Chapter 10

Matrices

Chapter questions

- ▶ What is a matrix?
- ▶ How is the order of a matrix defined?
- ▶ How are the positions of the elements of a matrix specified?
- ▶ How do we use matrices to represent information and solve practical problems?
- ▶ How do we use matrices to represent a network diagram?
- ▶ What are the rules for adding and subtracting matrices?
- ▶ How do we multiply a matrix by a scalar?
- ▶ What is the method for multiplying a matrix by another matrix?
- ▶ How do we form and use permutation, communication and dominance matrices?
- ▶ How can your CAS calculator be used to do matrix operations?

Matrix algebra was first studied in England in the middle of the nineteenth century. Matrices are now used in many areas of science and business: for example, in physics, medical research, encryption and internet search engines.

In this chapter we will show how addition and multiplication of matrices can be defined and how matrices can be used to describe the relationship between people, businesses and sporting teams.

In Chapter 11 we will see how matrices can be used to represent networks.

10A What is a matrix?

Learning intentions

- To be able to state the order of a given matrix.
- To be able to describe the location of an element in a matrix.
- To be able to determine the transpose of a matrix.
- To be able to define and recognise diagonal, symmetric and triangular matrices.
- To be able to define and recognise identity matrices.

A **matrix** (plural **matrices**) is a rectangular array or table of numbers or symbols, arranged in rows and columns. We form a matrix from data in the following way.

The table of data shown below displays the heights, weights, ages and pulse rates of eight students.

Name	*Height*	*Weight*	*Age*	*Pulse rate*
Mahdi	173	57	18	86
Dave	179	58	19	82
Jodie	167	62	18	96
Simon	195	84	18	71
Kate	173	64	18	90
Pete	184	74	22	78
Mai	175	60	19	88
Tran	140	50	34	70

$$D = \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 167 & 62 & 18 & 96 \\ 195 & 84 & 18 & 71 \\ 173 & 64 & 18 & 90 \\ 184 & 74 & 22 & 78 \\ 175 & 60 & 19 & 88 \\ 140 & 50 & 34 & 70 \end{bmatrix}$$

If we extract the numbers from the table and enclose them in square brackets, we form a matrix. We might call this matrix D (for data matrix). We use capital letters A, B, C, etc. to name matrices.

Rows and columns

Rows and columns are the building blocks of matrices. We number rows from the top down: row 1, row 2, etc. Columns are numbered from the left across: column 1, column 2, etc.

$$D = \begin{array}{ll} & \begin{array}{cccc} & & \text{Col. 3} & \end{array} \\ \begin{array}{l} \\ \text{Row 2} \\ \\ \\ \\ \\ \\ \\ \end{array} & \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 167 & 62 & 18 & 96 \\ 195 & 84 & 18 & 71 \\ 173 & 64 & 18 & 90 \\ 184 & 74 & 22 & 78 \\ 175 & 60 & 19 & 88 \\ 140 & 50 & 34 & 70 \end{bmatrix} \end{array}$$

Order of a matrix

In its simplest form, a matrix is just a rectangular array (rows and columns) of numbers.

The **order** (or size) of matrix D is said to be 8×4, read '8 by 4' because it has *eight rows* and *four columns*.

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Order of a matrix

Order of a matrix = number of rows × number of columns

The numbers, or entries, in the matrix are called **elements**.

The number of elements in a matrix is determined by its order. For example, the matrix D has order 8×4 and the the number of elements in matrix D is 32 ($8 \times 4 = 32$).

Row matrices

Matrices come in many shapes and sizes. For example, from this same set of data, we could have formed the matrix called K

$$K = \begin{bmatrix} 173 & 64 & 18 & 90 \end{bmatrix}$$

This matrix has been formed from just one row of the data: the data values for Kate. Because it only contains *one row* of numbers, it is called a **row matrix** (or **row vector**). It is a 1×4 matrix: one row by four columns. It contains $1 \times 4 = 4$ elements.

Column matrices

Equally, we could form a matrix called H (for height matrix). This matrix is formed from just one column of the data, the heights of the students.

Because it only contains *one column* of numbers, it is called a **column matrix** (or **column vector**). This is an 8×1 matrix: eight rows by one column. It contains $8 \times 1 = 8$ elements.

$$H = \begin{bmatrix} 173 \\ 179 \\ 167 \\ 195 \\ 173 \\ 184 \\ 175 \\ 140 \end{bmatrix}$$

Example 1

State the order of each of the following matrices.

a $\begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 7 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 5 & 8 & 9 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 1 \\ 4 \\ 4 \\ 9 \end{bmatrix}$ **d** $\begin{bmatrix} 1 & 4 & 4 & 5 \\ 4 & 3 & 4 & 6 \\ 4 & 3 & 2 & 1 \\ 9 & 1 & 0 & 7 \end{bmatrix}$

Solution

a 3 rows and 2 columns.
Order is 3×2

b 1 row and 5 columns
Order is 1×5

c 4 rows and 1 column.
Order is 4×1

d 4 rows and 4 columns.
Order is 4×4

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Switching rows and columns: the transpose of a matrix

If you switch the rows and columns in a matrix you have what is called the **transpose** of the matrix.

For example, the *transpose* of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

The transpose of a row matrix is a column matrix and vice versa.

For example, the *transpose* of the matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

The symbol we use to indicate the transpose of a matrix is T.

Thus, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Note: The transpose of a 3×2 matrix is a 2×3 matrix because the rows and columns are switched.

Example 2 The transpose of a matrix

a Write down the transpose of $\begin{bmatrix} 7 & 4 \\ 8 & 1 \end{bmatrix}$.

b Write down the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}^T$.

c If $A = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$, write down the matrix A^T.

Explanation

a The transpose of the matrix is obtained by switching (interchanging) its rows and columns.

b The symbol T is an instruction to transpose the matrix.

c The symbol T is an instruction to transpose matrix A.

Solution

a $\begin{bmatrix} 7 & 8 \\ 4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}$

c $A^T = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

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Square matrices

As a final example, we could form a matrix we call M (for males). This matrix contains only the data for the males. As this matrix has four rows and four columns, it is a 4×4 matrix. It contains $4 \times 4 = 16$ elements.

$$M = \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 195 & 84 & 18 & 71 \\ 184 & 74 & 22 & 78 \end{bmatrix}$$

A matrix with an *equal* number of *rows* and *columns* is called a **square matrix**.

Example 3 Matrix facts

For each of the matrices below, write down its type, order and the number of elements.

Solution

Matrix	*Type*	*Order*	*No. of elements*
$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 4 \\ 2 & -1 & 6 \end{bmatrix}$	Square matrix rows = columns	3×3 3 rows, 3 cols.	9 $3 \times 3 = 9$
$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	Column matrix single column	3×1 3 rows, 1 col.	3 $3 \times 1 = 3$
$C = \begin{bmatrix} 3 & 1 & 0 & 5 & -3 & 1 \end{bmatrix}$	Row matrix single row	1×6 1 row, 6 cols.	6 $1 \times 6 = 6$

Diagonal, symmetric and triangular matrices

Some square matrices occur so often in practice that they have their own names.

Diagonal matrices

A square matrix has two diagonals: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

In practice, the diagonal going downwards from left to right in the matrix (coloured red) turns out to be more important than the other diagonal (coloured blue), so we give it a special name: the *leading diagonal*.

A square matrix is called a *diagonal matrix* if all of the elements off the leading diagonal are zero. The elements on the leading diagonal may or may not be zero.

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The matrices opposite are all diagonal matrices: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Identity matrices

Diagonal matrices in which each element in the diagonal is 1 are of special importance. They are called identity or unit matrices and have their own name and symbol (I).

Every order of square matrix has its own **identity matrix**, three of which are shown opposite.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Symmetric matrices

A symmetric matrix is a square matrix that is unchanged by transposition (switching rows and columns). In a symmetric matrix, the elements above the leading diagonal are a mirror image of the elements below the diagonal. Three are shown.

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 1 & 5 & 7 \\ 4 & 5 & 3 & 8 \\ 6 & 7 & 8 & 5 \end{bmatrix}$$

Triangular matrices

Triangular matrices come in two types:

1. An upper triangular matrix is a square matrix in which all elements below the leading diagonal are zeros.
2. A lower triangular matrix is a square matrix in which all elements above the leading diagonal are zeros.

Examples of triangular matrices are shown.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 6 & 5 & 4 & 0 \\ 0 & 9 & 8 & 7 \end{bmatrix}$$

upper triangular matrix   lower triangular matrix

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Example 4 **Types of matrices**

Consider the following square matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 7 \\ 5 & 7 & 2 \end{bmatrix}$$

Write down:

a the upper triangular matrices

b the identity matrix

c the diagonal matrices

d the symmetric matrices.

Explanation	Solution
a All the elements below the leading diagonal are 0.	$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
b Elements in the leading diagonal are all 1 and the other elements 0.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
c All elements other than those in the leading diagonal are zero.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
d The matrix must be its own transpose.	$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 7 \\ 5 & 7 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Some notation

In some situations, we talk about a matrix and its elements without having specific numbers in mind. We can do this as follows.

For the matrix A, which has n rows and m columns, we write:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix}$$

a_{23}

row number column number

Thus:

- a_{21} represents the element in the second row and the first column
- a_{12} represents the element in the first row and the second column
- a_{22} represents the element in the second row and the second column
- a_{mn} represents the element in the mth row and the nth column.

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Example 5 Identifying the elements in a matrix

For the matrices A and B, opposite, write down the values of:

a a_{12} **b** a_{21} **c** a_{33} **d** b_{31}.

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -1 & 0 & 4 \\ 2 & -2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Explanation	Solution
a a_{12} is the element in the first row and the second column of A.	$a_{12} = 5$
b a_{21} is the element in the second row and the first column of A.	$a_{21} = -1$
c a_{33} is the element in the third row and the third column of A.	$a_{33} = 6$
d b_{31} is the element in the third row and the first column of B.	$b_{31} = 1$

In some instances, there is a rule connecting the value of each matrix with its row and column number. In such circumstances, it is possible to construct this matrix knowing this rule and the order of the matrix.

Example 6 Constructing a matrix given a rule for its *ij*th term

A is a 3×2 matrix. The element in row i and column j is given by $a_{ij} = i + j$. Construct the matrix.

Explanation

1 The matrix is square and will have the form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Use the rule

$$a_{ij} = i + j$$

to generate the elements one by one. For example $a_{32} = 3 + 2 = 5$

2 Write down the matrix.

Solution

$$\begin{array}{cc} & \begin{array}{cc} j=1 & j=2 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \\ i=3 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \end{array}$$

where

$a_{11} = 1 + 1 = 2 \quad a_{12} = 1 + 2 = 3$

$a_{21} = 2 + 1 = 3 \quad a_{22} = 2 + 2 = 4$

$a_{31} = 3 + 1 = 4 \quad a_{32} = 3 + 2 = 5$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$$

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Entering a matrix into a CAS calculator

Later in this chapter, you will learn about matrix arithmetic: how to add, subtract and multiply matrices. While it is possible to carry out these tasks by hand, for all but the smallest matrices this is very tedious. Most matrix arithmetic is better done with the help of a CAS calculator. However, before you can perform matrix arithmetic, you need to know how to enter a matrix into your calculator.

CAS 1: How to enter a matrix on the TI-Nspire CAS

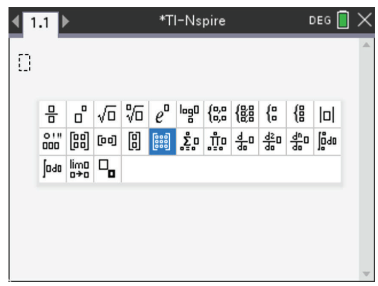
Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and determine its transpose (A^T).

Steps

1 Press ctrl + N. Select **Add Calculator**.

2 Press (templates key) and use the cursor ▼ ▶ arrows to highlight the matrix template shown. Press enter.

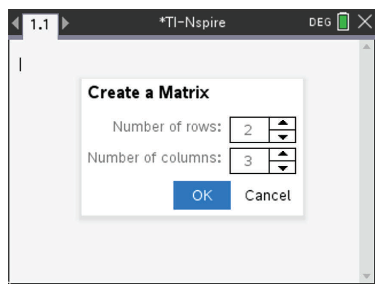
Note: **Math Templates** can also be accessed by pressing ctrl + menu **>Templates**.



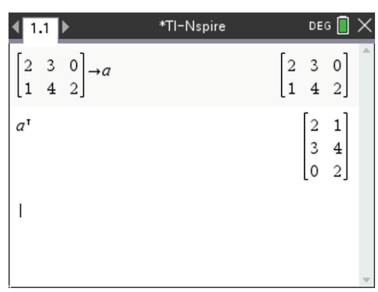
3 Use the ▼ arrow to select the **Number of rows** required (number of rows in this example is 2).

Press tab to move to the next entry and repeat for the **Number of columns** (the number of columns in this example is 3).

Press tab to highlight **OK**; press enter.



4 Type the values into the matrix template. Use tab to move to the required position in the matrix to enter each value. When the matrix has been completed, press tab to move outside the matrix, press ctrl var, followed by A. Press enter. This will store the matrix as the variable a.



5 When you type ***A*** (or ***a***) it will paste in the matrix $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Press enter to display.

6 To find a^T, type in **a** (for matrix A) and then menu **>Matrix & Vector>Transpose** **>** enter as shown.

Note: Superscript T can also be accessed from the symbols palette (ctrl (book key)).

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CAS 1: How to enter a matrix using the ClassPad

Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and determine its transpose (A^T).

Steps

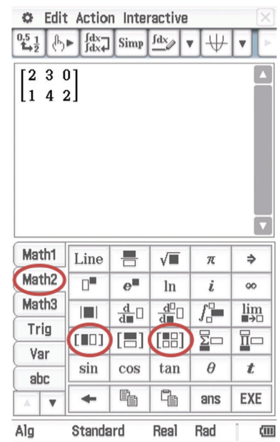
1 **a** Open the **Main** ($\sqrt{\alpha}$) application Press Keyboard to display the soft keyboard.

b Select the Math2 keyboard.

2 Tap the 2×2 matrix icon, followed by the 1×2 matrix icon. This will add a third column and create a 2×3 matrix.

3 Type the values into the matrix template.

Note: Tap at each new position to enter the new value or use the black cursor key on the hard keyboard to navigate to the new position.



