{}, \qquad H_{n+1} = H_n + \boxed{}$$

Example 9 **4** The following recurrence relation can be used to model a simple interest investment of \$2000, paying interest at the rate of 3.8% per annum.

$$V_0 = 2000, \qquad V_{n+1} = V_n + 76$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of the investment after 3 years is \$2228.

b Use your calculator to determine how many years it takes for the value of the investment to first be worth more than \$3000.

5 The following recurrence relation can be used to model a simple interest loan of \$7000 with interest charged at the rate of 7.4% per annum.

$$V_0 = 7000, \qquad V_{n+1} = V_n + 518$$

In the recurrence relation, V_n is the value of the loan after n years.

a Use the recurrence relation to find the value of the loan after 1, 2 and 3 years.

b Use your calculator to determine how many years it takes for the value of the loan to first have a value of more than \$10 000.

6 The following recurrence relation can be used to model a simple interest investment. In the recurrence relation, V_n is the value of the investment after n years.

$$V_0 = 15\,000, \qquad V_{n+1} = V_n + 525$$

a **i** What is the principal of this investment?

ii How much interest is earned each year?

iii Calculate 525 as a percentage of 15 000 to find the annual interest rate of this investment.

b State how many years it takes for the value of the investment to first exceed \$30 000.

Modelling flat-rate depreciation with recurrence relations

Example 10 **7** Fernando purchased a cherry picker for \$82 000 in 2022. The cherry picker depreciates by 15% of its purchase price each year.

Let C_n be the value of the cherry picker n years after it was purchased.

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a Calculate the value of D using the interest rate and the rule $D = \frac{r}{100}V_0$ to find the amount of depreciation each year.

b Model this simple investment using a recurrence relation of the form

$$C_0 = \text{starting value}, \qquad C_{n+1} = C_n - D$$

8 Wendy purchases a new chair for her dental surgery for \$2800. The chair depreciates by 8% of its purchase price each year.

a Show that the total amount, in dollars, that Wendy's chair will depreciate by each year is \$244.

b Let W_n be the value of Wendy's chair after n years. State the value of W_0.

c Complete the recurrence relation, in terms of W_0, W_{n+1} and W_n, that would model the investment over time by filling in the boxes below.

$$W_0 = \boxed{}, \qquad W_{n+1} = W_n + \boxed{}$$

Example 11

9 The following recurrence relation can be used to model the depreciation of a computer with purchase price \$2500 and annual depreciation of \$400.

$$V_0 = 2500, \qquad V_{n+1} = V_n - 400$$

In the recurrence relation, V_n is the value of the computer after n years.

a Use the recurrence relation to find the value of the computer after 1, 2 and 3 years.

b Use your calculator recursively to determine how many years it takes for the value of the computer to first be worth less than \$1000.

10 The following recurrence relation can be used to model the depreciation of a car purchased for \$23 000 and depreciated at 3.5% of its original value each year.

$$V_0 = 23\,000, \qquad V_{n+1} = V_n - 805$$

In the recurrence relation, V_n is the value of the car after n years.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b Determine how many years it takes for the value of the car to first be worth less than \$10 000.

11 The following recurrence relation can be used to model the depreciation of a television. In the recurrence relation, V_n is the value of the television after n years.

$$V_0 = 1500, \qquad V_{n+1} = V_n - 102$$

a **i** What is the purchase price of this television?

ii What is the depreciation of the television each year?

iii What is the annual percentage depreciation of the television?

b Use your calculator to determine the value of the television after 8 years.

c If the owner of the television decides to discard the television once it is first worth less than \$100, determine how long the owner will own the television before discarding it.

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Modelling unit-cost depreciation with recurrence relations

Example 12

12 A minibus was purchased for $32 600 to take passengers to and from the airport. The minibus depreciates by $10 on every round trip that it takes.

Let M_n be the value of the minibus after n round trips.

a State the starting value, M_0 of the minibus.

b Model the value of the minibus using a recurrence relation of the form

$$M_0 = \text{starting value}, \qquad M_{n+1} = M_n - D$$

13 The following recurrence relation can be used to model the depreciation of a printer with purchase price $450 and depreciation of 5 cents for every page printed.

$$V_0 = 450, \qquad V_{n+1} = V_n - 0.05$$

In the recurrence relation, V_n is the value of the printer after n pages are printed.

a Write the first five terms of the sequence.

b Use your calculator to find the value of the printer after 20 pages are printed.

14 The following recurrence relation can be used to model the depreciation of a delivery van with purchase price $48 000 and depreciation by $200 for every 1000 kilometres travelled.

$$V_0 = 48\,000, \qquad V_{n+1} = V_n - 200$$

In the recurrence relation, V_n is the value of the delivery van after n lots of 1000 kilometres are travelled.

a Use the recurrence relation to find the value of the van after 1000, 2000 and 3000 kilometres.

b Use your calculator to determine the value of the van after 15 000 kilometres.

c Use your calculator to determine how many kilometres it takes for the value of the van to reach $43 000.

15 Jasmine owns a cafe that sells juices. The commercial blender, purchased for $1440, depreciates in value using the unit cost method.

The rate of depreciation is $0.02 per juice that is produced. The recurrence relation that models the year-to-year value, in dollars, of the blender is

$$B_0 = 1440, \qquad B_{n+1} = B_n - 144$$

a Calculate the number of juices that the blender produces each year.

b Determine how many juices the blender can produce before its value becomes 0.

c Use your calculator to find the value of the blender after 36 000 juices have been produced.

d The recurrence relation above could also represent the value of the blender depreciating at a flat rate. What annual flat rate percentage of depreciation is represented?

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Exam 1 style questions

16 A coffee machine was purchased for $720.
After five years the coffee machine has a value of $586. On average, 670 coffees were made each year during those five years.
The value of the coffee machine was depreciated using a unit cost method of depreciation. The depreciation in the value of the coffee machine, per coffee made, is, in cents, closest to

A 2 **B** 3 **C** 4 **D** 5 **E** 6

17 The value of a tandoori oven is depreciated using the flat rate method and can be modelled using the following recurrence relation where T_n is the value of the oven after n years.

$$T_0 = 4500, \qquad T_{n+1} = T_n - 405$$

The annual depreciation rate is closest to

A 8% **B** 8.5% **C** 9% **D** 9.5% **E** 10%

18 Jane purchased a motorbike for $5500. She will depreciate the value of her motorbike by a flat rate of 10% of the purchase price per annum.
A recurrence relation that Jane can use to determine the value of the motorbike, V_n, after n years is

A $V_0 = 5500, \qquad V_{n+1} = V_n + 550$
B $V_0 = 5500, \qquad V_{n+1} = V_n - 550$
C $V_0 = 5500, \qquad V_{n+1} = 0.9V_n$
D $V_0 = 5500, \qquad V_{n+1} = 1.1V_n$
E $V_0 = 5500, \qquad V_{n+1} = 0.2(V_n - 550)$

7C Using an explicit rule for linear growth or decay

Learning intentions

- ▶ To be able to convert a recurrence relation to an explicit rule.
- ▶ To be able to model a simple interest investment using an explicit rule.
- ▶ To be able to use a rule to determine the value of a simple interest loan or investment.
- ▶ To be able to model flat rate depreciation of an asset using an explicit rule.
- ▶ To be able to use a rule for the flat rate depreciation of an asset.
- ▶ To be able to use an explicit rule for unit cost depreciation.

While we can generate as many terms of a sequence as we like through repeated addition and subtraction, the process can be tedious and so instead a rule can be used.

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Consider the example of investing \$2000 in a simple interest investment paying 5% per annum. If we let V_n be the value of the investment after n years, we can use the following recurrence relation to model this investment:

$$V_0 = 2000, \qquad V_{n+1} = V_n + 100$$

Using this recurrence relation we can write out the sequence of terms generated as follows:

$$\begin{aligned}
V_0 &= 2000 && = V_0 + 0 \times 100 && \text{(no interest paid yet)}\\
V_1 &= V_0 + 100 && = V_0 + 1 \times 100 && \text{(after 1 year of interest paid)}\\
V_2 &= V_1 + 100 = (V_0 + 100) + 100 && = V_0 + 2 \times 100 && \text{(after 2 years of interest paid)}\\
V_3 &= V_2 + 100 = (V_0 + 2 \times 100) + 100 && = V_0 + 3 \times 100 && \text{(after 3 years of interest paid)}\\
V_4 &= V_3 + 100 = (V_0 + 3 \times 100) + 100 && = V_0 + 4 \times 100 && \text{(after 4 years of interest paid)}
\end{aligned}$$

and so on.

Following this pattern, after n years of interest has been added, we can write:

$$V_n = 2000 + n \times 100$$

This rule can be used to determine the value after n iterations in the sequence. For example, using this rule, the value of the investment after 15 years would be:

$$V_{15} = 2000 + 15 \times 100 = \$3500$$

Explicit rule for linear growth

For a recurrence rule for linear growth of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n + D \qquad (D \text{ constant})$$

the value of the term V_n in the sequence generated by this recurrence relation is:

$$V_n = V_0 + nD$$

Explicit rule for linear decay

In general, for a recurrence rule for linear decay of the form:

$$V_0 = \text{initial value}, \qquad V_{n+1} = V_n - D \qquad (D \text{ constant})$$

the value of the term V_n in the sequence generated by this recurrence relation is:

$$V_n = V_0 - nD$$

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Example 13 **Converting a recurrence relation to an explicit rule**

Write down a rule for V_n for each of the following recurrence relations. Calculate V_{10} for each case.

a $V_0 = 8, \quad V_{n+1} = V_n + 3$

b $V_0 = 400, \quad V_{n+1} = V_n - 12$

c $V_0 = 30, \quad V_{n+1} = V_n - 7$

Explanation	Solution
a **1** Identify the starting value.	$V_0 = 8$
2 Identify the common difference, D.	$D = 3$
3 Write the rule for V_n, noting that this is an example of **linear growth**.	$V_n = 8 + 3n$
4 Calculate V_{10}.	$V_{10} = 8 + 3 \times 10 = 38$
b **1** Identify the starting value.	$V_0 = 400$
2 Identify the common difference, D.	$D = 12$
3 Write the rule for V_n, noting that this is an example of **linear decay**.	$V_n = 400 - 12n$
4 Calculate V_{10}.	$V_{10} = 400 - 12 \times 10 = 280$
c **1** Identify the starting value.	$V_0 = 30$
2 Identify the common difference, D.	$D = 7$
3 Write the rule for V_n, noting that this is an example of **linear decay**.	$V_n = 30 - 7n$
4 Calculate V_{10}.	$V_{10} = 30 - 7 \times 10 = -40$

These general rules can be applied to simple interest investments and loans, flat rate depreciation and unit cost depreciation.

Using a rule for simple interest loans or investments

Simple interest loans and investments are examples of linear growth so we use the rule

$V_n = V_0 + nD$, where $D = \dfrac{r}{100} \times V_0$.

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Example 14 Modelling simple interest investments

Amie invests \$3000 in a simple interest investment with interest paid at the rate of 6.5% per year.

Use a rule to find the value of the investment after 10 years.

Explanation	Solution
1 Identify the starting value.	$V_0 = 3000$
2 Identify the common difference, D.	$D = \frac{6.5}{100} \times 3000 = 195$
3 Write the rule for V_n, noting that this is an example of **linear growth**.	$V_n = 3000 + 195n$
4 Calculate V_{10}.	$V_{10} = 3000 + 195 \times 10 = 4950$

Example 15 Using a rule to determine the value of a simple interest investment

The following recurrence relation can be used to model a simple interest investment:

$$V_0 = 3000, \quad V_{n+1} = V_n + 260$$

where V_n is the value of the investment after n years.

a What is the principal of the investment? How much interest is added each year?

b Write down the rule for the value of the investment after n years.

c Use a rule to find the value of the investment after 15 years.

d Use a rule to find when the value of the investment first exceeds \$10 000.

Explanation	Solution
a These values can be read directly from the recurrence relation.	**a** Principal: \$3000 Amount of interest = \$260
b Start with the general rule: $V_n = V_0 + nD$ and substitute $V_0 = 3000$ and $D = 260$.	**b** $V_n = 3000 + n \times 260$ $= 3000 + 260n$
c Substitute $n = 15$ into the rule to calculate V_{15}.	**c** $V_{15} = 3000 + 260 \times 15$ $= \$6900$
d Substitute $V_n = 10\,000$ into the rule, and solve for n. Write your conclusion. Note: Because the interest is only paid into the account after a whole number of years, any decimal answer will need to be *rounded up* to the next whole number.	**d** $10\,000 = 3000 + 260n$ so $7000 = 260n$ or $n = 7000/260$ $= 26.92\ldots$ years The value of the investment will first exceed \$10 000 after 27 years.

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Using a rule for flat rate depreciation of assets

Flat rate depreciation is an example of linear decay so we use the rule $V_n = V_0 - nD$.

Example 16 Modelling flat rate depreciation of an asset using an explicit rule

A photocopier costs \$6000 when new. Its value depreciates at the flat rate of 17.5% per year. Write a rule and use this to find its value after 4 years.

Explanation	Solution
1 Identify the starting value.	$V_0 = 6000$
2 Identify the common difference, D.	$D = \frac{17.5}{100} \times 6000 = 1050$
3 Write the rule for V_n, noting that this is an example of **linear decay**.	$V_n = 6000 - 1050n$
4 Calculate V_4.	$V_4 = 6000 - 1050 \times 4 = 1800$ The value after 4 years is \$1800.

Example 17 Using a rule for the flat rate depreciation of an asset

The following recurrence relation can be used to model the flat rate of depreciation of a set of office furniture:

$$V_0 = 12\,000, \quad V_{n+1} = V_n - 1200$$

where V_n is the value of the furniture after n years.

a What is the initial value of the furniture? How much does the furniture decrease by each year?

b Write down the rule for the value of the investment after n years.

c Use a rule to find the value of the investment after 6 years.

d How long does it take for the furniture's value to decrease to zero?

Explanation	Solution
a These values can be read directly from the recurrence relation.	**a** Initial value: \$12 000 Depreciation = \$1200 each year
b Start with the general rule $V_n = V_0 - nD$ and substitute $V_0 = 12\,000$ and $D = 1200$.	**b** $V_n = 12\,000 - n \times 1200$ $= 12\,000 - 1200n$
c Use the rule to calculate V_6.	**c** $V_6 = 12\,000 - 1200 \times 6$ $= \$4800$
d Substitute $V_n = 0$, and solve for n. Write your conclusion.	**d** $0 = 12\,000 - n \times 1200$ so $n = 10$ The value of the furniture will depreciate to zero after 10 years.

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Using a rule for unit cost depreciation of assets

Example 18 Using an explicit rule for unit cost depreciation

A hairdryer in a salon was purchased for $850. The value of the hairdryer depreciates by 25 cents for every hour it is in use.

Let V_n be the value of the hairdryer after n hours of use.

a Write down a rule to find the value of the hairdryer after n hours of use.
b What is the value of the hairdryer after 50 hours of use?
c On average, the salon will use the hairdryer for 17 hours each week. How many weeks will it take for the value of the hairdryer to halve?
d The hairdryer has a scrap value of $100 before it is disposed of. Find the number of hours of use before this occurs.

Explanation	Solution
a **1** Identify the values of V_0 and D.	$V_0 = 850$ and $D = 0.25$
2 Write down the rule for the value of the hairdryer after n hours of use.	$V_n = 850 - 0.25n$
b **1** Decide the value of n and substitute into the rule.	After 50 hours of use, $n = 50$. $V_{50} = 850 - 0.25 \times 50$ $V_{50} = 837.50$
2 Write your answer.	After 50 hours of use, the hairdryer has a value of $837.50.
c **1** Halving the value of the hairdryer means it will have a value of $425.	Solve $V_n = 425$
2 Write down the rule, with the value of the hairdryer, $V_n = 425$.	$425 = 850 - 0.25n$
3 Solve the equation for n.	$0.25n = 850 - 425$ $0.25n = 425$ $n = 1700$
4 Divide by 17 as the hairdryer is used for 17 hours each week.	Number of weeks = 100
5 Write your answer.	After 100 weeks, the hairdryer is expected to **halve** in value.
d Solve for $V_n = 100$.	$100 = 850 - 0.25n$ $0.25n = 750$ $n = 3000$
Write your answer.	The hairdryer can be used for 3000 hours before it reaches its scrap value.

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Skill-sheet

Exercise 7C

Writing an explicit rule from a linear recurrence relation

Example 13

1 Write down a rule for A_n for each of the following recurrence relations. In each case calculate A_{20}.

a $A_0 = 4, \quad A_{n+1} = A_n + 2$

b $A_0 = 10, \quad A_{n+1} = A_n - 3$

c $A_0 = 5, \quad A_{n+1} = A_n + 8$

d $A_0 = 300, \quad A_{n+1} = A_n - 18$

Using a rule for simple interest loans and investments

Example 14

2 Webster borrows \$5000 from a bank at an annual simple interest rate of 5.4%.

a Let V_n be the value of the loan after n years. State the starting value, V_0.

b Determine how much interest is charged each year in dollars.

c Write down a rule for the value of the loan, V_n, after n years.

d Use your rule to find how much Webster will owe the bank after 9 years.

3 Anthony borrows \$12 000 from a bank at an annual simple interest rate of 7.2%.

a Let V_n be the value of the loan after n years. State the starting value, V_0.

b Determine how much interest is charged each year in dollars.

c Write down a rule for the value of the loan, V_n, after n years.

d Use your rule to find how much Anthony will owe the bank after 9 years.

Example 15

4 The value of a simple interest loan after n years, V_n, can be calculated using the rule $V_n = 8000 + 512n$.

a What is the principal of this loan?

b How much interest is charged every year in dollars?

c Use the rule to find:

i the value of the loan after 12 years

ii when the value of the loan first doubles in value.

5 The value of a simple interest investment after n years, V_n, can be calculated using the rule $V_n = 2000 + 70n$.

a What is the principal of this investment?

b How much interest is earned every year in dollars?

c Use the rule to find:

i the value of the investment after 6 years

ii when the value of the initial investment will first double in value.

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Using a rule for flat rate depreciation of assets

Example 16

6 A computer is purchased for \$5600 and is depreciated at a flat rate of 22.5% per year.

- **a** State the starting value.
- **b** Determine the annual depreciation in dollars.
- **c** Write down a rule for the value of the computer, V_n, after n years.
- **d** Find the value of the computer after 3 years.

7 A machine costs \$7000 new and depreciates at a flat rate of 17.5% per annum. The machine will be written off when its value is \$875.

- **a** State the starting value.
- **b** Determine the annual depreciation in dollars.
- **c** Write down a rule for the value of the machine, V_n, after n years.
- **d** Determine the number of full years that the machine will be used (that is, has a value greater than zero).

Example 17

8 The value of a sewing machine after n years, V_n, can be calculated from the rule $V_n = 1700 - 212.5n$.

- **a** What is the purchase price of the sewing machine?
- **b** By how much is the value of the sewing machine depreciated each year in dollars?
- **c** Use the rule to find the value of the sewing machine after 4 years.
- **d** Find its value after 7 years.
- **e** Determine the number of years it takes for the sewing machine to be worth nothing.

9 The value of a harvester after n years, V_n, can be calculated from the rule $V_n = 65000 - 3250n$.

- **a** What is the purchase price of the harvester?
- **b** By how much is the value of the harvester depreciated each year in dollars?
- **c** What is the annual percentage depreciation for the harvester?
- **d** Use the rule to find the value of the harvester after 7 years.
- **e** How long does it take the harvester to reach a value of \$29 250?

Using a rule for unit cost depreciation of assets

Example 18

10 The value of a taxi after n kilometres, V_n, can be calculated from the rule $V_n = 29000 - 0.25n$.

- **a** What is the purchase price of the taxi?
- **b** By how much is the value of the taxi depreciated per kilometre of travel?
- **c** What is the value of the taxi after 20 000 kilometres of travel?
- **d** Find how many kilometres have been travelled if the taxi is valued at \$5000.