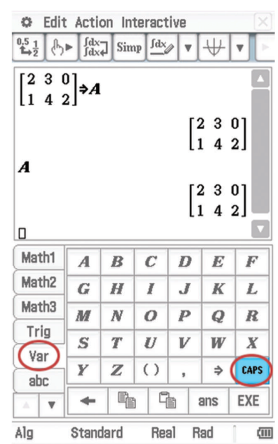
4 To assign the matrix the variable name ***A***:

a move the cursor to the very right-hand side of the matrix

b tap the variable assignment key ⇒ followed by var CAPS ***A***

c press EXE to confirm your choice.

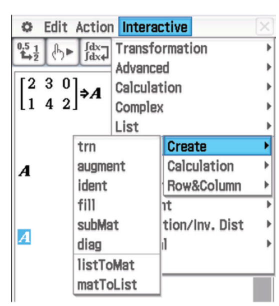
Note: Until it is reassigned, ***A*** will represent the matrix as defined above.

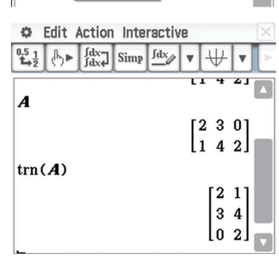


5 To calculate the transpose matrix $\boldsymbol{A}^T$:

a type and highlight ***A*** (by swiping with the stylus)

b select **Interactive** from the menu bar, tap **Matrix-Create** and then tap **trn**.





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Skill-sheet

Exercise 10A

Order of a matrix

Example 1

1 State the order of each of the following matrices.

a $\begin{bmatrix} 1 & 5 & 9 \\ 3 & 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 7 & 6 & 12 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 6 \\ 3 & 4 \\ 11 & 8 \end{bmatrix}$ **d** $\begin{bmatrix} 18 \\ 7 \\ 1 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 4 & 4 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix}$

2 Write down the order of the following matrices:

a $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$

3 How many elements are there in a

a 2×6 matrix? **b** 3×5 matrix? **c** 7×4 matrix?

4 A matrix has 12 elements. What are its possible orders? (There are six.)

The transpose of a matrix

Example 2

5 Determine the transpose of each matrix

a $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & 1 & 0 & 7 \\ 8 & 9 & 1 & 5 \end{bmatrix}$

Types of matrices and their elements

Example 3

6 For each of the following matrices, state whether it is a column, row or square matrix, and state the order and the number of elements.

a $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$ **c** $\begin{bmatrix} 7 & 11 & 2 & 1 \end{bmatrix}$

Example 4

7 Consider the following square matrices.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$$

Identify:

a the upper triangular matrices **b** the identity matrix

c the diagonal matrices **d** the symmetric matrices.

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Example 5

8 Complete the sentences below that relate to the following matrices.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 6 \\ -1 & 0 \\ 1 & 3 \\ 4 & -4 \end{bmatrix} \quad E = \begin{bmatrix} 4 & 3 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

a The square matrices are ☐ and ☐.
b Matrix B has ☐ rows.
c The row matrix is ☐.
d The column matrix is ☐.
e Matrix D has ☐ rows and ☐ columns.
f The order of matrix E is ☐ × ☐.
g The order of matrix A is ☐ × ☐.
h The order of matrix B is ☐ × ☐.
i The order of matrix D is ☐ × ☐.
j There are ☐ elements in matrix E.
k There are ☐ elements in matrix A.

l $a_{14} =$ ☐ **m** $b_{31} =$ ☐ **n** $c_{11} =$ ☐ **o** $d_{41} =$ ☐
p $e_{22} =$ ☐ **q** $d_{32} =$ ☐ **r** $b_{11} =$ ☐ **s** $c_{12} =$ ☐

Constructing a matrix given a rule for its *ij*th term

Example 6

9 B is a 3×2 matrix. The element in row i and column j is given by $b_{ij} = i \times j$. Construct the matrix.

10 C is a 4×1 matrix. The element in row i and column j is given by $c_{ij} = i + 2j$. Construct the matrix.

11 D is a 3×2 matrix. The element in row i and column j is given by $d_{ij} = i - 3j$. Construct the matrix.

12 E is a 1×3 matrix. The element in row i and column j is given by $e_{ij} = i + j^2$. Construct the matrix.

13 F is a 2×2 matrix. The element in row i and column j is given by $f_{ij} = (i + j)^2$. Construct the matrix.

Entering a matrix into a CAS calculator and determining the transpose

14 Enter the following matrices into your calculator and determine the transpose.

a $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ **b** $C = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$ **c** $E = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ **d** $F = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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Exam 1 style questions

15 Consider the following four matrices.

■ $\begin{bmatrix} 5 & 0 & 5 & 0 \\ 0 & 5 & 7 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ ■ $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ ■ $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ ■ $\begin{bmatrix} 5 & 0 & 5 & 0 \\ 0 & 5 & 7 & 5 \\ 0 & 8 & 5 & 0 \\ 9 & 0 & 0 & 5 \end{bmatrix}$

How many of these matrices are diagonal matrices?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

16 The element in row i and column j of matrix A is a_{ij}. A is a 3×3 matrix. It is constructed using the rule $m_{ij} = 3i + 2j$. A is

A $\begin{bmatrix} 5 & 3 & 5 \\ 1 & 4 & 7 \\ 6 & 7 & 15 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 5 & 9 & 10 \\ 7 & 11 & 12 \\ 9 & 13 & 14 \end{bmatrix}$ **D** $\begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 13 & 15 \end{bmatrix}$

17 Consider the matrix A, where $A = \begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{bmatrix}$. The element in row i and column j of matrix A is a_{ij}. The elements in matrix A are determined by the rule

A $a_{ij} = 4 + j$ **B** $a_{ij} = 2i + 3j$ **C** $a_{ij} = i + j + 3$

D $a_{ij} = 3i + 2j$ **E** $a_{ij} = 2i - j + 2$

18 The matrix $\begin{bmatrix} 1 & 3 & 5 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is an example of

A a diagonal matrix. **B** an identity matrix. **C** a symmetric matrix.

D an upper triangular matrix. **E** a lower triangular matrix.

19 The element in row i and column j of matrix A is a_{ij}. The elements in matrix A are determined using the rule $a_{ij} = 2i + j$. Matrix A could be

A $\begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$ **D** $[5]$ **E** $\begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 6 \\ 7 & 8 & 7 \end{bmatrix}$

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10B Using matrices to represent information

Learning intentions

- To be able to represent data given in a table by a matrix.
- To be able to represent a network diagram with a matrix.

At the start of this chapter we used a matrix to store numerical information in a data table. Matrices can also be used to carry codes that encrypt credit-card numbers for internet transmission or to carry all the information needed to solve sets of simultaneous equations. A less obvious application is using matrices to represent network diagrams.

Using a matrix to represent data tables

The numerical information in a data table is frequently presented in rows and columns. As such, it is a relatively straightforward process to convert this information into matrix form.

Example 7 Representing information in a table by a matrix

The table opposite shows the three types of membership of a local gym and the number of males and females enrolled in each. Construct a matrix to display the numerical information in the table.

	Gym membership		
Gender	*Weights*	*Aerobics*	*Fitness*
Males	16	104	86
Females	75	34	94

Explanation

1 Draw a blank (2×3) matrix.
Label the rows M for male and F for female.
Label the columns W for weights, A for aerobics and F for fitness.

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner of the table.

Solution

$$\begin{array}{cc} & \begin{array}{ccc} W & A & F \end{array} \\ \begin{array}{c} M \\ F \end{array} & \begin{bmatrix} & & \\ & & \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} W & A & F \end{array} \\ \begin{array}{c} M \\ F \end{array} & \begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix} \end{array}$$

Example 8 Entering a credit card number into a matrix

Convert the 16-digit credit card number: 4454 8178 1029 3161 into a 2×8 matrix, listing the digits in pairs, one under the other. Ignore the spaces.

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Explanation

1 Write out the sequence of numbers.

Note: Writing the number down in groups of four (as on the credit card) helps you keep track of the figures.

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner of the table.

Solution

4454 8178 1029 3161

$$\begin{bmatrix} 4 & 5 & 8 & 7 & 1 & 2 & 3 & 6 \\ 4 & 4 & 1 & 8 & 0 & 9 & 1 & 1 \end{bmatrix}$$

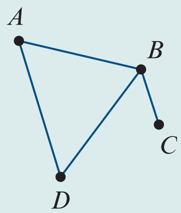
Using matrices to represent network diagrams

A less obvious use of matrices is to represent the information contained in network diagrams. Network diagrams consist of a series of numbered or labelled points joined in various ways. They are a powerful way of representing and studying things as different as friendship networks, airline routes, electrical circuits and road links between towns.

Example 9 Representing a network diagram by a matrix

Represent the network diagram shown opposite as a 4×4 matrix M, where the:

- matrix element = 1 if the two points are joined by a line
- matrix element = 0 if the two points are not connected.



Explanation

1 Draw a blank 4×4 matrix, labelling the rows and columns A, B, C, D to indicate the points.

Solution

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

2 Fill in the elements of the matrix row by row, starting at the top left-hand corner:

- $m_{11} = 0$ (no line joining point A to itself)
- $m_{12} = 1$ (a line joining points A and B)
- $m_{13} = 0$ (no line joining points A and C)
- $m_{14} = 1$ (a line joining points A and D)

and so on until the matrix is complete.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Note: If a network contains no 'loops' (lines joining points to themselves) the elements in the leading diagonal will always be zero. Knowing this can save a lot of work.

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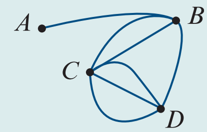
Example 10 Interpreting a matrix representing a network diagram

The diagram opposite shows the roads connecting four towns: town A, town B, town C, and town D. This diagram has been represented by a 4×4 matrix, M. The elements show the number of roads between each pair of towns.



$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

a In the matrix M, $m_{24} = 1$. What does this tell us?

b In the matrix M, $m_{34} = 3$. What does this tell us?

c In the matrix M, $m_{41} = 0$. What does this tell us?

d What is the sum of the elements in row 3 of matrix M and what does this tell us?

e What is the sum of all the elements of matrix M and what does this tell us?

Explanation

a There is one road between town B and town D.

b There are three roads between town D and town C.

c There is no road between town D and town A.

d 5: the total number of roads between town C and the other towns in the network.

e 14: the total number of different ways you can travel between towns.

Note: For each road, there are two ways you can travel; for example, from town A to town B ($a_{12} = 1$) and from town B to town A ($a_{21} = 1$).

Exercise 10B

Representing a table of data in matrix form

Example 7

1 The table opposite gives the number of residents, TVs and computers in three households.

Household	*Residents*	*TVs*	*Computers*
A	4	2	1
B	6	2	3
C	2	1	0

Use the table to:

a construct a matrix to display the numbers in the table. What is its order?

b construct a row matrix to display the numbers in the table relating to household B. What is its order?

c construct a column matrix to display the numbers in the table relating to computers. What is its order? What does the sum of its elements tell you?

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2 The table opposite gives the yearly car sales for two car dealers.

Car sales	*Small*	*Medium*	*Large*
Honest Joe's	24	32	11
Super Deals	32	34	9

Use the table to:

a construct a matrix to display the numbers in the table. What is its order?

b construct a row matrix to display the numbers in the table relating to Honest Joe's. What is its order?

c construct a column matrix to display the numerical information in the table relating to small cars. What is its order? What does the sum of its elements tell you?

3 Four exporters A, B, C and D sell refrigerators (R), dishwashers (D), microwave ovens (M) and televisions (T). The sales in a particular month can be represented by a matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} R & D & M & T \end{array} \\ \begin{bmatrix} 120 & 95 & 370 & 250 \\ 430 & 380 & 950 & 900 \\ 60 & 50 & 150 & 100 \\ 200 & 100 & 470 & 50 \end{bmatrix} \end{array}$$

a What is the order of the matrix?

b Construct a row matrix to display the exports of B. What is its order? What does the sum of the elements of this matrix tell you?

c Construct a column matrix to display the numerical information in the table relating to microwave ovens. What is its order?

4 At a certain school there are 200 girls and 110 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 117 in Year 8, 135 and 98 in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11, and 28 and 33 in Year 12. Summarise this information in a matrix.

Example 8

5 Convert the 16-digit credit card number 3452 8279 0020 3069 into a 2×8 matrix. List the digits in pairs, one under the other. Ignore any spaces.

6 The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5
Player B points 8, rebounds 2, assists 3
Player C points 4, rebounds 1, assists 1
Player D points 14, rebounds 8, assists 6
Player E points 0, rebounds 1, assists 2

Express this information in a 5×3 matrix.

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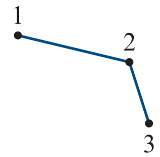
Representing the information in a network diagram in matrix form

Example 9

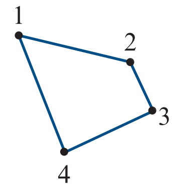
7 Represent each of the following network diagrams by a matrix A using the rules:

- matrix element = 1 if points are joined by a line
- matrix element = 0 if points are not joined by a line.

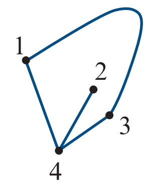
a



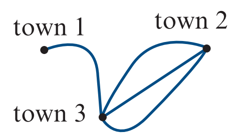
b



c



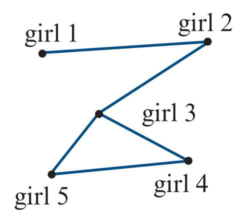
8 The diagram opposite shows the roads interconnecting three towns: town 1, town 2 and town 3. Represent this diagram with a 3×3 matrix where the elements represent the number of roads between each pair of towns.



Example 10

9 The network diagram opposite shows a friendship network between five girls: girl 1 to girl 5.

This network has been represented by a 5×5 matrix, F, using the rule:



- element = 1 if the pair of girls are friends
- element = 0 if the pair of girls are not friends.

$$F = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a In the matrix F, $f_{34} = 1$. What does this tell us?

b In the matrix F, $f_{25} = 0$. What does this tell us?