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11 A car is valued at \$35 400 at the start of the year, and at \$25 700 at the end of that year. During that year, the car travelled 25 000 kilometres.

a Find the total depreciation of the car in that year in dollars.

b Find the depreciation per kilometre for this car.

c Using $V_0 = 35\,400$, write down a rule for the value of the car, V_n, after n kilometres.

d How many kilometres have been travelled if the car has a value of \$6688?

12 A printing machine costing \$110 000 has a scrap value of \$2500 after it has printed 4 million pages.

a Find:

i the unit cost of using the machine

ii the value of the machine after printing 1.5 million pages

iii the annual depreciation of the machine if it prints 750 000 pages per year.

b Find the value of the machine after 5 years if it prints, on average, 750 000 pages per year.

c How many pages has the machine printed by the time the value of the machine is \$70 053?

Exam 1 style questions

13 The value of a bicycle, purchased for \$3800, is depreciated by 10% per annum using the flat rate method.
Recursive calculations can determine the value of the bicycle after n years, B_n.
Which one of the following recursive calculations is **not** correct?

A $V_0 = 3800$

B $V_1 = 0.9 \times 3800$

C $V_2 = 0.9 \times 3420$

D $V_3 = 0.9 \times 3080$

E $V_4 = 0.9 \times 2770.20$

14 An asset is purchased for \$4280.
The value of the asset after n time periods, V_n, can be determined using the rule

$$V_n = 4280 + 25n$$

A recurrence relation that also models the value of this asset after n time periods is

A $V_0 = 4280,\ V_{n+1} = V_n + 25n$

B $V_0 = 4280,\ V_{n+1} = V_n - 25n$

C $V_0 = 4280,\ V_{n+1} = V_n + 25$

D $V_0 = 4280,\ V_{n+1} = V_n - 25$

E $V_0 = 4280,\ V_{n+1} = 25V_n + 4280$

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7D Modelling geometric growth and decay

Learning intentions
- To be able to graph the terms of a geometric sequence.
- To be able to model compound interest with a recurrence relation.
- To be able to model reducing balance depreciation with recurrence relations.
- To be able to use reducing balance depreciation with recurrence relations.

A recurrence model for geometric growth and decay

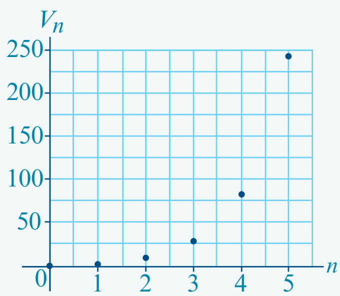
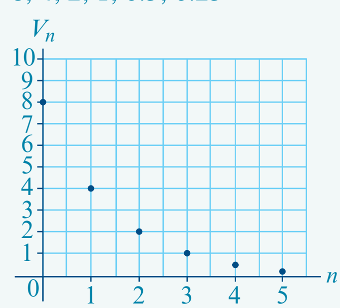
Geometric growth or **decay** occurs when quantities increase or decrease by the same percentage at regular intervals. For example, a sequence that starts with 3 and doubles is given by 3, 6, 12, 24, . . . and can be written as a recurrence relation $V_0 = 3, V_{n+1} = 2V_n$.

Example 19 Graphing the terms in a geometric sequence

For each recurrence relation, state the rule, find the first 6 terms and then plot each point on a graph.

a $V_0 = 1, \quad V_{n+1} = 3V_n$

b $V_0 = 8, \quad V_{n+1} = 0.5V_n$

Explanation	Solution
a **1** Convert to words.	Starting value = 1 Next value = 3 × current value
2 Multiply each term by 3 to find the next term.	1, 3, 9, 27, 81, 243
3 Plot each of the points on the axis.	
b **1** Convert to words.	Starting value = 8 Next value = 0.5 × current value
2 Multiply each term by 0.5 to find the next term.	8, 4, 2, 1, 0.5, 0.25
3 Plot each of the points on the axis.	

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As can be seen from the previous example, the first recurrence relation generates a sequence whose successive terms grow geometrically, while the second recurrence relation decays geometrically.

Modelling geometric growth and decay

As a general rule, if R is a positive constant, a recurrence relation rule of the form:

- $V_{n+1} = RV_n$ for $R > 1$, can be used to model *geometric growth*.
- $V_{n+1} = RV_n$ for $R < 1$, can be used to model *geometric decay*.

Compound interest investments and loans

More common than simple interest is **compound interest** where any interest that is earned after one time period is added to the principal and then contributes to the earning of interest in the next time period. This means that the value of the investment grows in ever increasing amounts, or grows geometrically, instead of by the same amount as in simple interest.

Consider an investment of \$5000 that pays 8% interest per annum, compounding yearly. This means that the investment's value increases by 8% each year.

We can model the investment with a recurrence relation as follows:

Let V_n be the value of the investment after n years. Initially the investment is worth \$5000 so $V_0 = \$5000$.

To find the rule between terms:

$$\begin{aligned}\text{next value} &= \text{current value } + 8\% \text{ of current value}\\ &= 108\% \text{ of current value}\\ &= 1.08 \times \text{ current value}\\ &= 1.08 \times V_n\end{aligned}$$

We now have a recurrence relation that we can use to model and investigate the growth of an investment over time. Compound interest loans and investments often accrue interest over periods of less than a year which we will consider at the end of this chapter.

A recurrence model for compound interest investments and loans that compound yearly

Let V_n be the value of the investment after n years.

Let r be the annual percentage interest rate.

The recurrence model for the value of the investment after n years is:

$$V_0 = \text{principal}, \qquad V_{n+1} = RV_n, \qquad \text{where } R = 1 + \frac{r}{100}$$

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Example 20 Modelling compound interest with a recurrence relation

The following recurrence relation can be used to model a compound interest investment of $2000 paying interest at the rate of 7.5% per annum.

$$V_0 = 2000, \qquad V_{n+1} = 1.075 \times V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of the investment after 3 years is $2484.59.

b Determine when the value of the investment will first exceed $2500.

Explanation

a **1** Write down the principal, V_0.

2 Use the recurrence relation to calculate V_1, V_2 and V_3 and round to the nearest cent.

b **1** Type '2000'. Press enter (or EXE).

2 Type **×1.075**.

3 Count how many times you press enter (or EXE) until the term value is greater than 2500.

4 Write your answer.

Solution

$V_0 = 2000$

$V_1 = 1.075 \times 2000 = 2150$

$V_2 = 1.075 \times 2150 = 2311.25$

$V_3 = 1.075 \times 2311.25 = 2484.59$

2000	2000.
2000. · 1.075	2150.
2150. · 1.075	2311.25
2311.25 · 1.075	2484.59375
2484.59375 · 1.075	2670.93828125

The investment will first exceed $2500 after 4 years.

Reducing balance depreciation

Reducing balance depreciation is another method of depreciation – one where the value of an asset decays geometrically. Each year, the value will be reduced by a percentage, r%, of the previous year's value.

A recurrence model for reducing balance depreciation

Let V_n be the value of the asset after n years.

Let r be the annual percentage depreciation rate.

The recurrence model for the value of the asset after n years is:

$$V_0 = \text{initial value}, \qquad V_{n+1} = RV_n, \qquad \text{where } R = 1 - \frac{r}{100}$$

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Example 21 Modelling reducing balance depreciation with recurrence relations

A sofa was purchased for $7500 and is depreciating at a reducing balance rate of 8.4% per annum. Write down a recurrence relation where V_n is the value of the sofa after n years.

Explanation	Solution
1 Identify the value of V_0.	$V_0 = 7500$
2 Calculate the value of R.	The depreciation rate is 8.4% per annum. $R = 1 - \frac{8.4}{100}$ so $R = 0.916$
3 Write your answer.	$V_0 = 7500, \quad V_{n+1} = 0.916 \times V_n$

Example 22 Using reducing balance depreciation with recurrence relations

The following recurrence relation can be used to model the value of office furniture with a purchase price of $9600, depreciating at a reducing-balance rate of 7% per annum.

$$V_0 = 9600, \quad V_{n+1} = 0.93 \times V_n$$

In the recurrence relation, V_n is the value of the office furniture after n years.

a Use the recurrence relation to find the value of the office furniture, correct to the nearest cent, after 1, 2 and 3 years.

b If the office furniture was initially purchased in 2023, at the end of which year will the value of the investment first be less than $7000?

Explanation	Solution
a 1 Write down the purchase price of the furniture, V_0.	$V_0 = 9600$
2 Use the recurrence relation to calculate V_1, V_2 and V_3. Use your calculator if you wish.	$V_1 = 0.93 \times 9600 = 8928$ $V_2 = 0.93 \times 8928 = 8303.04$ $V_3 = 0.93 \times 8303.04 = 7721.83$

b Steps

1 Type **9600** and press [enter] or [EXE].

2 Type **× 0.93**.

3 Count how many times you press [enter] until the term value is less than 7000.

9600	9600.
9600. · 0.93	8928.
8928. · 0.93	8303.04
8303.04 · 0.93	7721.8272
7721.8272 · 0.93	7181.299296
7181.299296 · 0.93	6678.708345

4 Write your answer.

The value of the furniture first drops below $7000 after 5 years. Thus, it is first worth less than $7000 at the end of 2028.

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Skill-sheet

Exercise 7D

Example 19

1 Generate and graph the first five terms of the sequences defined by the recurrence relations.

a $V_0 = 2, \quad V_{n+1} = 2V_n$

b $V_0 = 3, \quad V_{n+1} = 3V_n$

c $V_0 = 100, \quad V_{n+1} = 0.1V_n$

Modelling compound interest with recurrence relations

Example 20

2 An investment of \$6000 earns compounding interest at the rate of 4.2% per annum. A recurrence relation that can be used to model the value of the investment after n years is shown below.

$$V_0 = 6000, \qquad V_{n+1} = 1.042V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to show that the value of the investment after 3 years is \$6788.20.

b Determine how many years it takes for the value of the investment to first exceed \$8000.

3 A loan of \$20 000 is charged compounding interest at the rate of 6.3% per annum. A recurrence relation that can be used to model the value of the loan after n years is shown below.

$$V_0 = 20\,000, \qquad V_{n+1} = 1.063V_n$$

In the recurrence relation, V_n is the value of the loan after n years.

a Use the recurrence relation to show that the value of the loan after 3 years is \$24 023.14.

b Determine how many years it takes for the value of the loan to first exceed \$30 000.

4 Sue invests \$5000 at a compounding rate of 6.8% per annum. Let V_n be the value of the investment after n years. This compound interest investment can be modelled by a recurrence relation of the form

$$V_0 = \text{ principal }, \qquad V_{n+1} = R \times V_n$$

a State the value of V_0.

b Determine R using $R = 1 + \frac{r}{100}$.

c Write down a recurrence relation for the investment.

d Find the value of the investment after 5 years.

e Find the total interest earned over 5 years.

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5 Jay takes out a loan of $18 000 at a compounding interest rate of 9.4% per annum.

a State the principal (starting value).

b Determine R using $R = 1 + \frac{r}{100}$.

c Let V_n be the value of the loan after n years. Write down a recurrence relation for this loan.

d Use the recurrence relation to find the value of the loan after 4 years.

e When will the loan first be valued at more than $25 000.

Modelling reducing balance depreciation with recurrence relations

Example 21

6 A motorcycle, purchased new for $9800, will be depreciated using a reducing balance depreciation method with an annual depreciation rate of 3.5%. Write a recurrence relation to model the value of the motorcycle using V_n to represent the value of the motorcycle after n years.

7 Let M_n be the value of a minibus after n years. Write down a recurrence relation for a minibus that was initially valued at $28 600 and is depreciated at a reducing-balance rate of 7.4% per annum.

Example 22

8 Office furniture was purchased new for $18 000. It will be depreciated using a reducing balance depreciation method with an annual depreciation rate of 4.5%. Let V_n be the value of the furniture after n years.

a Write a recurrence relation to model the value of the furniture, V_n.

b Use the recurrence relation to find the value of the furniture after each of the first 5 years. Write the values of the terms of the sequence correct to the nearest cent.

c What is the value of the furniture after 3 years?

d What is the total depreciation of the furniture after 5 years?

9 A wedding gown was purchased new for $4 000. The value of the wedding gown depreciates using a reducing balance depreciation method with an annual depreciation rate of 4.1%. Let W_n be the value of the wedding dress after n years.

a Write a recurrence relation to model the value of the wedding dress, W_n.

b Calculate the value of the wedding dress after three years.

c Determine the total amount of depreciation of the dress after five years.

10 A new computer server was purchased for $13 420. The value of the computer server depreciates using a reducing-balance depreciation method with an annual depreciation rate of 11.2%. Let S_n be the value of the server after n years.

a Write a recurrence relation to model the value of the server, S_n.

b Use the recurrence relation to find the value of the server after each of the first 5 years. Write the values of the terms of the sequence correct to the nearest cent.

c What is the value of the server after 5 years?

d What is the depreciation of the server in the third year?

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Exam 1 style questions

11 The value of a compound interest investment, in dollars, after n years, V_n can be modelled by the recurrence relation shown below.

$$V_0 = 80\,000, \qquad V_{n+1} = 1.02 \times V_n$$

The interest rate, per annum, for this investment is

A 0.02%
B 0.2%
C 2%
D 1.02%
E 102%

12 Raymond invests $6000 in an account that pays interest compounding annually. The balance of his investment after n years, R_n, can be determined using the recurrence relation.

$$R_0 = 6000, \qquad R_{n+1} = 1.0384 \times R_n$$

The total interest earned by Raymond's investment after the first three years is closest to

A $230.40
B $248.43
C $469.64
D $469.65
E $718.08

13 Mana invests $28 000 at an interest rate of 6.2% per annum, compounding annually. Her investment will first be more than double its original value after

A 1 year
B 2 years
C 10 years
D 11 years
E 12 years

14 Giles purchases a jet ski for $18 000. The value of the jet ski is depreciated each year by 10% using the reducing balance method. In the third year, the jet ski will depreciate in value by

A $1312.20
B $13122
C $1458
D $11809.80
E $14580

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7E Using an explicit rule for geometric growth or decay

Learning intentions

- To be able to write explicit rules for geometric growth and decay.
- To be able to use an explicit rule to find the value of an investment after n years.
- To be able to calculate the value and total depreciation of an asset after a period of reducing balance depreciation.
- To be able to use a calculator to solve geometric growth and decay problems.

As with linear growth and decay, we can derive a rule to calculate any term in a geometric sequence directly.

Assume \$2000 is invested in a compound interest investment paying 5% per annum, compounding yearly. Let V_n be the value of the investment after n years, giving the following recurrence relation to model this investment:

$$V_0 = 2000, \qquad V_{n+1} = 1.05V_n$$

Using this recurrence relation we can write out the sequence of terms generated as follows:

$$V_0 = 2000$$
$$V_1 = 1.05V_0$$
$$V_2 = 1.05V_1 = 1.05(1.05V_0) = 1.05^2 V_0$$
$$V_3 = 1.05V_2 = 1.05(1.05^2 V_0) = 1.05^3 V_0$$
$$V_4 = 1.05V_3 = 1.05(1.05^3 V_0) = 1.05^4 V_0$$

and so on.

Following this pattern, after n years of interest are added, we have:

$$V_n = 1.05^n V_0$$

With this rule, we can now find the value of the investment for any specific year. For example, using this rule, the value of the investment after 18 years would be:

$$V_{18} = 1.05^{18} \times 2000 = \$4813.24 \text{ (to the nearest cent)}$$

A rule for individual terms of a geometric growth and decay sequence

For a geometric growth or decay recurrence relation

$$V_0 = \text{starting value}, \qquad V_{n+1} = RV_n$$

the value after n iterations is given by the rule:

$$V_n = R^n \times V_0$$

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Exactly the same rule will work for both growth and decay, noting that $R > 1$ is used for growth and $R < 1$ for decay.

Example 23 Writing explicit rules for geometric growth and decay

Write down a rule for the value of V_n in terms of n for each of the following. Use the rule to find the value of V_6.

a $V_0 = 5, \quad V_{n+1} = 4V_n$

b $V_0 = 10, \quad V_{n+1} = 0.5V_n$

Explanation	Solution
a 1 $V_0 = 5, R = 4$	$V_n = 4^n \times 5$
2 Substitute $n = 6$.	$V_6 = 4^6 \times 5 = 20480$
b 1 $V_0 = 10, R = 0.5$	$V_n = 0.5^n \times 10$
2 Substitute $n = 6$.	$V_6 = 0.5^6 \times 10 = 0.15625$

Explicit rules for compound interest loans and investments and reducing balance depreciation

Since compound interest loans and investments *increase* over time, the value of R is greater than 1 and can be found using the interest rate of $r\%$ per annum using the formula $R = 1 + \frac{r}{100}$.

Compound interest loans and investments

Let V_0 be the amount borrowed or invested (principal).
Let r be the annual percentage interest rate, with interest compounding annually.
The value of a compound interest loan or investment after n years, V_n, is given by the rule

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

Example 24 Using a rule to find the value of an investment after *n* years

The rule for the value of the investment after n years, V_n, is shown below.

$$V_n = 1.09^n \times 10\,000$$

a State how much money was initially invested.

b Find the annual interest rate for this investment.

c Find the value of the investment after 4 years, correct to the nearest cent.

d Find the amount of interest earned over the first 4 years, correct to the nearest cent.

e Find the amount of interest earned in the fourth year, correct to the nearest cent.

f Determine if the investor has doubled their money within 10 years.

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Explanation	Solution
a Recall the form of the direct rule $V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$. Read off V_0.	\$10 000
b Since $R = 1.09 = 1 + \frac{r}{100}$	$r = 9$ The annual interest rate is 9%.
c **1** Substitute $n = 4$ into the rule for the value of the investment.	$V_4 = 1.09^4 \times 10\,000$ $V_4 = 14\,115.816\ldots$
2 Write your answer, rounded to the nearest cent.	After 4 years, the value of the investment is \$14 115.82, correct to the nearest cent.
d To find the total interest earned in 4 years, subtract the principal from the value of the investment after 4 years.	Amount of interest = \$14 115.82 – \$10 000 = \$4115.82 After 4 years, the amount of interest earned is \$4115.82.
e **1** Calculate V_3 to the nearest cent.	$V_3 = 1.09^3 \times 10\,000$ $V_3 = 12\,950.29$ (nearest cent)
2 Calculate $V_4 - V_3$.	$V_4 - V_3 = 14115.82 - 12950.29$ $= 1165.53$
3 Write your answer. Note: An alternate method is to calculate 9% of V_3.	Interest of \$1165.53 was earned in the fourth year.
f Calculate V_{10} and compare this to double the principal.	We require $V_n = 2 \times V_0 = 20\,000$. Note $V_{10} = 1.09^{10} \times 10\,000 = 23\,673.64$ Since 23 673.64 > 20 000, the investor has doubled their money within 10 years.

With reducing balance depreciation, the value of an asset *declines* over time. The value of R can be found using the formula $R = 1 - \frac{r}{100}$ where r% is the annual depreciation rate.

Reducing balance depreciation

Let V_0 be the purchase price of the asset.

Let r be the annual percentage rate of depreciation.

The value of an asset after n years, V_n, is given by the rule

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

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Example 25 **Calculating the value and total depreciation of an asset after a period of reducing balance depreciation**

A machine costs $9500 to buy, and decreases in value with reducing balance depreciation of 20% each year. A recurrence relation that can be used to model the value of the machine after n years, V_n, is shown below.

$$V_0 = 9500, \qquad V_{n+1} = 0.8 \times V_n$$

a Write down the rule for the value of the machine after n years.

b Use the rule to find the value of the machine after 8 years. Write your answer, correct to the nearest cent.

c Calculate the total depreciation of the machine after 8 years.

Explanation	Solution
a **1** Write down the values of V_0 and R.	$V_0 = 9500$ $R = 1 - \frac{20}{100} = 0.8$
2 Write down the rule.	$V_n = R^n \times V_0$ $V_n = 0.8^n \times 9500$
b **1** Substitute $n = 8$ into the rule.	$V_8 = 0.8^8 \times 9500$ $V_8 = 1593.835\ldots$
2 Write your answer, rounding as required.	After 8 years, the value of the machine is $1593.84, correct to the nearest cent.
c To find the total depreciation after 8 years, subtract the value of the machine after 8 years from the original value of the machine. Write your answer.	Depreciation = $9500 − $1593.84 = $7906.16 After 8 years, the machine has depreciated by $7906.16.