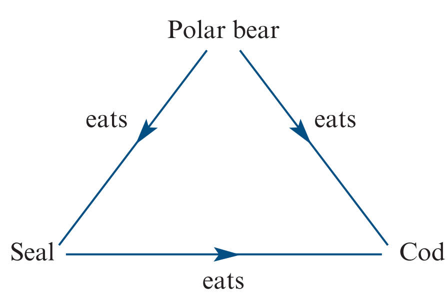
c What is the sum of the elements in row 3 of the matrix and what does this tell us?

d Which girl has the least friends? The most friends?

10 **a** The diagram below shows a 'food web' for polar bears (P), seals (S) and cod (C).

The matrix F below has been set up to represent the information in the diagram.

$$F = \begin{array}{c} \\ P \\ S \\ C \end{array} \begin{array}{c} \begin{array}{ccc} P & S & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$



i What does the '1' in column C, row P, of matrix F represent?

ii What does the column of zeroes in matrix F represent?

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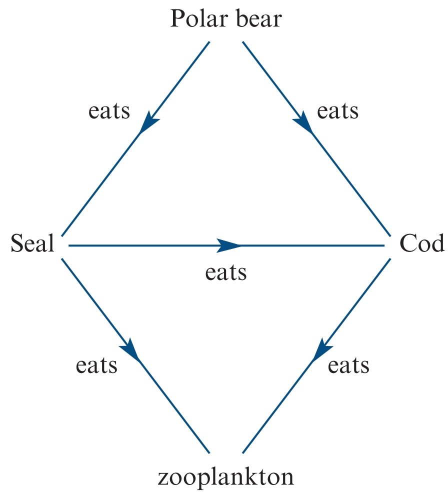
b The diagram below shows a 'food web' for polar bears (P), seals (S) cod (C) and zooplankton (Z).
Complete the matrix W to represent the information in the diagram.

$$W = \begin{array}{c} \\ P \\ S \\ C \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} P & S & C & Z \end{array} \\ \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \end{array}$$



Exam 1 style questions

11 The matrix below shows the number of boys and girls at a large school.

$$\begin{array}{r} \\ \textit{Year 10} \\ \textit{Year 11} \\ \textit{Year 12} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Boys} & \textit{Girls} \end{array} \\ \begin{bmatrix} 157 & 163 \\ 148 & 154 \\ 139 & 145 \end{bmatrix} \end{array}$$

How many girls are there in Year 11?

A 157 **B** 148 **C** 154 **D** 139 **E** 145

12 A small chain of delicatessans with shopnames Allbright, Bunchof, Crisp, Delic and Elite (A, B, C, D, E) each sell the products Feta, Goatmilk, Haloumi, Insalata and Jam (F, G, H, I, J). The number of weekly sales of each product at each of the shops is shown in the matrix below.

$$P = \begin{array}{c} \\ F \\ G \\ H \\ I \\ J \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 34 & 40 & 52 & 106 & 27 \\ 42 & 154 & 38 & 55 & 68 \\ 136 & 145 & 11 & 44 & 77 \\ 136 & 147 & 43 & 72 & 111 \\ 139 & 140 & 66 & 56 & 145 \end{bmatrix} \end{array}$$

Find which product had the highest weekly sales at any single one of the shops. The name of this product and the shop is

A Feta at Allbright **B** Goatmilk at Bunchof **C** Jam at Elite
D Insalata at Bunchof **E** Haloumi at Allbright

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10C Matrix arithmetic: addition, subtraction and scalar multiplication

Learning intentions

- To be able to establish when two matrices are equal.
- To be able to recognise when matrix addition and subtraction are defined and to peform these operations.
- To be able to undertake scalar multiplication.
- To be able to identify zero matrices.
- To be able to undertake addition, subtraction and multiplication by a scalar with a CAS calculator.
- Using the operations of addition, subtraction and scalar multiplication of matrices in practical situations.

Equality of two matrices

Equal matrices have the same order and each corresponding element is identical in value. It is not sufficient for the two matrices to contain an identical set of numbers; they must also be in the same positions.

For example:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is equal to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ because the corresponding elements are equal.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is *not* equal to $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ because the numbers are in different positions.

Example 11

Given that

$$\begin{bmatrix} 2 & a \\ 8 & b+1 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 20 & 7 \end{bmatrix}$$

find the value of a and the value of b.

Solution

$a = 10$ and $b + 1 = 7$ which implies $b = 6$.

Matrix addition and subtraction

Adding and subtracting matrices

If two matrices are of the same order (have the same number of rows and columns), they can be added (or subtracted) by adding (or subtracting) their corresponding elements.

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Example 12 Adding two matrices

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ Find $A + B$.

Explanation

1 As the two matrices have the same order, 2×3, they can be added.

2 Add corresponding elements.

Solution

$$A + B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 3+2 & 0+3 \\ 1+2 & 4+(-2) & 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 3 \\ 3 & 2 & 3 \end{bmatrix}$$

Likewise, if we have two matrices of the same order (same number of rows and columns), we can subtract the two matrices by subtracting their corresponding elements.

Example 13 Subtracting two matrices

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ Find $A - B$.

Explanation

1 As the two matrices have the same order, 2×3, they can be subtracted.

2 Subtract corresponding elements.

Solution

$$A - B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 3-2 & 0-3 \\ 1-2 & 4-(-2) & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -3 \\ -1 & 6 & 1 \end{bmatrix}$$

Multiplying matrices by a number (scalar multiplication)

Mutiplying a matrix by a number has the effect of multiplying each element in the matrix by that number.

Multiplying a matrix by a number is called **scalar multiplication**, because it has the effect of scaling the matrix by that number. For example, multiplying a matrix by 2 doubles each element in the matrix.

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Example 14 Scalar multiplication

$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$ Find $3A$ and $0.5C$.

Explanation

Multiplying a matrix by a number has the effect of multiplying each element by that number.

Solution

$$3\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 0 \\ 3 & 12 & 6 \end{bmatrix}$$

$$0.5\begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

The zero matrix

If $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $X - Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

A matrix of any order with *all zeros* is known as a **zero matrix**. The symbol O is used to represent a zero matrix. The matrices below are all examples of zero matrices.

$$O = [0], \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 15 The zero matrix

$G = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$ $H = \begin{bmatrix} 9 & 0 \\ -6 & 3 \end{bmatrix}$ Show that $3G - 2H = O$.

Solution

$$\begin{aligned} 3G - 2H &= 3 \times \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} - 2 \times \begin{bmatrix} 9 & 0 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ -12 & 6 \end{bmatrix} - \begin{bmatrix} 18 & 0 \\ -12 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 18 - 18 & 0 - 0 \\ -12 - (-12) & 6 - 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \therefore 3G - 2H &= O \end{aligned}$$

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Using a CAS calculator to perform matrix addition, subtraction and scalar multiplication

For small matrices, it is usually quicker to add, subtract or multiply a matrix by a number (scalar multiplication) by hand. However, if dealing with larger matrices, it is best to use a CAS calculator.

CAS 2: How to add, subtract and scalar multiply matrices using the TI-Nspire CAS

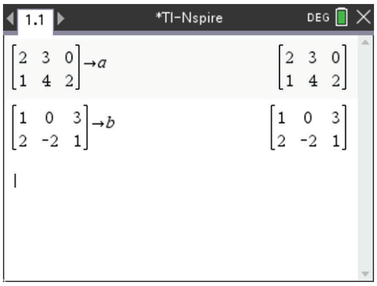
If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $3A - 2B$

Steps

1 Press ctrl + N. Select **Add Calculator**.

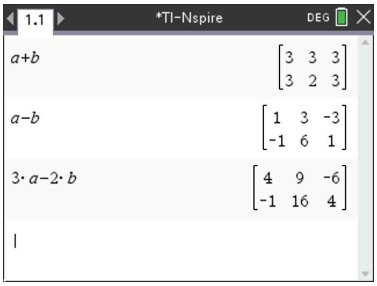
2 Enter and store the matrices A and B into your calculator.



a To determine $A + B$, type ***a*** **+** ***b***.
Press enter to evaluate.

b To determine $A - B$, type ***a*** **–** ***b***.
Press enter to evaluate.

c To determine $3A - 2B$, type **3*a*** **–** **2*b***.
Press enter to evaluate.



CAS 2:How to add, subtract and scalar multiply matrices with the ClassPad

If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $3A - 2B$

Steps

1 Enter the matrices A and B into your calculator using the var keyboard.

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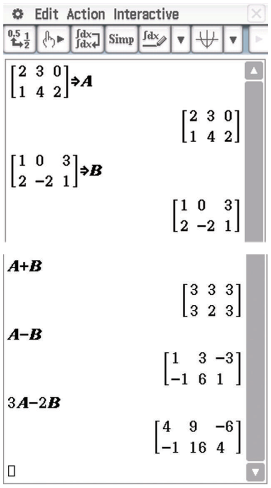
2 a To calculate $A + B$, type ***A*** + ***B*** and then press EXE to evaluate.

b To calculate $A - B$, type ***A*** – ***B*** and then press EXE to evaluate.

c To calculate $3A - 2B$, type **3*A*** – **2*B*** and then press EXE to evaluate.



Example 16 Processing data using addition, subtraction and scalar multiplication

The sales data for two used car dealers, Honest Joe's and Super Deals, are displayed below.

Car sales	*2014*			*2015*		
	Small	*Medium*	*Large*	*Small*	*Medium*	*Large*
Honest Joe's	24	32	11	26	38	16
Super Deals	32	34	9	35	41	12

Solution

a Construct two matrices, A and B, to represent the sales data for 2014 and 2015 separately.

$$A = \begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$

b Construct a new matrix $C = A + B$. What does this matrix represent?

$$C = A + B$$
$$= \begin{bmatrix} 24 & 32 & 11 \\ 32 & 34 & 9 \end{bmatrix} + \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$
$$= \begin{bmatrix} 50 & 70 & 27 \\ 67 & 75 & 21 \end{bmatrix}$$

Matrix C represents the total sales for 2014 and 2015 for the two dealers.

c Construct a new matrix, $D = B - A$. What does this matrix represent?

$$D = B - A$$
$$= \begin{bmatrix} 2 & 6 & 5 \\ 3 & 7 & 3 \end{bmatrix}$$

Matrix D represents the increase in sales from 2014 and 2015 for the two dealers.

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d Both dealers want to increase their 2015 sales by 50% by 2016. Construct a new matrix $E = 1.5B$. Explain why this matrix represents the planned sales figures for 2016.

$$E = 1.5B = 1.5 \times \begin{bmatrix} 26 & 38 & 16 \\ 35 & 41 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 \times 26 & 1.5 \times 38 & 1.5 \times 16 \\ 1.5 \times 35 & 1.5 \times 41 & 1.5 \times 12 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 57 & 24 \\ 52.5 & 61.5 & 18 \end{bmatrix}$$

Forming the scalar product 1.5B multiplies each element by 1.5. This has the effect of increasing each value by 50%.

Skill-sheet

Exercise 10C

Equality of two matrices

Example 11

1 Given that

$$\begin{bmatrix} 4 & a+2 \\ 6 & b-1 \end{bmatrix} = \begin{bmatrix} 3 & 11 \\ 10 & 6 \end{bmatrix}$$

find the value of a and the value of b.

Matrix addition, subtraction and scalar multiplication

Example 12
Example 13

2 The questions below relate to the following six matrices. Computations will be quicker if done by hand.

$$A = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

a Which matrices are equal?

b Which matrices have the same order?

c Which matrices can be added or subtracted?

d Compute each of the following, where possible.

i $A + B$ **ii** $D + E$ **iii** $C - F$ **iv** $A - B$ **v** $E - D$

Example 14

vi $3B$ **vii** $4F$ **viii** $3C + F$ **ix** $4A - 2B$ **x** $E + F$

Example 15

3 Let $G = \begin{bmatrix} 8 & 0 \\ -4 & 2 \end{bmatrix}$ and $H = \begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$. Show that $2G - 8H = O$.

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4 Evaluate each of the folowing giving your answer as a single matrix.

a $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

b $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

c $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 2 \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

d $\begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \end{bmatrix}$

e $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

f $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

g $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

h $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

5 Use a calculator to evaluate the following.

a $2.2 \times \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 1.1 \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

b $\begin{bmatrix} 1.2 & 0.2 \\ 4.5 & 3.3 \end{bmatrix} - 3.5 \times \begin{bmatrix} 0.4 & 4 \\ 1 & 2 \end{bmatrix}$

c $5 \times \begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 2 \times \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 0.5 & 0 & -2 \end{bmatrix}$

d $0.8 \times \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 0 & -1 & 2 \end{bmatrix} + 0.2 \times \begin{bmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$

6 Find the values of x, y, z and w in the following. $2\begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 12 & 6 \end{bmatrix}$

Applications of matrix addition, subtraction and scalar multiplication

Example 16

7 The number of DVDs sold in a company's city, suburban and country stores for each 3-month period in a year is shown in the table.

	DVD sales (thousands)			
Store location	*Jan–March*	*April–June*	*July–Sept*	*Oct–Dec*
City	2.4	2.8	2.5	3.4
Suburban	3.5	3.4	2.6	4.1
Country	1.6	1.8	1.7	2.1

a Construct four 3×1 matrices A, B, C, and D that show the sales in each of the three-month periods during the year.

b Evaluate $A + B + C + D$. What does the sum $A + B + C + D$ represent?

8 The numbers of females and males enrolled in three different gym programs for 2014 and 2015, *Weights*, *Aerobics* and *Fitness*, are shown in the table.

	2014			*2015*		
Gym membership	*Weights*	*Aerobics*	*Fitness*	*Weights*	*Aerobics*	*Fitness*
Females	16	104	86	24	124	100
Males	75	34	94	70	41	96

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a Construct two matrices, A and B, which represent the gym memberships for 2014 and 2015 separately.

b Construct a new matrix $C = A + B$. What does this matrix represent?

c Construct a new matrix $D = B - A$. What does this matrix represent? What does the negative element in this matrix represent?

d The manager of the gym wants to double her 2015 membership by 2018. Construct a new matrix E that would show the membership in 2018 if she succeeds with her plan. Evaluate.