Using a CAS calculator

As well as finding the value of an investment or loan with compound interest or the value of an asset with reducing balance depreciation, it is also possible to find how long it will take for an investment to reach a particular value.

While this can be done using trial and error, it is also possible to solve these types of problems using a CAS calculator. Similarly, it is also possible to find the annual rate of interest or depreciation which will lead to a particular value in a given number of years.

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Example 26 Using a calculator to solve geometric growth and decay problems to find n

How many years will it take for an investment of \$5000, paying compound interest at 6% per annum, to grow above \$8000? Write your answer correct to the nearest year.

Explanation	Solution
1 Write down the values of V_0, V_n and R.	$V_0 = 5000, \quad R = 1 + \frac{6}{100} = 1.06$ $V_n = 8000$
2 Substitute into the rule for the particular term of a sequence.	$V_n = R^n \times V_0$ $8000 = 1.06^n \times 5000$
3 Solve this equation for n using a CAS calculator.	$\text{solve}(8000 = (1.06)^n \cdot 5000, n)$ $n = 8.06611354799$
4 Write your answer, rounding up as interest is paid at the end of the year. After 8 years, the value is \$7969.24.	The value of the investment will grow above \$8000 after 9 years.

Example 27 Using a calculator to solve geometric growth and decay problems to find r

An industrial weaving company purchased a new loom at a cost of \$56 000. It has an estimated value of \$15 000 after 10 years of operation. If the value of the loom is depreciated using a reducing balance method, what is the annual rate of depreciation? Write your answer correct to one decimal place.

Explanation	Solution
1 Write down the values of V_0, V_n, R and n.	$V_0 = 56\,000,\ V_n = 15\,000,\ n = 10$ $R = 1 - \frac{r}{100}$
2 Substitute into the rule for the nth term of a sequence.	$V_n = R^n \times V_0$ $V_{10} = \left(1 - \frac{r}{100}\right)^{10} \times V_0$ $15\,000 = \left(1 - \frac{r}{100}\right)^{10} \times 56\,000$
3 Solve this equation for r using a CAS calculator. Note: there are two answers. Choose the more appropriate of the two.	$\text{solve}\left(15000 = \left(1 - \frac{r}{100}\right)^{10} \cdot 56000, r\right)$ $r = 12.3422491484$ or $r = 187.657750852$
4 Write your answer.	The annual rate of depreciation is 12.3%, correct to one decimal place.

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Exercise 7E

Writing explicit rules for geometric recurrence relations

Example 23

1 Write down a rule for V_n in terms of n for each of the following recurrence relations. Use each rule to find the value of V_4.

a $V_0 = 6, \quad V_{n+1} = 2V_n$

b $V_0 = 10, \quad V_{n+1} = 3V_n$

c $V_0 = 1, \quad V_{n+1} = 0.5V_n$

d $V_0 = 80, \quad V_{n+1} = 0.25V_n$

Using a rule for compound interest loans and investments

Example 24

2 The value of an investment earning compound interest every year is modelled using the recurrence relation:

$$V_0 = 3000, \qquad V_{n+1} = 1.1V_n.$$

a **i** How much money was invested?

ii What is the annual interest rate for this investment?

b Write down a rule for the value of the investment after n years, V_n.

c Use the rule to find the value of the investment after 5 years. Round your answer to the nearest cent.

3 The value of a loan that is charged compound interest every year is modelled using the recurrence relation:

$$V_0 = 2000, \qquad V_{n+1} = 1.06V_n.$$

a **i** How much money was borrowed?

ii What is the annual interest rate for this loan?

b Write down a rule for the value of the loan after n years, V_n.

c Use the rule to find the value of the loan after 4 years. Round your answer to the nearest cent.

d If the loan is fully repaid after 6 years, what is the total interest that is paid? Round your answer to the nearest cent.

4 Pacey invests $8000 in an account earning 12.5% compound interest each year. Let V_n be the value of the investment after n years.

a Write down a rule for the value of Pacey's investment after n years.

b Use the rule to find the value of the investment after 3 years. Round your answer to the nearest cent.

c How much interest has been earned after 3 years? Round your answer to the nearest cent.

d How much interest was earned in the third year of the investment? Round your answer to the nearest cent.

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Using a rule for reducing balance depreciation

Example 25

5 The value of a stereo system depreciating annually using reducing balance depreciation is modelled using the recurrence relation $V_0 = 1200, \quad V_{n+1} = 0.88V_n$.

a **i** What is the purchase price of the stereo system?

ii At what percentage rate is the stereo system being depreciated?

b Write down a rule for the value of the stereo system after n years.

c Find the value of the stereo system after 7 years. Round your answer to the nearest cent.

6 A car was purchased for $38 500 and depreciates at a rate of 9.5% per year, using a reducing balance depreciation method. Let V_n be the value of the car after n years.

a Write down a rule for the value of the car after n years.

b Find the value of the car after 5 years. Round your answer to the nearest cent.

c What is the total depreciation of the car over 5 years? Round your answer to the nearest cent.

Using a CAS calculator to solve geometric growth and decay problems

Example 26

7 Sarah invested $3500 at 6.75% per annum, compounding annually.
If the investment now has a value of $5179.35, for how many years was it invested?

8 After how many years would an investment of $200 invested at 4.75% per annum, compounding annually, first exceed a value of $20 000?

Example 27

9 An investment of $1000 has grown to $1601.03 after 12 years invested at r% per annum compound interest. Find the value of r to the nearest whole number.

10 What annual reducing balance depreciation rate would cause the value of a car to drop from $8000 to $6645 in 3 years? Give your answer to the nearest percent.

11 How much money must you deposit in a compounding interest investment at a rate of 6.8% per annum if you require $12 000 in 4 years' time? Round your answer to the nearest cent.

12 A machine has a book value after 10 years of $13 770.
If it depreciated at a reducing balance rate of 8.2% per annum, what was the initial value of the machine? Round your answer to the nearest cent.

Exam 1 style questions

13 A ute had an initial value of $68 000 and was depreciated using the reducing balance method. After five years, it had a value of $37 971.60.
The annual rate of depreciation was closest to

A 9% **B** 10% **C** 11% **D** 55% **E** 56%

14 Amber invests $15 000 at an interest rate of 5.8% per annum, compounding annually. After how many years will her investment first be more than double its original value?

A 1 **B** 2 **C** 10 **D** 12 **E** 13

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7F Interest rates over different time periods and effective interest rates

Learning intentions

- ▶ To be able to convert nominal (annual) interest rates to compounding period interest rates.
- ▶ To be able to model loans with different compounding periods using recurrence relations.
- ▶ To be able to model investments with different compounding periods using recurrence relations.
- ▶ To be able to compare loans and investments using effective interest rates.
- ▶ To be able to calculate effective interest rates using a CAS calculator.

Compound interest rates are usually quoted as annual rates, or interest rates per annum. This rate is called the **nominal interest rate** for the investment or loan. Despite this, interest can be calculated and paid according to a different time period, such as monthly. The time period which compound interest is calculated and paid upon is called the **compounding period**.

Interest rate conversions

The interest rate for the compounding period is calculated based on the following:

- 12 equal months in every year (even though some months have different numbers of days)
- 4 quarters in every year (a quarter is equal to 3 months)
- 26 fortnights in a year (even though there are slightly more than this)
- 52 weeks in a year (even though there are slightly more than this)
- 365 days in a year (ignore the existence of leap years).

A nominal interest rate is converted to a compounding period interest rate by *dividing* by these numbers, which we will refer to with the letter p.

Example 28 Converting nominal interest rates to compounding period interest rates

An investment account will pay interest at the rate of 4.68% per annum. Convert this interest rate to each of the following rates:

a monthly **b** fortnightly **c** quarterly.

Explanation	Solution
a Divide by $p = 12$.	Monthly interest rate $= \frac{4.68}{12} = 0.39\%$
b Divide by $p = 26$.	Fortnightly interest rate $= \frac{4.68}{26} = 0.18\%$
c Divide by $p = 4$.	Quarterly interest rate $= \frac{4.68}{4} = 1.17\%$

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Recurrence relations with different compounding periods

An annual interest rate can be converted to a compounding period interest rate and then used in a recurrence relation to model a compounding investment or loan.

To do this, we update our definition of the growth multiplier, R, for compound interest loans and investments as follows:

$$R = 1 + \frac{r}{100 \times p}$$

where r is the annual nominal interest rate and p is the number of compounding periods in each year. If compounding is annual, we use $p = 1$.

Example 29 **Recurrence relations with different compounding periods**

Brian borrows \$5000 from a bank. He will pay interest at the rate of 4.5% per annum.

Let V_n be the value of the loan after n compounding periods.

Write down a recurrence relation to model the value of Brian's loan if interest is compounded:

a yearly **b** quarterly **c** monthly.

Explanation	Solution
a 1 Define the variable V_n. The compounding period is *yearly*.	Let V_n be the value of Brian's loan after n years.
2 Determine the value of R where $r = 4.5$ and $p = 1$.	The interest rate is 4.5% per annum. $R = 1 + \frac{4.5}{100 \times 1} = 1.045$
3 Write the recurrence relation.	$V_0 = 5000, \quad V_{n+1} = 1.045V_n$
b 1 Define the variable V_n. The compounding period is *quarterly*.	Let V_n be the value of Brian's loan after n quarters.
2 Determine the value of R, where $r = 4.5$ and $p = 4$.	The interest rate is 4.5% per annum. $R = 1 + \frac{4.5}{100 \times 4} = 1.01125$
3 Write the recurrence relation.	$V_0 = 5000, \quad V_{n+1} = 1.01125V_n$
c 1 Define the variable V_n. The compounding period is *monthly*.	Let V_n be the value of Brian's loan after n months.
2 Determine the value of R where $r = 4.5$ and $p = 12$.	The interest rate is 4.5% per annum. $R = 1 + \frac{4.5}{100 \times 12} = 1.00375$
3 Write the recurrence relation.	$V_0 = 5000, \quad V_{n+1} = 1.00375V_n$

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Example 30 **Modelling an investment that compounds monthly using recurrence relations**

A principal value of \$10 000 is invested in an account earning compound interest monthly at the rate of 9% per annum.