Exam 1 style questions

9 $\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} =$

A $\begin{bmatrix} -2 & 2 \\ -3 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 4 \\ -3 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} -2 & 4 \\ -6 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ -6 & 0 \end{bmatrix}$

10 If $M = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, then $2M - 2N =$

A $\begin{bmatrix} 0 & 0 \\ -9 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ -6 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 12 & -2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ 6 & -1 \end{bmatrix}$

11 The table below shows information about two matrices A and B.

	Order	Rule
A	3×3	$a_{ij} = ij$
B	3×3	$b_{ij} = i + j$

The sum $A + B$ is

A $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 5 & 7 \\ 3 & 4 & 5 \\ 3 & 6 & 19 \end{bmatrix}$

D $\begin{bmatrix} 3 & 2 & 7 \\ 3 & 4 & 5 \\ 14 & 15 & 16 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & 5 & 7 \\ 5 & 8 & 11 \\ 7 & 11 & 15 \end{bmatrix}$

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10D Matrix arithmetic: the product of two matrices

Learning intentions

- To be able to establish when multiplication of two matrices can be undertaken.
- To be able to undertake the multiplication of matrices.
- To be able to understand that the order of multiplication of matrices is important.
- To be able to apply multiplication of matrices to practical problems.
- To be able to use your CAS calculator to undertake multiplication of matrices.
- To be able to understand the role of 'summing matrices'.
- To be able to work with calculations involving powers of matrices.

The process of multiplying two matrices involves both multiplication and addition. The process can be illustrated using Australian Rules football scores.

An illustration of matrix multiplication

Two teams, the Ants and the Bulls, play each other in a game of Australian rules football. At the end of the game:

- the Ants had scored 11 goals 5 behinds
- the Bulls had scored 10 goals 9 behinds.

Now calculate each team's score in points:

- one goal = 6 points
- one behind = 1 point.

Thus we can write:

$11 \times 6 + 5 \times 1 = 71$ points
$10 \times 6 + 9 \times 1 = 69$ points

Matrix multiplication follows the same pattern.

$$\begin{array}{lc} & \begin{array}{cc}\text{Goals} & \text{Behinds}\end{array} \\ \begin{array}{l}\text{Ants score:}\\ \text{Bulls score:}\end{array} & \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix} \end{array} \times \begin{array}{c}\text{Point values} \\ \begin{bmatrix} 6 \\ 1 \end{bmatrix}\end{array} = \begin{bmatrix} 11 \times 6 + 5 \times 1 \\ 10 \times 6 + 9 \times 1 \end{bmatrix} = \begin{array}{c}\text{Final points} \\ \begin{bmatrix} 71 \\ 69 \end{bmatrix}\end{array}$$

The order of matrices and matrix multiplication

Look at the **order** of each of the matrices involved in the **matrix multiplication** below.

$$\begin{array}{lc} & \begin{array}{cc}\text{Goals} & \text{Behinds}\end{array} \\ \begin{array}{l}\text{Ants score:}\\ \text{Bulls score:}\end{array} & \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix} \end{array} \times \begin{array}{c}\text{Point values} \\ \begin{bmatrix} 6 \\ 1 \end{bmatrix}\end{array} = \begin{bmatrix} 11 \times 6 + 5 \times 1 \\ 10 \times 6 + 9 \times 1 \end{bmatrix} = \begin{array}{c}\text{Final points} \\ \begin{bmatrix} 71 \\ 69 \end{bmatrix}\end{array}$$

Order of matrices: 2×2 $\quad 2 \times 1$ $\quad 2 \times 1$

Thus, multiplying a 2×2 matrix by a 2×1 matrix gives a 2×1 matrix.

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Two observations can be made here:

1 To perform matrix multiplication, the *number of columns in the first matrix* (2) needs to be the same as the *number of rows in the second matrix* (2). For example, if there were three columns in the first matrix, there would not be enough elements in the second matrix to complete the multiplication. When this happens, we say that matrix multiplication is not defined.

2 The final result of multiplying the two matrices is a 2×1 matrix. For each row in the first matrix, there will be a row in the product matrix (there are two rows). For each column in the second matrix, there will be a column in the product matrix (there is one column).

These observations can be generalised to give two important rules for matrix multiplication.

Rule 1: Condition for matrix multiplication to be defined

Matrix multiplication of two matrices requires the *number of columns*in the *first matrix* to *equal* the *number of rows* in the *second matrix*.

That is, if A is of order $m \times n$ and B is of order $r \times s$, then the product AB is only defined if $n = r$.

For example, if $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, then:

- $AB = \underset{2 \times 3}{\begin{bmatrix} 3 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}} \times \underset{3 \times 1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}$ is *defined*: columns in $A(3)$ = rows in $B(3)$

- $BC = \underset{3 \times 1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} \times \underset{2 \times 2}{\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}}$ is *not defined*: columns in $B(1) \neq$ rows in $C(2)$.

Example 17 **Is a matrix product defined?**

$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Which of the following matrix products are defined?

a AB **b** BC **c** AC

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Explanation	Solution
a Write down the matrix product. Under each matrix, write down its order (columns × rows)	$AB = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}$; not defined order: $2 \times 2 \quad 1 \times 2$
b The matrix product is defined if the number of columns in matrix 1 = the number of rows in matrix 2.	$BC = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; defined order: $1 \times 2 \quad 2 \times 1$
c Write down your conclusion.	$AC = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; defined order: $2 \times 2 \quad 2 \times 1$

Once we know that two matrices can be multiplied, we can use the order of the two matrices to determine the order of the resulting matrix.

Rule 2: Determining the order of the product matrix

If two matrices can be multiplied, then the *product matrix* will have the same *number of rows* as the *first matrix* and the same *number of columns* as the *second matrix*.

That is, if A is of order $m \times n$ and B is of order $n \times s$, then AB will be of order $m \times s$.

For example, if $A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, then:

- $AB = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is *defined* and will be of order 2×1.

$$2 \times \underbrace{3 \qquad 3}_{\text{equal}} \times 1 \qquad\qquad 2 \times 1$$

- $AB = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ is *defined* and will be of order 2×3.

$$2 \times \underbrace{2 \qquad 2}_{\text{equal}} \times 3 \qquad\qquad 2 \times 3$$

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Example 18 **Determining the order of a matrix product**

$$A = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix},\ B = \begin{bmatrix} 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The following matrix products are defined. What is their order?

a BA **b** BC **c** AC

Explanation

1 Write down the matrix product. Under each matrix, write down its order rows × columns.

2 The order of the product matrix is given by rows in matrix 1 × columns in matrix 2.

3 Write down the order.

Solution

a $BA = \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}$; order of BA 1×2

order: $(1 \times 2)\ (2 \times 2)$

b $BC = \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; order of BC 1×1

order: $(1 \times 2)\ (2 \times 1)$

c $AC = \begin{bmatrix} 6 & 0 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; order of AC 2×1

order: $(2 \times 2)\ (2 \times 1)$

Order of multiplication is important when multiplying matrices

You might have noticed in Example 17 that while the matrix product BA was defined, the matrix product AB in Example 18 was not defined. Order is important in matrix multiplication. For example, if we have two matrices, M and N, and form the products MN and NM, frequently the products will be different. We will return to this point when in the next section.

Determining matrix products

For large matrices, the process of matrix multiplication is complex and can be error prone and tedious to do by hand. Fortunately, CAS calculators will do it for us, and that is perfectly acceptable.

However, before we show you how to use a CAS calculator to multiply matrices, we will illustrate the process by multiplying a row matrix by a column matrix and a rectangular matrix by a column matrix *by hand*.

In terms of understanding matrix multiplication, and using this knowledge to solve problems later in this module, these are the two most important worked examples in the chapter.

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Example 19 **Multiplying a row matrix by a column matrix (by hand)**

Evaluate the matrix product AB, where $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

Explanation

1 Write down the matrix product and, above each matrix, write down its order. Use this information to determine whether the matrix product is defined and its order.

Solution

$$AB = \underset{1 \times 3}{\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}}$$

Defined: the number of columns in A equals the number of rows in B.

The order of AB is 1×1.

2 To determine the matrix product:

a multiply each element in the row matrix by the corresponding element in the column matrix

b add the results

c write down your answer.

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$= [1 \times 2 + 3 \times 4 + 2 \times 1]$$

$$= [16]$$

$\therefore AB = [16]$

Example 20 **Multiplying a rectangular matrix by a column matrix (by hand)**

Evaluate the matrix product AB, where $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Explanation

1 Write down the matrix product and, above each matrix, write down its order. Use this information to determine whether the matrix product is defined and its order.

Solution

$$AB = \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

Defined: the number of columns in A equals the number of rows in B.

The order of AB is 2×1.

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2 To determine the matrix product: **a** multiply each element in the row matrix by the corresponding element in the column matrix **b** add the results **c** write down your answer.	$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times 2 + 0\times 3 \\ 2\times 2 + 3\times 3 \end{bmatrix}$ $= \begin{bmatrix} 2 \\ 13 \end{bmatrix}$ $\therefore AB = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$

Using a CAS calculator to multiply two matrices

In principle, if you can multiply a row matrix by a column matrix, you can work out the product between any two matrices, provided it is defined. However, because you have to do it for every possible row/column combination, it soon gets beyond even the most patient and careful person. For that reason, in practice we use technology to do the calculation for us.

We will illustrate how to use a calculator to multiply matrices by evaluating the matrix product in the football score example given earlier.

CAS 3: How to multiply two matrices using the TI-Nspire CAS

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, find the matrix CD.

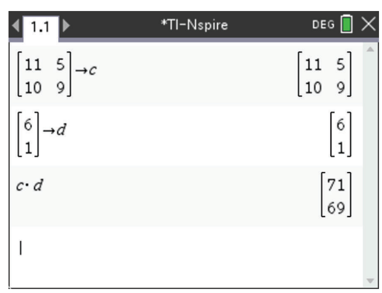
Steps

1 Press ctrl + N. Select **Add Calculator**.

2 Enter and store the matrices C and D into your calculator.

3 To calculate matrix CD, type $\boldsymbol{c} \times \boldsymbol{d}$. Press enter to evaluate.

Note: You must put a multiplication sign between the c and d.



CAS 3:How to multiply two matrices using the ClassPad

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ find the matrix product CD.

Steps

1 Enter the matrices C and D into your calculator.

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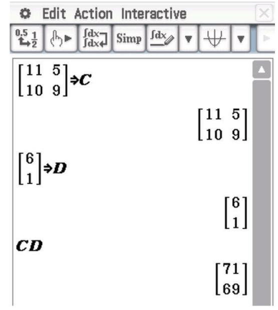
2 To calculate $C \times D$, type ***CD*** and then press EXE to evaluate.



Applications of the product of two matrices

The summing matrix

A row or column matrix in which all the elements are 1 is called a *summing matrix*.

The matrices opposite are all examples of summing matrices.

The rules for using a summing matrix to sum the rows and columns of a matrix follow.

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 1\end{bmatrix} \quad \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Using matrix multiplication to sum the rows and columns of a matrix

- To *sum* the *rows* of an $m \times n$ matrix, *post-multiply* the matrix by an $n \times 1$ summing matrix.
- To *sum the columns* of an $m \times n$ matrix, *pre-multiply* the matrix by a $1 \times m$ summing matrix.

Example 21 **Using matrix multiplication to sum the rows and columns of a matrix**

Use matrix multiplication to generate a matrix that:

a displays

i row sums of the matrix $\begin{bmatrix}1 & 3 & 0\\2 & 6 & 7\\3 & 0 & 1\end{bmatrix}$ **ii** column sums of the matrix $\begin{bmatrix}1 & 3 & 0\\2 & 6 & 7\\3 & 0 & 1\end{bmatrix}$

b displays the column sums of the matrix $\begin{bmatrix}2 & 5 & -1 & -3 & 4\\0 & 6 & 2 & -2 & 3\end{bmatrix}$.

Explanation

a **i** To sum the rows of a 3×3 matrix, *post*-multiply a 3×1 summing matrix.

Solution

$$\begin{bmatrix}1 & 3 & 0\\2 & 6 & 7\\3 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\15\\4\end{bmatrix}$$

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ii To sum the columns of a 3×3 matrix, *pre*-multiply a 1×3 summing matrix.

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 7 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 8 \end{bmatrix}$$

b To sum the columns of a 2×5 matrix, *pre*-multiply a 1×2 summing matrix.

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & -1 & -3 & 4 \\ 0 & 6 & 2 & -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 11 & 1 & -5 & 7 \end{bmatrix}$$

Example 22 shows a practical application of matrix multiplication.

Example 22 A practical application of matrix multiplication

$$E = \begin{array}{c} \begin{bmatrix} 25 \\ 40 \end{bmatrix} \end{array}\begin{array}{l} Walk \\ Run \end{array}$$

Matrix E gives the energy in kilojoules consumed per minute when walking and running.

$$T = \begin{array}{cc} [20 & 40] \\ Walk & Run \end{array}$$

Matrix T gives the times (in minutes) a person spent walking and running in a training session.

Compute the matrix product TE and show that it gives the total energy consumed during the training session.

Solution

$$T \times E = \begin{bmatrix} 20 & 40 \end{bmatrix}\begin{bmatrix} 25 \\ 40 \end{bmatrix} = [20 \times 25 + 40 \times 40] = [2100]$$

The total energy consumed is:

20 minutes × 25 kJ/minute + 40 minutes × 40 kJ/minute = 2100 kJ

This is the value given by the matrix product TE.

You could work out the energy consumed on the training run for one person just as quickly without using matrices. However, the advantage of using a matrix formulation is that, with the aid of a calculator, you could almost as quickly have worked out the energy consumed by 10 or more runners, all with different times spent walking and running.

Matrix powers

Now that we can multiply matrices, we can also determine the **power of a matrix**. This is an important tool when we meet communication and dominance matrices in the next section and transition matrices in the next chapter.

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The power of a matrix

Just as we define

2^2 as 2×2,

2^3 as $2 \times 2 \times 2$,

2^4 as $2 \times 2 \times 2 \times 2$ and so on,

we define the various powers of matrices as

A^2 as $A \times A$,

A^3 as $A \times A \times A$,

A^4 as $A \times A \times A \times A$ and so on.

Only square matrices can be raised to a power.