Let V_n be the value of the investment after n months.

a Calculate the growth multiplier, R.

b Write down a recurrence relation for the value of the investment after n months.

c Write down a rule for the value of the investment after n months.

d Use this rule to find the value of the investment after 4 years.

Explanation	Solution
a Since interest compounds monthly, $p = 12$.	$R = 1 + \frac{9}{100 \times 12} = 1.0075$
b Substitute V_0 and R to form the recurrence relation.	$V_0 = 10\,000, \quad V_{n+1} = 1.0075V_n$
c Substitute $R = 1.0075$ and $V_0 = 10\,000$ into the rule to find the rule for V_n.	$V_n = 1.0075^n \times 10\,000$
d Substitute $n = 48$ (4 years = 48 months) into the rule.	$V_{48} = 1.0075^{48} \times 10\,000$ $= \$14314.05$

Effective interest rates

When interest compounds with different compounding periods, the total amount of interest earned in one year differs. To compare loans and investments, we can calculate the **effective interest rate** which is the percentage the value increases in one year.

$$\text{effective rate} = \frac{\text{Total interest in one year}}{\text{Principal}} \times 100\%$$

For example, if \$5000 is invested paying a nominal rate of 4.8% per annum and interest compounds quarterly, the value at the end of the year is \$5244.35 so the interest is \$244.35. Using the formula,

$$\begin{aligned}\text{effective rate} &= \frac{244.35}{5000} \times 100\% \\ &= 4.89\%\end{aligned}$$

In contrast, carrying out the same procedure when interest is compounding monthly gives an effective interest rate of 4.91% due to a value of \$5245.35 at the end of the year and hence interest of \$245.35. Thus, calculating effective interest rates with different compounding periods provides us with a good way of comparing loans and investments.

In order to calculate the effective interest rates for different loans or investments, we can use the following rule.

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Effective interest rate

The effective interest rate of a loan or investment is the interest earned after one year expressed as a percentage of the amount borrowed or invested.

Let:

- r be the nominal interest rate per annum
- r_{eff} be the effective annual interest rate
- n be the number of times the interest compounds each year.

The effective annual interest rate is given by: $r_{\text{eff}} = \left(\left(1 + \dfrac{r}{100 \times n}\right)^n - 1\right) \times 100\%$

Note: n is used here in line with the VCAA formula sheet and should not be confused with other usages of n in this chapter.

Example 31 Comparing loans and investments with effective interest rates

Brooke would like to borrow \$20 000 that she will repay entirely after one year. She is deciding between two loan options:

- option A: 5.95% per annum, compounding weekly
- option B: 6% per annum, compounding quarterly.

a Calculate the effective interest rate for each investment.

b Which investment option is the best and why?

Explanation	Solution
a 1 Decide on the values of r and n for each option.	A: $r = 5.95\%$ and $n = 52$ B: $r = 6\%$ and $n = 4$
2 Apply the effective interest rate rule.	A: $r_{\text{eff}} = \left(\left(1 + \dfrac{5.95}{100 \times 52}\right)^{52} - 1\right) \times 100\% = 6.13\%$ B: $r_{\text{eff}} = \left(\left(1 + \dfrac{6}{100 \times 4}\right)^{4} - 1\right) \times 100\% = 6.14\%$
b Compare the effective interest rates.	Brooke is borrowing money, so the best option is the one with the lowest effective interest rate. She will pay less interest with option A.

While the effective interest rate can be calculated manually, the CAS calculator can also be used to quickly perform the calculation. To do this, the nominal interest rate and the number of compounding periods in a year are required.

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Example 32 Calculating effective interest rates using a CAS calculator

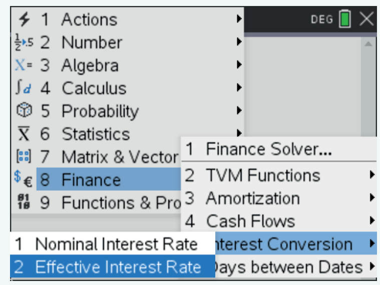
Marissa has $10 000 to invest. She chooses an account that will earn compounding interest at the rate of 4.5% per annum, compounding monthly.

Use a CAS calculator to find the effective interest rate for this investment, correct to three decimal places.

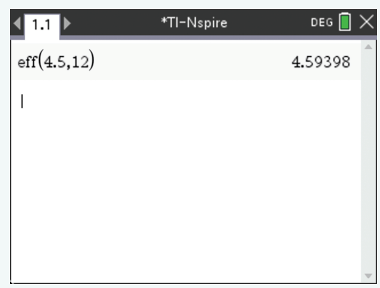
Explanation

Steps

1 Press menu and then select
8: Finance ▶
5: Interest Conversion ▶
2: Effective interest rate
to paste in the **eff(…)** command.
The parameters of this function are eff(nominal rate, number of times the interest compounds each year).

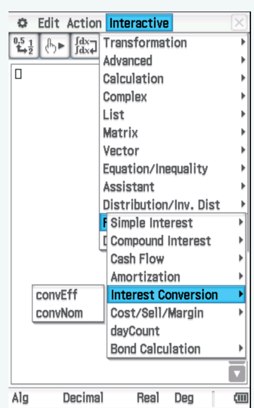


2 Enter the nominal rate (4.5) and number of times the interest compounds each year (12) into the function, separated by a comma. Press enter to get the effective interest rate.

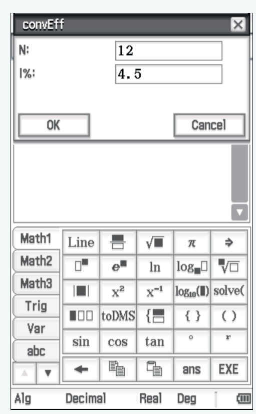


Steps

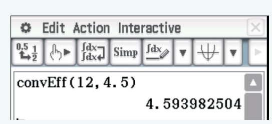
1 Select **Interactive**, **Financial**, **Interest Conversion, ConvEff**.



2 Enter the number of times the interest compounds each year (12) and the nominal rate (4.5) as shown .



3 Press **EXE** to get the effective interest rate. Write your answer.



Solution

The effective interest rate for this investment is 4.594%.

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Exercise 7F

Interest rate conversions

Example 28

1 Convert each of the annual interest rates below to an interest rate for the given time period. Write your answers correct to two decimal places.

a 4.8% per annum to monthly
b 8.3% per annum to quarterly
c 10.4% per annum to fortnightly
d 7.4% per annum to weekly
e 12.7% per annum to daily

2 Convert each of the interest rates below to an annual interest rate.

a 0.54% monthly
b 1.45% quarterly
c 0.57% fortnightly
d 0.19% weekly
e 0.022% daily

Recurrence relations with different compounding periods

Example 29

3 Verity borrows $8000 from a bank. She will pay interest at a rate of 4.8% per annum. Let V_n be the value of the loan after n compounding periods. Write down a recurrence relation to model the value of Verity's loan if interest is compounded:

a yearly
b quarterly
c monthly

Example 30

4 A principal value of $20 000 is invested in an account earning compound interest of 6% per annum, compounding monthly.

a Write down a recurrence relation for the value of the investment, V_n, after n months.
b Write down a rule for V_n in terms of n.
c Use this rule to find the value of the investment after 5 years (60 months). Round your answer to the nearest cent.

5 A principal value of $8 000 is invested in an account earning compound interest quarterly at the rate of 4.8% per annum.

a Write down a recurrence relation for the value of the investment, V_n, after n quarters.
b Write down a rule for V_n in terms of n.
c Use this rule to find the value of the investment after 3 years (12 quarters). Round your answer to the nearest cent.

6 Wayne invests $7600 with a bank. He will be paid interest at the rate of 6% per annum, compounding monthly. Let V_n be the value of the investment after n months.

a Write a recurrence relation to model Wayne's investment.
b Write down a rule for V_n in terms of n.
c How much is Wayne's investment worth after 5 months? Round your answer to the nearest cent.
d After how many months will Wayne's investment first double in value?

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7 Jessica borrows $3500 from a bank. She will be charged compound interest at the rate of 8% per annum, compounding quarterly. Let V_n be the value of the loan after n quarters.

a Write a recurrence relation to model the value of Jessica's loan.

b If Jessica pays back everything she owes to the bank after 1 year, how much money will she pay? Round your answer to the nearest cent.

Comparing loans and investments with effective interest rates

Example 31

8 Brenda invests $15 000 in an account earning (nominal) compound interest of 4.6% per annum.

a Calculate the effective interest rate for the current investment when interest compounds quarterly, correct to two decimal places.

b Calculate the effective rate for this investment when interest compounds monthly, correct to two decimal places.

c Should Brenda choose quarterly or monthly compounding?

9 Stella borrows $25 000 from a bank and pays nominal compound interest of 7.94% per annum.

a Calculate the effective rate for the current loan when interest compounds fortnightly, correct to two decimal places.

b Calculate the effective rate for this loan when interest compounds monthly, correct to two decimal places.

c Should Stella choose fortnightly or monthly compounding?

10 Luke is considering a loan of $35 000. His bank has two compound interest rate options:

A: 8.3% per annum, compounding monthly

B: 7.8% per annum, compounding weekly.

a Calculate the effective interest rate for each of the loan options. Round your answers to two decimal places.

b Calculate the amount of interest Luke would pay in the first year for each of the loan options. Round your answers to the nearest cent.

c Which loan should Luke choose and why?

11 Sharon is considering investing $140 000. Her bank has two compound interest investment options:

A: 5.3% per annum, compounding monthly

B: 5.5% per annum, compounding quarterly.

a Calculate the effective interest rate for each of the investment options. Round your answers to two decimal places.

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b Calculate the amount of interest Sharon would earn in the first year for each of the investment options. Give your answer to the nearest dollar.

c Which investment option should Sharon choose and why?