Example 23 Evaluating matrix expressions involving powers

If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, determine:

a $2A + B^2 - 2C$ **b** $(2A - B)^2 - C^2$ **c** $AB^2 - 3C^2$

Explanation	Solution
1 Write down the matrices.	$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
2 Enter the matrices A, B and C into your calculator.	$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \rightarrow a$ $\begin{bmatrix} 1 & 0 \\ 2. & -1. \end{bmatrix}$
	$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \rightarrow b$ $\begin{bmatrix} -1. & 1. \\ 2. & 1. \end{bmatrix}$
	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow c$ $\begin{bmatrix} 0. & 1. \\ 1. & 1. \end{bmatrix}$
	$2 \cdot a + b^2 - 2 \cdot c$ $\begin{bmatrix} 5. & -2. \\ 2. & -1. \end{bmatrix}$
	$(2 \cdot a - b)^2 - c^2$ $\begin{bmatrix} 6. & -1. \\ -1. & 5. \end{bmatrix}$
	$a \cdot b^2 - 3 \cdot c^2$ $\begin{bmatrix} 0. & -3. \\ 3. & -9. \end{bmatrix}$

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3 Type in each of the expressions as written, and press to evaluate. Write down your answer.

a $2A + B^2 - 2C = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix}$

b $(2A + B)^2 - C^2 = \begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix}$

c $AB^2 - 3C^2 = \begin{bmatrix} 0 & -3 \\ 3 & -9 \end{bmatrix}$

Note: For CAS calculators you must use a multiplication sign between a and b^2 in the last example, otherwise it will be read as variable $(ab)^2$.

Skill-sheet

Exercise 10D

Matrix multiplication

Example 17

1 The questions below relate to the following five matrices.

$$A = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

a Which of the following matrix products are defined?

i AB **ii** BA **iii** AC **iv** CE
v EC **vi** EA **vii** DB **viii** CD

Example 19
Example 20

b Compute the following products by hand.

i AB **ii** CE **iii** DB **iv** AD

c Enter the five matrices into your calculator and compute the following matrix expressions.

i AB **ii** EC **iii** $AB - 3CE$ **iv** $2AD + 3B$

2 Evaluate each of the following matrix products *by hand*.

a $\begin{bmatrix} 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \end{bmatrix} =$ **b** $\begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$ **c** $\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} =$

d $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ **e** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$ **f** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} =$

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3 Evaluate each of the following matrix products using a CAS calculator.

a $\begin{bmatrix} 0.5 \\ -1.5 \\ 2.5 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} =$

b $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} =$

c $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}$

d $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 2 & 1 \\ -1 & 4 & 2 \\ -2 & 1 & 2 \end{bmatrix} =$

Using summing matrices to sum the rows and columns of matrices

Example 21

4 For the matrix opposite, write down a matrix that can be used to: $\begin{bmatrix} 2 & 5 \\ -1 & 1 \\ 9 & 3 \end{bmatrix}$.

a sum its rows **b** sum its columns

5 Show how the matrix $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ can be used to sum the rows of $\begin{bmatrix} 7 & 1 & 2 \\ 1 & 2 & 2 \\ 8 & 1 & 4 \end{bmatrix}$.

6 Show how the matrix $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ can be used to sum the columns of $\begin{bmatrix} 9 & 0 & 2 \\ 1 & 7 & 3 \\ 8 & 3 & 4 \end{bmatrix}$.

7 Use matrix multiplication to construct a matrix that:

a displays row sums of the matrix $\begin{bmatrix} 2 & 4 & 1 & 7 & 8 \\ 1 & 9 & 0 & 0 & 2 \\ 3 & 4 & 3 & 3 & 5 \\ 2 & 1 & 1 & 1 & 7 \\ 5 & 3 & 6 & 7 & 9 \end{bmatrix}$

b displays the column sums of the matrix $\begin{bmatrix} 4 & 5 & 1 & 2 & 1 \\ 0 & 3 & 4 & 5 & 1 \\ 4 & 2 & 1 & 7 & 9 \end{bmatrix}$.

Practical applications of matrix multiplication

Example 22

8 Six teams play an indoor soccer competition.
If a team:

- wins, it scores two points
- draws, it scores one point
- loses, it scores zero points.

This is summarised in the points matrix opposite.

$$P = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{array}{l} \textit{Win} \\ \textit{Draw} \\ \textit{Lose} \end{array}$$

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The results of the competition are summarised in the results matrix. Work out the final points score for each team by forming the matrix product RP.

$$R = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} W & D & L \end{array} \\ \begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Team 1} \\ \text{Team 2} \\ \text{Team 3} \\ \text{Team 4} \\ \text{Team 5} \\ \text{Team 6} \end{array}$$

9 Four people complete a training session in which they walked, jogged and ran at various times.
The energy consumed in kJ/minute when walking, jogging or running is listed in the energy matrix opposite.
The time spent in each activity (in minutes) by four people is summarised in the time matrix opposite. Work out the total energy consumed by each person, by forming the matrix product TE.

$$E = \begin{bmatrix} 25 \\ 40 \\ 65 \end{bmatrix} \begin{array}{l} \text{Walk} \\ \text{Jog} \\ \text{Run} \end{array}$$

$$T = \begin{array}{c} \begin{array}{ccc} W & J & R \end{array} \\ \begin{bmatrix} 10 & 20 & 30 \\ 15 & 20 & 25 \\ 20 & 20 & 20 \\ 30 & 20 & 10 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Person 1} \\ \text{Person 2} \\ \text{Person 3} \\ \text{Person 4} \end{array}$$

10 A manufacturer sells three types of fruit straps, A, B and C, through outlets at two shops, Energy (E) and Nourishing (N).
The number of fruit straps sold per month at each shop is given by the matrix Q.

$$Q = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 25 & 34 & 19 \\ 30 & 45 & 25 \end{bmatrix} \end{array} \begin{array}{l} \\ E \\ N \end{array}$$

a Write down the order of matrix Q.

The matrix P, shown opposite, gives the selling price, in dollars, of each type of fruit strap A, B and C.

$$P = \begin{bmatrix} 2.50 \\ 1.80 \\ 3.20 \end{bmatrix} \begin{array}{l} A \\ B \\ C \end{array}$$

b **i** Evaluate the matrix M, where $M = QP$.
ii What information do the elements of matrix M provide?

c Explain why the matrix PQ is not defined.

11 Matrix X shows the number of cars of models a and b bought by four dealers A, B, C, D. Matrix Y shows the cost in dollars of cars a and b. Find XY and explain what it represents.

$$X = \begin{array}{r} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad Y = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix} \begin{array}{l} a \\ b \end{array}$$

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12 It takes John 5 minutes to drink a milk shake which costs \$2.50, and 12 minutes to eat a banana split which costs \$3.00.

a Find the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in fast-food economics.

b Two friends join John. Find $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

13 The final grades for Physics and Chemistry are made up of three components: tests, practical work and exams. Each semester, a mark out of 100 is awarded for each component. Wendy scored the following marks in the three components for Physics:

Semester 1 tests 79, practical work 78, exam 80

Semester 2 tests 80, practical work 78, exam 82

a Represent this information in a 2×3 matrix.

To calculate the final grade for each semester, the three components are weighted: tests are worth 20%, practical work is worth 30% and the exam is worth 50%.

b Represent this information in a 3×1 matrix.

c Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in the three components for Chemistry:

Semester 1 tests 86, practical work 82, exam 84

Semester 2 tests 81, practical work 80, exam 70

d Calculate Wendy's final grade for Chemistry in each semester.

Students who gain a total score of 320 or more for Physics and Chemistry over the two semesters are awarded a Certificate of Merit in Science.

e Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to re-mark her Semester 2 Chemistry exam, hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

f How many extra marks on the exam does she need?

Powers of matrices

Example 23

14 If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, determine A^2, A^3, A^4 and A^7.

15 If $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, determine A^4, A^5, A^6 and A^7.

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16 If $B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ use your calculator to find B^3.

17 If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, evaluate:

a $A + 2B - C^2$ **b** $AB - 2C^2$ **c** $(A + B + 2C)^2$

d $4A + 3B^2 - C^3$ **e** $(A - B)^3 - C^3$

Exam 1 style questions

18 The matrix product $\begin{bmatrix} 6 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is equal to

A $[30]$ **B** $\begin{bmatrix} 18 \\ 12 \\ 0 \end{bmatrix}$ **C** $[39]$ **D** $3 \times \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **E** $2 \times \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

19 Matrix $P = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 1 & 2 \end{bmatrix}$ and matrix $Q = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 6 \end{bmatrix}$

Matrix $R = P \times Q$. Element r_{31} is determined by the calculation

A $1 \times 1 + 4 \times 2$ **B** $1 \times 1 + 4 \times 0$ **C** $1 \times 1 + 2 \times 2$

D $4 \times 1 + 5 \times 0$ **E** $4 \times 2 + 5 \times 1$

20 There are two types of chocolate boxes, Minty Chews and Orange Delight, available in a shop. The cost, in dollars, to purchase a box is shown in the table below.

Chocolate	Cost($)
Minty Chews	6
Orange Delight	8

Liam is doing all his Christmas shopping by buying chocolate boxes. He buys 7 boxes of Minty Chews and 9 boxes of Orange Delight. The total cost in dollars of these chocolates can be determined by which one of the following calculations?

A $[7] \times \begin{bmatrix} 6 & 8 \end{bmatrix}$ **B** $\begin{bmatrix} 7 & 9 \end{bmatrix} \times \begin{bmatrix} 6 & 8 \end{bmatrix}$ **C** $\begin{bmatrix} 7 \\ 9 \end{bmatrix} \times \begin{bmatrix} 6 & 8 \end{bmatrix}$

D $\begin{bmatrix} 7 & 9 \end{bmatrix} \times \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ **E** $\begin{bmatrix} 9 & 7 \end{bmatrix} \times \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

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10E Matrix inverse, the determinant and matrix equations

Learning intentions

- To be able to detemine by multiplication when two matrices are inverses.
- To be able to determine the determinant of a 2×2 matrix.
- To be able to determine the inverse of a 2×2 matrix.
- To be able to use a CAS calculator to determine the inverse and determinant of an $n \times n$ matrix.
- To be able to solve simple matrix equations.

The inverse matrix A^{-1}

So far, you have been shown how to add, subtract and multiply matrices, but what about dividing them? As you might expect matrix division, like matrix multiplication, is a more complicated process than its equivalent process for dividing numbers.

The starting point for matrix division is the **inverse matrix**. You will see why as we proceed.

The inverse matrix A^{-1}

The inverse of a square matrix A is called A^{-1}.

The inverse matrix has the property $AA^{-1} = A^{-1}A = I$.

Having defined the inverse matrix, two questions immediately come to mind. Does the inverse of a matrix actually exist? If so, how can we calculate it?

First we will demonstrate that at least some matrices have inverses. We can do this by showing that two matrices, which we will call A and B, have the property $AB = I$ and $BA = I$, where I is the **identity matrix**. If this is the case, we can then say that $B = A^{-1}$.

Example 24 Demonstrating that two matrices are inverses

Show that the matrices $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ are inverses.

Explanation	Solution
1 Write down A and B.	$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

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2 Form the product AB and evaluate. You can use your calculator to speed things up if you wish.

$$AB = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 5 + 3\times(-3) & 2\times(-3) + 3\times 2 \\ 3\times 5 + (-5)\times 3 & 3\times(-3) + 5\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore AB = I$

3 Form the product BA and evaluate. You can use your calculator here to speed things up if you wish.

$$BA = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 2 + (-3)\times 3 & 5\times 3 + (-3)\times 5 \\ (-3)\times 2 + 2\times 3 & (-3)\times 3 + 2\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore BA = I$

4 Write down your conclusion.

Because $AB = I$ and $BA = I$, we conclude that A and B are inverses.

While Example 24 clearly demonstrates that the matrices $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ both have an inverse, many square matrices do *not* have inverses. To see why, we need to introduce another new matrix concept, the **determinant**, and see how it relates to finding the inverse of a square matrix. To keep things manageable, we will restrict ourselves initially to 2×2 matrices.

The determinant of a matrix

The determinant of a 2×2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of matrix A is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$$

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Example 25 **Finding the determinant of a 2×2 matrix**

Find the determinant of the matrices:

a $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ **c** $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$

Solution

1 Write down the matrix and use the rule $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$.

2 Evaluate.

a $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $\therefore \det(A) = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 3 = 1$

b $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$ $\therefore \det(B) = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 0$

c $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ $\therefore \det(C) = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 4 = -2$

From Example 25, we can see that the determinant of a matrix is a number that can take on both positive and negative values as well as being zero. For a matrix to have an inverse, its determinant must be non-zero.

How to determine the inverse of a 2×2 matrix

Normally you will use a calculator to determine the inverse of a matrix, but we need to do the following example by hand to show you why some square matrices do not have an inverse. To do this we first need to consider the rule for finding the determinant of a 2×2 matrix.

The rule for finding the inverse of a 2×2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its inverse, A^{-1}, is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad - bc \neq 0$; that is, provided $\det(A) \neq 0$.

The most important thing about this rule is that it shows immediately why you cannot calculate an inverse for some matrices. These are the matrices whose determinant is zero.

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Example 26 Using the rule to find the inverse of a 2×2 matrix

Find the inverse of the following matrices.

a $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

Explanation

1 Write down the matrix and use the rule $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$ to evaluate the determinant.

Use the rule $A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ to evaluate A^{-1}.

2 Write down the matrix and use the rule $\det(B) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} =$ $a \times d - b \times c.$ to evaluate the determinant.

Solution

a

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - 3 \times 2 = 2$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1.5 & 1 \end{bmatrix}$$

b

$$B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\therefore \det(B) = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 0$$

$\det(B) = 0$

$\therefore B$ does not have an inverse.

Using a CAS calculator to determine the determinant and inverse of an $n \times n$ matrix

There are rules for finding the inverse of a square matrix of any size, but in practice we tend to use a calculator. The same goes for calculating determinants, although the inverse and determinant of a 2×2 matrix are often computed more quickly by hand.

CAS 4: How to find the determinant and inverse of a matrix using the TI-Nspire CAS

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find $\det(A)$ and A^{-1}.

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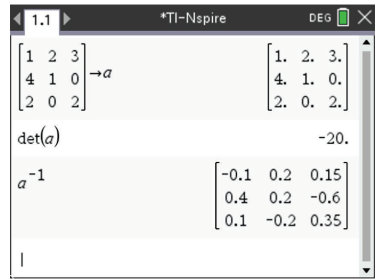
Steps

2 Press ctrl + N. Select **Add Calculator**.

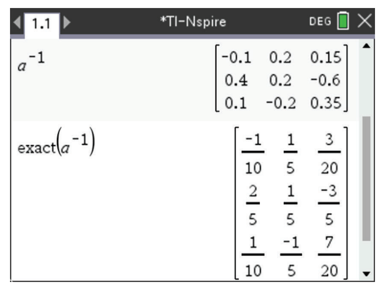
3 Enter the matrix A into your calculator.

4 To calculate det(A), type **det(*a*)** and press enter to evaluate.

Note: det() can also be accessed using menu>**Matrix & Vector>Determinant**.



4 To calculate the inverse matrix A^{-1} type **a ∧ − 1** and press enter to evaluate. If you want to see the answer in fractional form, enter as **exact (a∧ −1)** and press enter to evaluate.



Note:

1 Long strings of decimals can be avoided by asking for an exact inverse. Type in **exact(a^{-1})**.

2 If the matrix has no inverse, the calculator will respond with the error message **Singular matrix**.

CAS 4: How to find the determinant and inverse of a matrix using the ClassPad

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find det($A$) and A^{-1}.

Steps

1 Enter the matrix A into your calculator.

Note: Change the status of the calculator to **Standard** for fractions to be displayed. Tapping on **Decimal** will change the calculator to **Standard**.

2 To calculate det(A):

a type and highlight ***A*** (by swiping with the stylus)

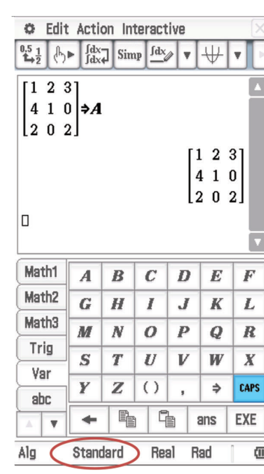
b select **Interactive** from the menu bar, tap **Matrix-Calculation**, then tap **det**.

3 To calculate the inverse matrix A^{-1}:

a type ***A***^**−1**

b press EXE to evaluate.

Note: If the matrix has no inverse, the calculator will respond with the message **Undefined**.



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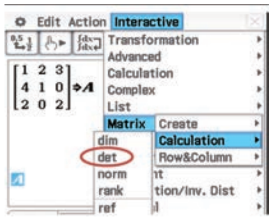

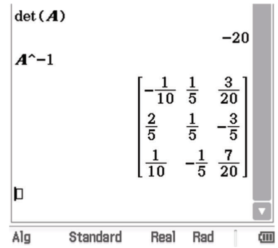

Solving matrix equations

In the following example the solution of simple matrix equations is illustrated. The techniques are analagous to those we use in solving linear equations while noting that there are some important differences such as multiplication of matrices is not commutative and to take care that operations are defined on the matrices involved.

Example 27

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad C = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$ $\quad D = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $E = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Solve each of the following matrix equations for X

a $B + X = C$ **b** $BX = C$ **c** $XB = C$

d $BX = D$ **e** $AX = E$ **f** $BX + \begin{bmatrix} 4 \\ 5 \end{bmatrix} = D$

Explanation

a 1 Write down B and C.

2 Form the equation $B + X = C$ and note that X must be a 2×2 matrix.

3 Calculate $X = C - B$.

Solution

$B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad C = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$

$B + X = C$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + X = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$$

$$X = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 \\ 1 & 4 \end{bmatrix}$$

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b **1** Form the equation $BX = C$ and note that X must be a 2×2 matrix. **2** Note that $B^{-1}B = I$, the 2×2 identity matrix. Calculate $X = B^{-1}C$.	$BX = C$ $B^{-1}BX = B^{-1}C$ $X = B^{-1}C$ $X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}$ $\therefore X = \begin{bmatrix} 12 & -8 \\ -16 & 15 \end{bmatrix}$
c **1** Form the equation $XB = C$ and note that X must be a 2×2 matrix. **2** Note that $BB^{-1} = I$, the 2×2 identity matrix. Calculate $X = CB^{-1}$.	$XB = C$ $XBB^{-1} = CB^{-1}$ $X = CB^{-1}$ $X = \begin{bmatrix} 8 & -1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ $\therefore X = \begin{bmatrix} 12 & -8 \\ -16 & 15 \end{bmatrix}$
d **1** Form the equation $BX = D$ and note that X must be a 2×1 matrix. **2** Calculate $X = B^{-1}D$.	$BX = D$ $B^{-1}BX = B^{-1}D$ $X = B^{-1}D$ $X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\therefore X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
e **1** Form the equation $AX = E$ and note that X must be a 3×1 matrix. **2** Calculate $X = A^{-1}E$.	$AX = E$ $A^{-1}AX = A^{-1}E$ $X = A^{-1}E$ $X = \begin{bmatrix} -2 & -1 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ $\therefore X = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

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f **1** Form the equation $BX + \begin{bmatrix} 4 \\ 5 \end{bmatrix} = D$ and note that X must be a 2×1 matrix.	$BX + \begin{bmatrix} 4 \\ 5 \end{bmatrix} = D$
2 Calculate $X = B^{-1}\left(D - \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right)$.	$BX = D - \begin{bmatrix} 4 \\ 5 \end{bmatrix}$
	$X = B^{-1}\left(D - \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right)$
	$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -2 \end{bmatrix}$
	$\therefore X = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Skill-sheet

Exercise 10E

Review of the properties of the identity matrix

1 **a** Write down the:

i 2×2 identity matrix **ii** 3×3 identity matrix **iii** 4×4 identity matrix.

b If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, show that $AI = IA = A$.

c If $C = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, show that $CI = IC = C$.

Demonstrating that one matrix is the inverse of the other

Example 24

2 Show that each of the following pairs of matrices are inverses by multiplying one by the other. Use a calculator if you wish.

a $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1.5 & 1 \end{bmatrix}$

c $\begin{bmatrix} 9 & 7 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} -3 & 7 \\ 4 & -9 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$

Calculating the determinant of a matrix

Example 25

3 Determine (by hand) the value of the determinant for each of the following matrices.

a $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}$ **c** $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ **d** $D = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$

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Calculating the inverse of a matrix

Example 26

4 Use a calculator to determine the inverse of each of the following matrices.

a $A = \begin{bmatrix} 1.1 & 2.2 \\ 0 & 3.0 \end{bmatrix}$ **b** $B = \begin{bmatrix} 0.2 & -0.1 \\ 10 & 4 \end{bmatrix}$ **c** $D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **d** $E = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Solving matrix equations

Example 27

5 Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ $C = \begin{bmatrix} -6 & -1 \\ 5 & 6 \end{bmatrix}$ $D = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $E = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$

Solve each of the following matrix equations for X

a $B + X = C$ **b** $BX = C$ **c** $XB = C$

d $BX = D$ **e** $AX = E$ **f** $BX + \begin{bmatrix} 7 \\ 6 \end{bmatrix} = D$

6 Find the 2×2 matrix A such that $A\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.

7 Suppose that A, B, C and X are 2×2 matrices and that both A and B have inverses. Solve the following for X:

a $AX = C$ **b** $ABX = C$ **c** $AXB = C$

d $A(X + B) = C$ **e** $AX + B = C$ **f** $XA + B = A$

8 If $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15\,000 \\ 20\,000 \\ 10\,000 \end{bmatrix}$ find x, y and z

9 Three crop sprays are manufactured by combining chemicals A, B and C as follows:

Spray P One barrel of spray P contains 1 unit of A, 3 units of B and 4 units of C.

Spray Q One barrel of spray Q contains 3 units each of A, B and C.

Spray R One barrel of spray R contains 2 units of A and 5 units of B.

To control a certain crop disease, a farmer requires 6 units of chemical A, 10 units of chemical B and 6 units of chemical C. How much of each type of spray should the farmer use?

10 A factory makes and assembles three products, P, Q and R, each requiring different quantities of three components, a, b and c. The following matrix A represents the required quantities of components for each product, and the matrix K represents the daily production of components at the factory.

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$$A = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix} \end{array} \quad \text{and} \quad K = \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 95 \\ 80 \\ 40 \end{bmatrix}$$

a Find the inverse of A.

b Assume that the factory uses all components that are produced. Find the rate of assembly of P, Q and R at the factory, expressed as number of products per day.

11 Bronwyn and Noel have a clothing warehouse in Summerville. They are supplied by three contractors: Brad, Flynn and Lina.
The matrix shows the number of dresses, pants and shirts that one worker, for each of the contractors, can produce in a week.

$$\begin{array}{r} \\ \text{Dresses} \\ \text{Pants} \\ \text{Shirts} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Brad} & \text{Flynn} & \text{Lina} \end{array} \\ \begin{bmatrix} 5 & 6 & 10 \\ 3 & 4 & 5 \\ 2 & 6 & 5 \end{bmatrix} \end{array}$$

The number produced varies because of the different equipment used by the contractors.
The warehouse requires 310 dresses, 175 pants and 175 shirts in a week. How many workers should each contractor employ to meet the requirement exactly?

Exam 1 style questions

12 The determinant of the matrix $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ is

A 4 **B** 0 **C** −4 **D** 1 **E** 2

13 The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ is

A −1 **B** $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

14 Consider the matrix equation

$$3 \times \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix} + X = \begin{bmatrix} 14 & 12 \\ 18 & 22 \end{bmatrix}$$

Which one of the following is the matrix X?

A $\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$

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10F Binary, permutation and communication matrices

Learning intentions

- To be able to identify and work with binary matrices, permutation matrices and communication matrices.

Binary matrices

The following matrices are examples of binary matrices.

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Binary matrices are at the heart of many practical matrix applications, including analysing communication systems and using the concept of dominance to rank players in sporting competitions.

Permutation matrices

A **permutation**[1] **matrix** is a square binary matrix in which there is only one '1' in each row and column.

The following matrices are examples of permutation matrices.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

An identity matrix is a special permutation matrix. A permutation matrix can be used to rearrange the elements in another matrix.

Example 28 **Applying a permutation matrix**

X is the column matrix $X = \begin{bmatrix} T \\ A \\ R \end{bmatrix}$. P is the permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

a Show that:

i pre-multiplying[2] X by P changes the matrix X to the matrix $Y = \begin{bmatrix} R \\ A \\ T \end{bmatrix}$

ii pre-multiplying X by P^2 leaves the matrix X unchanged.

b What can be deduced about P^2 from the result in **a ii**?

---

[1] The word 'permutation' means a rearrangement a group of objects, in this case the elements of a matrix, into a different order.

[2] When we form the matrix product $A \times B$, we say that we are pre-multiplying by A.

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Explanation

a **i** Form the matrix product PX. To find the first entry in the resulting column matrix note that the 1 of the permutation matrix is in the third column. of the first row. We obtain

$$0 \times T + 0 \times A + 1 \times R = R$$

To find the second entry in the resulting column matrix note that the 1 of the permutation matrix is in the second column of the second row and so on.

ii Form the matrix product P^2X.

b To leave the matrix X unchanged, P^2 must be an identity matrix.

Solution

$$PX = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ A \\ R \end{bmatrix} = \begin{bmatrix} R \\ A \\ T \end{bmatrix}$$

$$P^2X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} T \\ A \\ R \end{bmatrix} = \begin{bmatrix} T \\ A \\ R \end{bmatrix}$$

P^2 is the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 29

Find a permutation matrix that takes the column matrix $\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$ to the column matrix $\begin{bmatrix} D \\ C \\ B \\ A \end{bmatrix}$

Explanation

- We want to move D from the fourth place to the first place. Place the 1 in the fourth column of the first row.
- We want to move C from the third place to the second place. Place the 1 in the third column of the second row.
- We want to move B from the second place to the third place. Place the 1 in the second column of the third row.
- We want to move A from the first place to the fourth place. Place the 1 in the first column of the fourth row.

Solution

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} D \\ C \\ B \\ A \end{bmatrix}.$$

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Inverses of permutation matrices

Every permutation matrix has an inverse and this inverse is the adjoint of the matrix. That is if P is a permutation matrix then P^{-1} exists and

$$P^{-1} = P^T$$

Example 30

A 4×4 permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ applied to a 4×1 column matrix A gives the column matrix $\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$. Determine the matrix A.

Solution

$$PA = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$$

Using your calculator.

$$\therefore A = P^{-1}\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = P^T\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} W \\ Z \\ X \\ Y \end{bmatrix}$$

Communication matrices

A **communication matrix** is a square binary matrix in which the 1s represent the links in a communication system.

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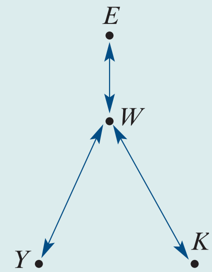
Example 31 Constructing a communication matrix

Eva, Wong, Yumi and Kim are students who are staying in a backpacker's hostel. Because they speak different languages they can have problems communicating. The situation they have to deal with is that:



- Eva speaks English only
- Yumi speaks Japanese only
- Kim speaks Korean only
- Wong speaks English, Japanese and Korean.

Conveniently, this information can be summarised in a network diagram, as shown above. In this diagram, the arrow linking Eva and Wong indicates that they can communicate directly because they both speak English.

The task is to construct a communication matrix.

Explanation

1 There are four people so a 4×4 matrix is needed. Label the columns and rows E, W, Y and K.

2 Label the rows 'Speaker' and the columns 'Receiver'.

3 Designate each element as a '1' or '0' according to the following rules:

- the element = 1 if two people can communicate directly because they speak the same language
- the element = 0 if two people *cannot* communicate directly because they do not have a common language.

The completed matrix is shown opposite.

Solution

$$\begin{array}{ccc} & & \textit{Receiver} \\ & & \begin{array}{cccc} E & W & Y & K \end{array} \\ \textit{Speaker} & \begin{array}{c} E \\ W \\ Y \\ K \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

There is little point in having a matrix representation of a communication system if we already have a network diagram. However, several questions that are not so easily solved with a network diagram can be answered using a communication matrix.

For example, we can see from the network diagram that Eva, who speaks only English, cannot communicate directly with Yumi, who speaks only Japanese. We call this a *one-step communication link*.

However, Eva can communicate with Yumi by sending a message via Wong, who speaks both English and Japanese. In the language of communication systems we call this a *two-step communication link*.

The power of the matrix representation is that *squaring* the communication matrix generates a matrix that identifies all possible *two-step communication links* in a communication

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network. For example, if we call the communication matrix C, we have:

$$C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{array}{c c} \begin{array}{cccc} E & W & Y & K \end{array} & \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} & \begin{array}{c} E \\ W \\ Y \\ K \end{array} \end{array}$$

We have the *two-step link*: Eva → Wong → Yumi.

That is, Eva can communicate with Yumi by using Wong as a go-between.