Calculating effective interest rates using the CAS

Example 32

12 Use your calculator to determine the effective annual interest rate, correct to two decimal places, for the following nominal rates and compounding periods.

a 6.2% per annum compounding monthly

b 8.4% per annum compounding daily

c 4.8% per annum compounding weekly

d 12.5% per annum compounding quarterly

13 An account increases by 7% in one year when interest compounds monthly. Find the annual interest rate correct to 2 decimal places.

Exam 1 style questions

14 Chung invests $3300 with a bank. He will be paid compound interest at the rate of 4.8% per annum, compounding monthly. If V_n is the value of Chung's investment after n months, a recurrence model for Chung's investment is

A $V_0 = 3300, \quad V_{n+1} = V_n + 4.8$

B $V_0 = 3300, \quad V_{n+1} = 4.8V_n$

C $V_0 = 3300, \quad V_{n+1} = 0.004V_n$

D $V_0 = 3300, \quad V_{n+1} = V_n + 158.40$

E $V_0 = 3300, \quad V_{n+1} = 1.004V_n$

15 An amount of $4700 is invested, earning compound interest at the rate of 6.8% per annum, compounding quarterly. The effective annual interest rate is closest to

A 6.80% **B** 6.97% **C** 6.98% **D** 7.02% **E** 7.03%

16 Isabella invests $5000 in an account that pays interest compounding monthly. After one year, the balance of the account is $5214.09.
The effective interest rate for this investment, rounded to two decimal places, is

A 0.35% **B** 0.42% **C** 3.50% **D** 4.20% **E** 4.28%

17 Maya invested $25 000 in an account at her bank with interest compounding monthly. After one year, the balance of Maya's account was $26 253.
The difference between the rate of interest per annum used by her bank and the effective annual rate of interest for Maya's investment is closest to

A 0.112% **B** 0.2% **C** 4.89%
D 4.9% **E** 5.012%

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Key ideas and chapter summary

Assign-ment

Sequence A **sequence** is a list of numbers or symbols written in succession, for example: 5, 15, 25, . . .

Term Each number or symbol that makes up a sequence is called a **term**.

Recurrence relation A relation that enables the value of the next term in a sequence to be obtained by one or more current terms. Examples include 'to find the next term, add two to the current term' and 'to find the next term, multiply the current term by three and subtract five'.

Modelling **Modelling** is the use of a mathematical rule or formula to represent or model real-life situations. Recurrence relations can be used to model situations involving the *growth* (increase) or *decay* (decrease) in values of a quantity.

Percentage growth and decay If a quantity grows by $r\%$ each year, then $R = 1 + \frac{r}{100}$.

If a quantity decays by $r\%$ each year, then $R = 1 - \frac{r}{100}$.

Principal The **principal** is the initial amount that is invested or borrowed.

Balance The **balance** is the value of a loan or investment at any time during the loan or investment period.

Interest The fee that is added to a loan or the payment for investing money is called the **interest**.

Simple interest **Simple interest** is a fixed amount of interest that is paid at regular time intervals. Simple interest is an example of linear growth.

Depreciation **Depreciation** is the amount by which the value of an item decreases after a period of time.

Scrap value **Scrap value** is the value of an item at which it is 'written off' or is considered no longer useful or usable.

Flat rate depreciation **Flat rate depreciation** is a constant amount that is subtracted from the value of an item at regular time intervals. It is an example of linear decay.

Unit cost depreciation **Unit cost depreciation** is depreciation that is calculated based on units of use rather than time. Unit-cost depreciation is an example of linear decay.

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Review

Compounding period Interest rates are usually quoted as annual rates (per annum). Interest is sometimes calculated more regularly than once a year, for example each quarter, month, fortnight, week or day. The time period for the calculation of interest is called the **compounding period**.

Compound interest When interest is added to a loan or investment and then contributes to earning more interest, the interest is said to compound. **Compound interest** is an example of geometric growth.

Reducing balance depreciation When the value of an item decreases as a percentage of its value after each time period, it is said to be depreciating using a reducing balance method. **Reducing balance depreciation** is an example of geometric decay.

Nominal interest rate A **nominal interest rate** is an annual interest rate for a loan or investment.

Effective interest rate The **effective interest rate** is the interest earned or charged by an investment or loan, written as a percentage of the original amount invested or borrowed. Effective interest rates allow loans or investments with different compounding periods to be compared. Effective interest rates can be calculated using the rule $r_{\text{eff}} = \left(\left(1 + \frac{r}{100 \times n}\right)^n - 1\right) \times 100\%$ where r is the nominal annual interest rate and n is the number of compounding periods in 1 year.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

7A **1** I can generate a sequence of terms recursively. ☐

See Example 1 and 2, and Exercise 7A Question 1 and 2

7A **2** I can generate a sequence of numbers with a calculator. ☐

See Example 3, and Exercise 7A Question 3

7A **3** I can generate a sequence from a recurrence relation. ☐

See Example 4, and Exercise 7A Question 4

7A **4** I can use a calculator to generate sequences from recurrence relations. ☐

See Example 5, and Exercise 7A Question 5

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7A **5** I can name terms in a sequence. ☐

See Example 6, and Exercise 7A Question 6

7B **6** I can graph the terms of a linear growth/decay sequence. ☐

See Example 7, and Exercise 7B Question 1

7B **7** I can model simple interest loans and investments with a recurrence relation. ☐

See Example 8, and Exercise 7B Question 2

7B **8** I can use a recurrence relation to analyse a simple interest investment. ☐

See Example 9, and Exercise 7B Question 4

7B **9** I can model flat rate depreciation using a recurrence relation. ☐

See Example 10, and Exercise 7B Question 7

7B **10** I can use a recurrence relation to analyse flat rate depreciation. ☐

See Example 11, and Exercise 7B Question 9

7B **11** I can model unit cost depreciation with a recurrence relation. ☐

See Example 12, and Exercise 7B Question 12

7C **12** I can convert a recurrence relation to an explicit rule. ☐

See Example 13, and Exercise 7C Question 1

7C **13** I can model a simple interest investment using an explicit rule. ☐

See Example 14, and Exercise 7C Question 2

7C **14** I can use a rule to determine the value of a simple interest investment. ☐

See Example 15, and Exercise 7C Question 4

7C **15** I can model flat rate depreciation of an asset with an explicit rule. ☐

See Example 16, and Exercise 7C Question 6

7C **16** I can use a rule for the flat rate depreciation of an asset. ☐

See Example 17, and Exercise 7C Question 8

7C **17** I can use an explicit rule for unit cost depreciation. ☐

See Example 18, and Exercise 7C Question 10

7D **18** I can graph the terms in a geometric sequence. ☐

See Example 19, and Exercise 7D Question 1

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Review

7D **19** I can model compound interest with a recurrence relation. ☐

See Example 20, and Exercise 7D Question 2

7D **20** I can model reducing balance depreciation with recurrence relations. ☐

See Example 21, and Exercise 7D Question 6

7D **21** I can use reducing balance depreciation with recurrence relations. ☐

See Example 22, and Exercise 7D Question 8

7E **22** I can write explicit rules for geometric growth and decay. ☐

See Example 23, and Exercise 7E Question 1

7E **23** I can use a rule to find the value of an investment after *n* years. ☐

See Example 24, and Exercise 7E Question 2

7E **24** I can calculate the value and total depreciation of an asset after a period of reducing balance depreciation. ☐

See Example 25, and Exercise 7E Question 5

7E **25** I can use a calculator to solve geometric growth and decay problems. ☐

See Example 26 and 27, and Exercise 7E Question 7 and Question 9

7F **26** I can convert nominal (annual) interest rates to compounding period interest rates. ☐

See Example 28, and Exercise 7F Question 1

7F **27** I can use recurrence relations to model loans with different compounding periods. ☐

See Example 29, and Exercise 7F Question 3

7F **28** I can model an investment that compounds monthly using recurrence relations. ☐

See Example 30, and Exercise 7F Question 4

7F **29** I can compare loans and investments using effective interest rates. ☐

See Example 31, and Exercise 7F Question 8

7F **30** I can calculate effective interest rates using a CAS calculator. ☐

See Example 32, and Exercise 7F Question 12

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Multiple-choice questions

1 Consider the following recurrence relation

$$V_0 = 5, \qquad V_{n+1} = V_n - 3$$

The sequence generated by this recurrence relation is

A 5, 15, 45, 135, 405, . . .
B 5, 8, 11, 14, 17, . . .
C 5, 2, −1, −4, −7, . . .
D 5, 15, 45, 135, 405, . . .
E 5, −15, 45, −135, 405, . . .

2 Consider the following recurrence relation

$$V_0 = 2, \qquad V_{n+1} = 2V_n + 8$$

The value of the term V_4 in the sequence generated by this recurrence relation is

A 12 **B** 18 **C** 32 **D** 72 **E** 152

3 Consider the following recurrence relation

$$V_0 = 5, \qquad V_{n+1} = 3V_n - 6$$

The value of the term V_3 in the sequence of numbers generated by this recurrence relation is

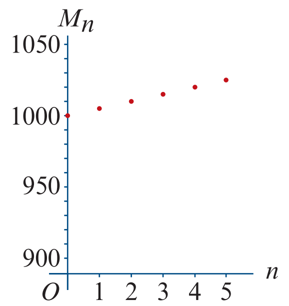
A 5 **B** 9 **C** 21 **D** 57 **E** 165

4 Brian has two trees in his backyard. Every month, he will plant three more trees. A recurrence relation for the number of trees, T_n, in Brian's backyard after n months is

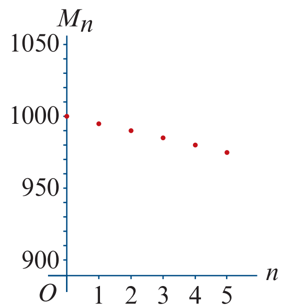
A $T_0 = 2, \qquad T_{n+1} = 3T_n$
B $T_0 = 2, \qquad T_{n+1} = 3T_n + 3$
C $T_0 = 2, \qquad T_{n+1} = T_n + 3$
D $T_0 = 2, \qquad T_{n+1} = T_n - 3$
E $T_0 = 2, \qquad T_{n+1} = 3T_n - 3$