However, if we use the same process to help us interpret the '1' in row E, column E, we will find that it represents the two-step link Eva → Wong → Eva. This is not a very useful thing to know. Two-step links that have the same sender and receiver are said to be **redundant communication links** because they do not contribute to the communication between different people.

Redundant communication links

A communication link is said to be redundant if the sender and the receiver are the same person.

All of the non-zero elements in the leading diagonal of a communication matrix, or its powers, represent redundant links in the matrix.

However, all of the remaining non-zero elements represent meaningful two-step communication links.

For example, the 1 in row Y, column K represents the two-step communication link that enables Yumi to send a message to Kim.

$$C^2 = \begin{array}{c c} \begin{array}{cccc} E & W & Y & K \end{array} & \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} & \begin{array}{c} E \\ W \\ Y \\ K \end{array} \end{array}$$

Finally, the total number of one and two-step links in a communication system, T, can be found by evaluating $T = C + C^2$.

$$T = C + C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{array}{c c} \begin{array}{cccc} E & W & Y & K \end{array} & \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} E \\ W \\ Y \\ K \end{array} \end{array}$$

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Analysing communication matrices

- A communication matrix (C) is a square binary matrix in which the 1s are used to identify the direct (one-step) links in the communication system.
- The number of two-step links in a communication system can be identified by squaring its communication matrix.
- The total number of one and two-step links in a communication system can be found by evaluating the matrix sum $T = C + C^2$.

These statements can be readily generalised to include the determination of three (or more) step links by evaluating the matrices C^3, C^4, etc. However, unless the communication networks are extremely large, most of the multi-step links identified will be redundant.

Note: In all cases, the diagonal elements of a communication matrix (or its power) represent redundant communication links.

Skill-sheet

Exercise 10F

Permutation matrices

1 Which of the following binary matrices are permutation matrices?

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 28

2 X is the row matrix: $X = \begin{bmatrix} \mathrm{S} & \mathrm{H} & \mathrm{U} & \mathrm{T} \end{bmatrix}$

P is the permutation matrix: $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

a What does matrix X change to if it is post-multiplied by P?

b For what value of n does XP^n first equal X?

Example 29

3 Find a permutation matrix that takes the column matrix $\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$ to the column matrix $\begin{bmatrix} C \\ D \\ A \\ B \end{bmatrix}$

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Inverses of permutation matrices

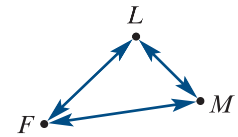
Example 30

4 A 4×4 permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ applied to a 4×1 column matrix A gives the column matrix $\begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix}$. Determine the matrix A.

Communication matrices

Example 31

5 Freya (F), Lani (L) and Mei (M) are close friends who regularly send each other messages. The direct (one-step) communication links between the friends are shown in the diagram opposite.

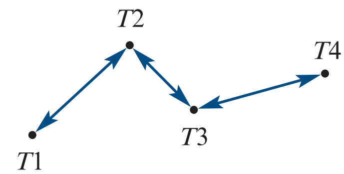


a Construct a communication matrix C from this diagram.

b Calculate C^2.

c How many different ways can Mei send a message to Freya?

6 Four fire towers $T1$, $T2$, $T3$ and $T4$, can communicate with one another as shown in the diagram opposite. In this diagram an arrow indicates that a direct channel of communication exists between a pair of fire towers.



For example, a person at tower 1 can directly communicate with a person in tower 2 and vice versa.

The communication matrix C can also be used to represent this information.

$$C = \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \square & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \square & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \end{array}$$

a Explain the meaning of a zero in the communication matrix.

b Which two towers can communicate directly with $T2$?

c Write down the values of the two missing elements in the matrix.

The matrix C^2 is shown opposite.

$$C^2 = \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \end{array}$$

d Explain the meaning of the 1 in row $T3$, column $T1$.

e How many of the two-step communication links shown in the matrix C^2 are redundant?

f Construct a matrix that shows the total number of one and two-step communication links between each pair of towers.

g Which of the four towers need a three-step link to communicate with each other?

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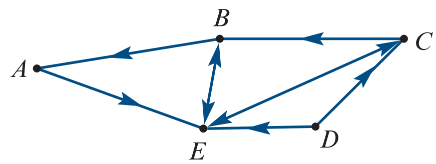
Representing a large network diagram by a matrix

7 Construct a 5×5 matrix to represent the communication network diagram opposite.



Exam 1 style questions

8 The matrix below shows how five people, Adam(A), Bertie(B), Catherine(C), David(D) and Evan (E), can communicate with each other.

$$\begin{array}{ccc} & & \textit{Receiver} \\ & & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \textit{Sender} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

A '1' in the matrix shows that the person named in that row can send a message directly to the person named in that column. Adam wants to send a message to David. This can be done through a sequence of communications formed from the five people. Which of the following is a possible sequence of communications to get the message from Adam to David?

A A, D **B** A, B, D **C** A, B, C, D **D** A, B, E, D **E** A, E, C, B, D

9 Matrix P is a 3×3 permutation matrix. Matrix Z is another matrix such that the matrix product $P \times Z$ is defined. This matrix product results in the third row becoming the first row, the second row becoming the third and the first row becoming the second row of matrix Z. The permutation matrix P is

A $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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10G Dominance matrices

Learning intentions

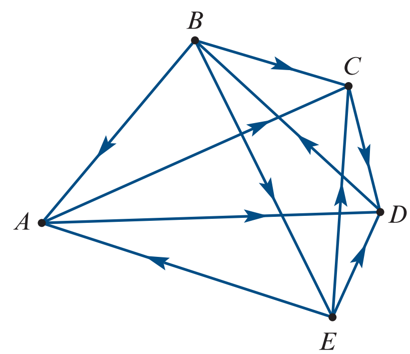
- To be able to construct and interpret dominance matrices.

In many group situations, certain individuals are said to be dominant. This is particularly true in sporting competitions. Problems of identifying dominant individuals in a group can be analysed using the same approach we used to analyse communication networks.

For example, five players – Anna, Birgit, Cas, Di and Emma – played in a **round-robin tournament**[3] of tennis to see who was the dominant (best) player.

The results were as follows:

- Anna defeated Cas and Di
- Birgit defeated Anna, Cas and Emma
- Cas defeated Di
- Di defeated Birgit
- Emma defeated Anna, Cas and Di.



We can use a network diagram to display the results graphically, as shown opposite. In this diagram, the arrow from *B to A* tells us that, when they played, Birgit defeated Anna.

Both Birgit and Emma had three wins each so there is a tie. How can we resolve this situation and see who is the best player? One way of doing this is to calculate a dominance score for each player. We do this by a constructing a series of dominance matrices.

One-step dominances

The first **dominance matrix**, D, records the number of one-step dominances between the players.

For example, Anna has a one-step dominance over Cas because, when they played, Anna *beat* Cas.

$$D = \begin{array}{ccc} & \begin{array}{ccccc} A & B & C & D & E \end{array} & \textit{Dominance} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \\ 1 \\ 1 \\ 3 \end{array} \end{array}$$

This matrix can be used to calculate a one-step dominance score for each player, by summing each of the rows of the matrix. According to this analysis, B and E are equally dominant with a dominance score of 3.

[3] A round-robin tournament is one in which each of the participants play each other once.

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Now let us take into account two-step dominances between players.

Two-step dominances

A two-step dominance occurs when a player beats another player who has beaten someone else. For example, Birgit has a two-step dominance over Di because Birgit defeated Cas who defeated Di.

Two-step dominances can be determined using the same technique used to obtain two-step links in a communication network. We simply square the one-step dominance matrix.

$$D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Dominance} \\ 2 \\ 6 \\ 1 \\ 3 \\ 4 \end{array}$$

The two-step dominances for these players are shown in matrix D^2. Reading across the 'B row'.

- The 1 in column A represents the two-step dominance 'Birgit defeated Emma who defeated Anna'.
- The 2 in column C represents the two-step dominances 'Birgit defeated Emma who defeated Cas' and 'Birgit defeated Anna who defeated Cas'
- The 3 in column D represents the three two-step dominances 'Birgit defeated Emma who defeated Di', 'Birgit defeated Anna who defeated Di' and 'Birgit defeated Cas who defeated Di'.
- In column E the 0 tells us that there are no two-step dominances for Birgit over Emma even though there was a one-step dominance.

We can combine the information contained in both D and D^2 by calculating a new matrix $T = D + D^2$.

$$T = D + D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Total} \\ 4 \\ 9 \\ 2 \\ 4 \\ 7 \end{array}$$

Using these total dominance scores:

- Birgit is the top-ranked player with a total dominance score of 9
- Eva is second with a total score of 7
- Anne and Di are equal third with a total score of 4
- Cas is the bottom-ranked player with a total score of 2.

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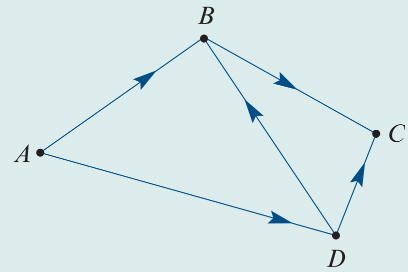
Example 32 Determining dominance

Four people, A, B, C and D, have been asked to form a committee to decide on the location of a new toxic waste dump.



From previous experience, it is known that:

- A influences the decisions of B and D
- B influences the decisions of C
- C influences the decisions of no one
- D influences the decisions of C and B.

a Use the graph to construct a dominance matrix that takes into account both one-step and two-step dominances.

b From this matrix, determine who is the most influential person on the committee.

Explanation | **Solution**

a Construct the one-step dominance matrix D.

$$\begin{array}{ccc} & \begin{array}{cccc} A & B & C & D \end{array} & \textit{One-step} \\ D = \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} 2\\ 1\\ 0\\ 2 \end{array} \end{array}$$

Construct the two-step dominance matrix D^2.

$$\begin{array}{ccc} & \begin{array}{cccc} A & B & C & D \end{array} & \textit{Two-step} \\ D^2 = \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} 3\\ 0\\ 0\\ 1 \end{array} \end{array}$$

Form the sum $T = D + D^2$.

$$\begin{array}{ccc} & \begin{array}{cccc} A & B & C & D \end{array} & \textit{Total} \\ T = D + D^2 = \begin{array}{c} A\\ B\\ C\\ D \end{array} & \begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} & \begin{array}{c} 5\\ 1\\ 0\\ 3 \end{array} \end{array}$$

b The person with the highest total dominance score is the most influential.

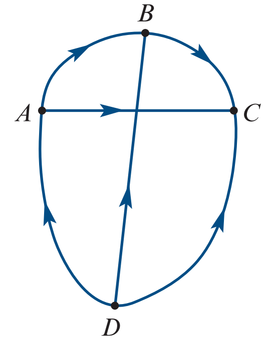
Person A is the most influential person with a total dominance score of 5.

Exercise 10G

Dominance matrices

Example 32

1 The results of a competition between teams A, B, C and D are displayed opposite. An arrow from D to C indicates that team D defeated team C.



a Construct a dominance matrix showing one-step dominance between the teams. Rank the teams according to one-step dominances.

b Construct a dominance matrix showing two-step dominances between the teams. Rank the teams, taking into account both one-step and two-step dominances.

2 Five students play each other at chess. The dominance matrix shows the winner of each game with a '1' and the loser or no match with a '0'. For example, row 2 indicates that B loses to A, D and E but beats C.

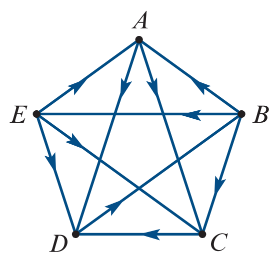
a Find the one-step dominance score for each student and use these to rank them.

b Calculate the two-step dominance matrix.

c Determine the matrix $T = D + D^2$ and use this matrix to rank the players.

$$\text{Winners} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \textit{Losers} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

3 Five friends – Ann, Bea, Cat, Deb and Eve – competed in a round-robin tennis tournament. The results were as follows:



- Ann defeated Cat and Deb
- Bea defeated Ann, Cat and Eve
- Cat defeated Deb
- Deb defeated Bea
- Eve defeated Ann, Cat and Deb.

Using this information:

a Construct a one-step dominance matrix, D.

b Construct a two-step dominance matrix, D^2.

c Use the dominance scores from the matrix $D + D^2$ to rank the five players.

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4 The following dominance matrix, M, gives the results of a series of squash matches between five friends, where $m_{ij} = 1$ if player i beat player j.

$$M = \begin{array}{c} \\ \text{Ash} \\ \text{Ben} \\ \text{Carl} \\ \text{Dot} \\ \text{Elle} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Ash} & \text{Ben} & \text{Carl} & \text{Dot} & \text{Elle} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

a How many matches were there?

b Describe the outcomes of the matches.

c Use the dominance scores from the matrix M to give a ranking of the players.

5 Five chess players – A, B, C, D and E – competed in a round-robin chess tournament. The results were as follows:

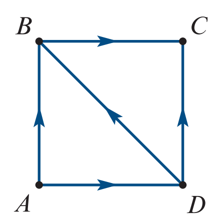
- A defeated B and D
- B defeated C and E
- C defeated A and D
- D defeated B
- E defeated A, C and D.

Using this information:

a Construct a one-step dominance matrix, M.

b Use the dominance scores from the matrix $M + M^2$ to rank the players.

6 A committee of four people – A, B, C and D – will decide on the location of a new toxic waste dump. From previous experience, it is known that:



- A influences the decisions of B and D
- B influences the decisions of C
- C influences the decisions of no one
- D influences the decisions of B and C.

Using this information:

a Construct a matrix that takes into account both one-step and two-step dominance.

b From this matrix, determine who is the most influential person on the committee.

7 The following table gives the results of the first round of games at a chess club.

Game	A vs B	C vs D	A vs D	B vs C	B vs D	A vs C
Winner	A	C	D	B	B	A

a Create a one-step dominance matrix, D.

b Create a ranking of the four players using D and D^2.

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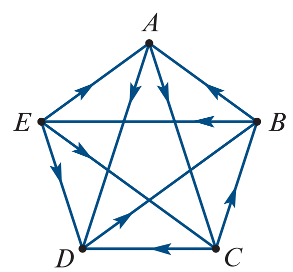
8 Four schools – A, B, C and D – compete in a round-robin hockey tournament. The results are summarised by the following dominance matrix:

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Create an ordering of the four schools using M, M^2 and M^3.