5 A graph that shows the value of a simple interest investment of \$1000, earning interest of \$5 per month is

A



B



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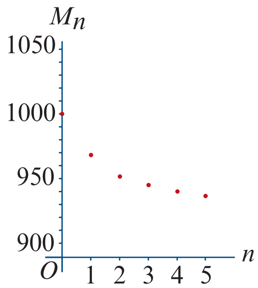
C



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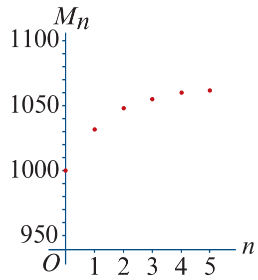
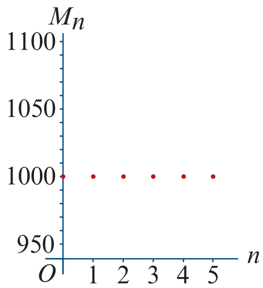


E



6 A car is depreciated using a unit cost depreciation method. It was purchased for \$18 990 and, after travelling a total of 20 000 kilometres, it has an estimated value of \$15 990. The depreciation amount, per kilometre, is

A \$0.15 **B** \$0.80 **C** \$0.95 **D** \$6.67 **E** \$3000

7 Arthur invests \$2000 with a bank. He will be paid simple interest at the rate of 5.1% per annum. If V_n is the value of Arthur's investment after n years, a recurrence relation for Arthur's investment is

A $V_0 = 2000, \quad V_{n+1} = V_n + 5.1$

B $V_0 = 2000, \quad V_{n+1} = 5.1V_n$

C $V_0 = 2000, \quad V_{n+1} = 0.051V_n + 102$

D $V_0 = 2000, \quad V_{n+1} = V_n + 102$

E $V_0 = 2000, \quad V_{n+1} = 1.051V_n + 2000$

8 An interest rate of 4.6% per annum is equivalent to an interest rate of

A 1.15% per quarter **B** 0.35% per month **C** 0.17% per week

D 0.17% per fortnight **E** \$0.39% per month

9 A sequence is generated from the recurrence relation

$$V_0 = 40, \quad V_{n+1} = V_n - 16$$

The rule for the value of the term V_n is

A $V_n = 40n - 16$ **B** $V_n = 40 - 16n$ **C** $V_n = 40n$

D $V_n = 40 + 16n$ **E** $V_n = 40n - 16$

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10 The recurrence relation that generates a sequence of numbers representing the value of a car n years after it was purchased is

$$V_0 = 18\,000, \qquad V_{n+1} = V_n - 1098$$

The car had a purchase price of \$18 000 and is being depreciated using

A flat rate depreciation at 6.1% of its value per annum
B flat rate depreciation at \$6.10 per kilometre travelled
C flat rate depreciation at \$1098 per kilometre travelled
D unit cost depreciation at \$6.10 per kilometre travelled
E unit cost depreciation at \$1098 per kilometre travelled

11 A computer is depreciated using a flat rate depreciation method. It was purchased for \$2800 and depreciates at the rate of 8% per annum. The amount of depreciation after 4 years is

A \$224 **B** \$448 **C** \$794 **D** \$896 **E** \$1904

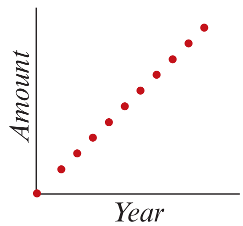
12 Sandra invests \$6000 in an account that pays interest at the rate of 4.57% per annum, compounding annually. The number of years it takes for the investment to exceed \$8000 is

A 5 **B** 6 **C** 7 **D** 8 **E** 9

13 The value of a machine is depreciating by 8% every year. The initial value is 2700. A recurrence relation model for the value of the machine after n years, P_n, is

A $P_0 = 2700, \quad P_{n+1} = 1.8 \times P_n$
B $P_0 = 2700, \quad P_{n+1} = 1.08 \times P_n$
C $P_0 = 2700, \quad P_{n+1} = 0.92 \times P_n$
D $P_0 = 2700, \quad P_{n+1} = 1 + 8 \times P_n$
E $P_0 = 2700, \quad P_{n+1} = 1.08 + P_n$

14 An investment of \$50 000 is compounding annually over a number of years. The graph that best represents the value of the investment at the end of each year is

A


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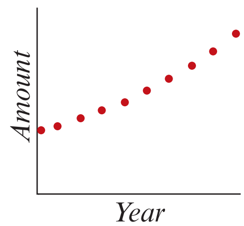

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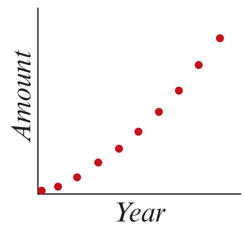

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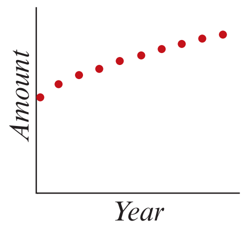

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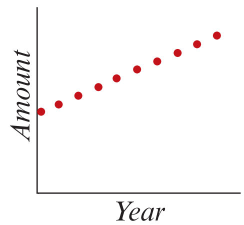

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15 An item is depreciated using a reducing balance depreciation method. The value of the item after n years, V_n, is modelled by the recurrence relation

$$V_0 = 4500, \quad V_{n+1} = 0.86V_n$$

The rule for the value of the item after n years is

A $V_n = 0.86^n \times 4500$
B $V_n = 1.86^n \times 4500$
C $V_n = (1 + 0.86)^n \times 4500$
D $V_n = 0.86 \times n \times 4500$
E $V_n = (1 - 0.86)^n \times 4500$

16 After 10 years, a compound interest investment of \$8000 earned a total of \$4000 in interest when compounding annually. The annual interest rate of this investment was closest to

A 2.5% **B** 4.14% **C** 5.03% **D** 7.2% **E** 50%

17 The interest rate on a compound interest loan is 12.6% per annum, compounding monthly. The value of the loan after n months, V_n, is modelled by the recurrence relation

$$V_0 = 400, \quad V_{n+1} = R \times V_n$$

The value of the growth multiplier, R, in this recurrence relation is

A 0.874 **B** 1.00 **C** 1.0105 **D** 1.126 **E** 2.05

18 An amount of \$2000 is invested, earning compound interest at the rate of 5.4% per annum, compounding quarterly. The effective annual interest rate is closest to

A 5.2% **B** 5.3% **C** 5.4% **D** 5.5% **E** 5.6%

19 A car was purchased for \$74 500. It depreciates in value at a rate of 8.5% per year, using a reducing balance depreciation method. The total depreciation of the car over 5 years is closest to

A \$4439
B \$26 718
C \$37 522
D \$47 782
E \$112 022

20 Sam invested \$6500 at 8.75% per annum with interest compounding monthly. If the investment now amounts to \$13 056, for how many years was it invested?

A 5 **B** 7 **C** 8 **D** 9 **E** 96

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Written response questions

1 Jack borrows \$20 000 from a bank and is charged simple interest at the rate of 9.4% per annum. Let V_n be the value of the loan after n years.

a Write down a recurrence relation for the value of Jack's loan after n years.

b Use the recurrence relation to model how much Jack will need to pay the bank after 5 years.

The bank decides to change the loan to a compound interest loan on a yearly basis, with an annual interest rate of 9.4%. Let W_n be the value of the loan after n years.

c Write a recurrence relation to model the value of Jack's loan.

d Write a rule for W_n in terms of n.

e Use the rule to find the value of the loan after 5 years. Round your answer to the nearest cent.

2 Ilana uses a personal loan to buy a dress costing \$300. Interest is charged at 18% per annum, compounding monthly.
If she repays the loan fully after 6 months, how much will she pay? Round your answer to the nearest cent.

3 Kelly bought her current car 5 years ago for \$22 500.
Let V_n be the value of Kelly's car after n years.

a If Kelly uses a flat rate depreciation of 12% per annum:

i write down a recurrence relation for the value of Kelly's car after n years

ii use the recurrence relation to find the current value of Kelly's car.

b If Kelly uses reducing value depreciation at 16% per annum:

i write down a recurrence relation for the value of Kelly's car after n years

ii use the recurrence relation to find the current value of Kelly's car using reducing balance depreciation. Round your answer to the nearest cent.

c On the same axes, sketch a graph of the value of Kelly's car against the number of years for both flat rate and reducing balance depreciation.

4 A commercial cleaner bought a new vacuum cleaner for \$650. The value of the vacuum cleaner decreases by \$10 for every 50 offices that it cleans.

a How much does the value of the vacuum cleaner depreciate when one office is cleaned?

b Give a recurrence relation for the value of the vacuum cleaner, V_n, after n offices have been cleaned.

c The cleaner has a contract to clean 10 offices, 5 nights a week for 40 weeks in a year. What is the value of the vacuum cleaner after 1 year?

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5 Meghan has \$5000 to invest.

Company A offers her an account paying 6.3% per annum simple interest.

Company B offers her an account paying 6.1% per annum compound interest.

a How much will she have in the account offered by company A at the end of 5 years?

b How much will she have in the account offered by company B at the end of 5 years? Round your answer to the nearest cent.

c Find the simple interest rate that company A should offer if the two investments are to have equal values after 5 years. Round your answer to one decimal place.

6 A sum of \$30 000 is borrowed at an interest rate of 9% per annum, compounding monthly.

Let V_n be the value of the loan after n months.

a Write a recurrence relation to model the value of this loan.

b Use the recurrence relation to find the value of the loan at the end of the first five months. Round your answer to the nearest cent.

c What is the value of the loan after 1 year? Round your answer to the nearest cent.

d If the loan is fully repaid after 18 months, how much money is paid? Round your answer to the nearest cent.

7 On the birth of his granddaughter, a man invests a sum of money at a rate of 11.65% per annum, compounding twice per year.

On her 21st birthday he gives all of the money in the account to his granddaughter.

If she receives \$2529.14, how much did her grandfather initially invest? Round your answer to the nearest cent.

8 Geoff invests \$18 000 in an investment account. After 2 years the investment account contains \$19 300.

If the account pays r% interest per annum, compounding quarterly, find the value of r, to one decimal place.

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