9 Five teams competed in a football tournament. Each team played every other team once. The results are shown in the diagram. (For example, the arrow pointing from A to C indicates that A defeated C.)

Create a rank order for the five teams by taking into consideration both one-step and two-step dominance.



Exam 1 style questions

10 Four teams, A, B, C and D, competed in a round-robin competition where each team played each of the other teams once. There were no draws. The results are shown in the matrix below.

$$\begin{array}{c} \\ \\ \\ \textit{winner} \\ \\ \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \textit{loser} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & z & 1 & 0 \end{bmatrix} \end{array}$$

The values of x, y and z are

A $x = 0, y = 0, z = 0$ **B** $x = 0, y = 1, z = q$ **C** $x = 1, y = 0, z = 0$

D $x = 1, y = 0, z = 1$ **E** $x = 1, y = 1, z = 1$

11 There are five hens in a coup. Their owner calls them Alpha, Beta, Gamma, Delta and Epsilon. There is a pecking order in the coop, and the following dominance matrix, M, was formed by the owner:

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$$M = \begin{array}{r} \\ \text{Alpha} \\ \text{Beta} \\ \text{Gamma} \\ \text{Delta} \\ \text{Epsilon} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Alpha} & \text{Beta} & \text{Gamma} & \text{Delta} & \text{Epsilon} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Based on the matrix $M + M^2$, which of the following best describes the pecking order in the coop?

A Beta, Epsilon, Delta, Alpha, Gamma
B Beta, Epsilon, Gamma, Delta, Alpha
C Beta, Epsilon, Delta, Gamma, Alpha
D Epsilon, Beta, Delta, Alpha, Gamma
E Epsilon, Beta, Delta, Gamma, Alpha

12 Four soccer teams, X, Y, Z and W, compete in a round-robin competition. In each game, there is a winner and a loser. To decide the winner of the tournament, the sum of the one-step dominance matrix, D, and the two-step dominance matrix, D^2, is found. This sum is

$$D + D^2 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}$$

Which one of the following is the correct one-step dominance for this tournament?

A $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
B $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
C $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$
E $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

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Key ideas and chapter summary

Assign-ment

Matrix A **matrix** is a rectangular array of numbers or symbols (elements) enclosed in brackets (plural: matrices).

Row matrix A **row matrix** contains a *single row* of elements.

Column matrix A **column matrix** contains a *single column* of elements.

Transpose The **transpose** of a matrix is obtained by interchanging its rows and columns.

Square matrix A **square matrix** has an **equal number of rows and columns**.

Zero matrix A **zero (null) matrix**, O, contains only zeros.

Order The **order** (or size) of a matrix is given by the number of rows and columns in that order.

Locating an element The location of each element in the matrix is specified by its row and column number in that order.

Equal matrices Matrices are *equal* when they have the *same order* and *corresponding elements* are *equal* in value.

Adding and subtracting matrices Two matrices of the same order can be added or subtracted, by adding or subtracting corresponding elements.

Scalar multiplication Multiplying a matrix by a number (**scalar multiplication**) multiplies every element in the matrix by that number.

Matrix multiplication **Matrix multiplication** is a process of multiplying rows by columns. To multiply a row matrix by a column matrix, each element in the row matrix is multiplied by each element in the column matrix and the results added. For example:

$$\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = [1 \times 4 + 0 \times 2 + 3 \times 5] = [19]$$

Power of a matrix The **power of a matrix** is defined in the same way as the powers of numbers: $A^2 = A \times A$, $A^3 = A \times A \times A$, and so on.
Only **square matrices** can be raised to a power.
A^0 is defined to be I, the **identity matrix**.

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Identity matrix An **identity matrix**, I, is a square matrix with 1s down the leading diagonal and zeros elsewhere.

Determinant The **determinant** of a *matrix*, A, is written as det(A).
Only **square matrices** have determinants.
If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ then $\det(A) = \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 3 = -5$
For higher order matrices, a calculator is used to calculate the determinant.

Inverse The **inverse** of a matrix, A, is written as A^{-1} and has the property that $AA^{-1} = A^{-1}A = I$.
Only **square matrices** have inverses.
The *inverse* of a matrix is *not defined* if $\det(A) = 0$.
A calculator is used to determine the inverse of a matrix.

Binary matrix A **binary matrix** is a matrix whose elements are either zeros or ones.

Permutation matrix A **permutation matrix** is a square binary matrix in which there is only a single 1 in each row and column.

Communication matrix A **communication matrix** is a square binary matrix in which the 1s represent direct (one-step) communication links.

Redundant communication link A communication link is said to be redundant if the sender and the receiver are the same people.

Round-robin tournament A **round-robin tournament** is one in which each of the participants plays every other competitor once.

Dominance matrix A **dominance matrix** is a square binary matrix in which the 1s represent one-step dominances between the members of a group.

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Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

10A **1** I can state the order of a given matrix. ☐

See Example 1, and Exercise 10A Question 1

10A **2** I can find the transpose of a matrix. ☐

See Example 2, and Exercise 10A Question 4

10A **3** I can identify a matrix as being a square, column or row matrix. ☐

See Example 3, and Exercise 10A Question 5

10A **4** I can identify a square matrix as being an identity or diagonal or symmetric or upper/lower triangular matrix. ☐

See Example 4, and Exercise 10A Question 6

10A **5** I can identify the a_{ij} term in a matrix A. ☐

See Example 5, and Exercise 10A Question 7

10A **6** I can construct a matrix given a rule for a_{ij}. ☐

See Example 6, and Exercise 10A Question 8

10A **7** I can enter a matrix into a CAS calculator. ☐

See CAS 1, and Exercise 10A Question 13

10B **8** I can enter information from a table into a matrix. ☐

See Example 7, and Exercise 10B Question 1

10B **9** I can represent a network diagram by a matrix. ☐

See Example 9, and Exercise 10B Question 4

10B **10** I can interpret a matrix representing a network diagram. ☐

See Example 10, and Exercise 10B Question 6

10C **11** I can recognise when two matrices are equal and use this to solve problems. ☐

See Example 11, and Exercise 10C Question 1

10C **12** I can add two matrices of the same order together. ☐

See Example 12, and Exercise 10C Question 2

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10C	**13**	I can subtract one matrix from another when they have the same order.	☐
		See Example 13, and Exercise 10C Question 2	
10C	**14**	I can multiply a matrix by a scalar.	☐
		See Example 14, and Exercise 10C Question 2	
10C	**15**	I recognise the role of the zero matrix and can undertake operations using the zero matrix.	☐
		See Example 15, and Exercise 10C Question 3	
10C	**16**	I can use addition, subtraction and scalar multiplication to process data.	☐
		See Example 16, and Exercise 10C Question 5	
10D	**17**	I can determine if the product of two given matrices is defined.	☐
		See Example 17, and Exercise 10D Question 1	
10D	**18**	II can determine the order of a matrix product.	☐
		See Example 18, and Exercise 10D Question 1	
10D	**19**	I can multiply a row matrix by a column matrix by hand.	☐
		See Example 19, and Exercise 10D Question 2	
10D	**20**	I can multiply a rectangular matrix by a column matrix by hand.	☐
		See Example 20, and Exercise 10D Question 2	
10D	**21**	I can use summing matrices to sum the rows or columns of a matrix.	☐
		See Example 21, and Exercise 10D Question 4	
10D	**22**	I can undertake multiplications of matrices to solve practical problems.	☐
		See Example 22, and Exercise 10D Question 8	
10D	**23**	I can evaluate matrix expressions involving powers.	☐
		See Example 23, and Exercise 10D Question 11	
10E	**24**	I can recognise that two matrices are inverses if their product is the identity matrix.	☐
		See Example 24, and Exercise 10E Question 2	
10E	**25**	I can evaluate the determinant of a 2×2 matrix by hand.	☐
		See Example 25, and Exercise 10E Question 3	
10E	**26**	I can calculate the inverse of a 2×2 matrix by hand.	☐
		See Example 26, and Exercise 10E Question 4	

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Review

10E **27** I can calculate the inverse and determinant of a $n \times n$ matrix using CAS. ☐

See CAS 4, and Exercise 10E Question 4

10E **28** I can solve simple matrix equations. ☐

See Example 27, and Exercise 10E Question 5

10F **29** I can use a permutation matrix to rearrange the elements of a column or row matrix. ☐

See Example 28, and Exercise 10F Question 2

10F **30** I can construct a permutation matrix to rearrange the elements of a column or row matrix in a given order. ☐

See Example 29, and Exercise 10F Question 3

10F **31** I can find the inverse of a permutation matrix. ☐

See Example 30, and Exercise 10F Question 4

10F **32** I can construct a communication matrix from information given in written form or a diagram. ☐

See Example 31, and Exercise 10F Question 5

10F **33** I can construct a dominance matrix from information given in written form or a diagram. ☐

See Example 30, and Exercise 10F Question 1

Multiple choice questions

The following matrices are needed for Questions 1 to 8.

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad W = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad Z = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

1 The row matrix is:

A U **B** V **C** W **D** X **E** Z

2 The square matrices are:

A U and V **B** X and Y **C** Y and W **D** U and Y **E** U, V and X

3 The order of matrix X is:

A 2×2 **B** 2×3 **C** 3×2 **D** 3×3 **E** 6

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Review

4 The following matrices can be added:

A U and V **B** V and W **C** X and Y **D** U and Y **E** none of the above

5 The following matrix product is *not* defined:

A WV **B** XZ **C** YV **D** XY **E** UY

6 $-2Y =$

A $\begin{bmatrix} 0 & -2 \\ 2 & -4 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -2 \\ -2 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} -2 & -1 \\ -4 & 2 \end{bmatrix}$

7 The order of matrix product XZ is:

A 1×3 **B** 2×1 **C** 3×1 **D** 3×2 **E** 3×3

8 $U^T =$

A $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

9 In the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 3 \\ -5 & -4 & 7 \end{bmatrix}$, the element $a_{23} =$

A -4 **B** -1 **C** 0 **D** 3 **E** 4

10 $2\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} =$

A $\begin{bmatrix} 5 & 0 \\ -4 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 0 \\ 4 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 0 \\ -2 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 6 & 0 \\ 1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$

11 $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} =$

A $\begin{bmatrix} 10 \end{bmatrix}$ **B** $\begin{bmatrix} 12 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$ **E** not defined

The following matrices are needed for Questions 12 to 16.

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} 43 \\ 45 \end{bmatrix} \quad W = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

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Review

12 The matrix that cannot be raised to a power is:

A U **B** V **C** W **D** X **E** Y

13 $\det(U) =$

A -2 **B** 0 **C** 1 **D** 2 **E** 4

14 $Y^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

15 $U^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

16 The matrix product that is defined is:

A UX **B** XY **C** VW **D** UW **E** WX

17 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$

A $\begin{bmatrix} 18 \end{bmatrix}$ **B** $\begin{bmatrix} 12 \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ **D** $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$

18 X is a 3×2 matrix. Y is a 2×3 matrix. Z is a 2×2 matrix. Which of the following matrix expressions is *not* defined?

A XY **B** YX **C** $XZ - 2X$ **D** $YX + 2Z$ **E** $XY - YX$

19 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$.

The matrix expression that displays the mean of the numbers 3, 5, 2, 4 is:

A $\frac{1}{4}(A + B)$ **B** $\frac{1}{2}(A + B)$ **C** $\frac{1}{4}B$ **D** $\frac{1}{4}AB$ **E** $\frac{1}{4}BA$

20 Consider the following four matrix expression.

$$\begin{bmatrix} 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$$

How many of these four matrix expressions are defined?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

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21 If both A and B are $m \times n$ matrices, where $m \neq n$, then $A + B$ is

A an $m \times n$ matrix **B** an $m \times m$ matrix **C** an $n \times n$ matrix

D a $2m \times 2n$ matrix **E** not defined

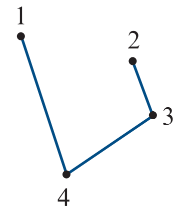
22 The matrix expression $\begin{bmatrix} 4 & 6 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -6 & -12 \\ 4 & -1 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

D $\begin{bmatrix} 13 & 24 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 24 \\ 0 & 3 \end{bmatrix}$

23 The diagram opposite is to be represented by a matrix, A, where:

- element = 1 if the two points are joined by a line
- element = 0 if the two points are not connected.



The matrix A is:

A $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

24 If matrix $M = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 7 & 2 \\ 9 & 6 \end{bmatrix}$ then the transpose matrix $M^T =$

A $\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 7 \\ 6 & 9 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 6 & 9 \\ 2 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 4 & 7 & 9 \\ 0 & 1 & 2 & 6 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 1 & 2 & 6 \\ 3 & 4 & 7 & 9 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & 4 & 0 & 1 \\ 2 & 6 & 7 & 9 \end{bmatrix}$

25 M is a 6×6 matrix. N is a 5×6 matrix. Which one of the following matrix expressions is defined?

A $NM - 2N$ **B** $M(MN)^{-1}$ **C** M^2N **D** N^TM **E** M^TN

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Review

26 Matrix A_1 is the 4×1 column matrix $\begin{bmatrix} C \\ B \\ A \\ D \end{bmatrix}$

A second 4×1 column matrix, A_2, contains the same elements as A_1, but the elements are ordered from top to bottom in alphabetical order. Matrix $A_2 = P \times A_1$, where P is a permutation matrix. Matrix P is

A $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

27 Four teams, X, Y, Z and W, compete in a round-robin competition. In each game, there is a winner and a loser. The sum of the one-step dominance matrix, D, and the two-step dominance matrix, D^2, is found. This sum is

$$D + D^2 = \begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix} \end{array}$$

In the first two games:

- team W defeated team X
- team Z defeated team W.

Which one of the following is the correct one-step dominance for this tournament?

A $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$ **B** $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \end{array}$ **C** $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$

D $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$ **E** $\begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$

28 The matrix $\begin{bmatrix} 1 & 5 & 3 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an example of a

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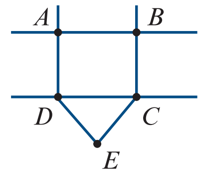
A symmetric matrix. **B** unit matrix. **C** triangular matrix.
D diagonal matrix. **E** communication matrix.

29 A, B, C, D and E are five intersections joined by roads, as shown in the diagram opposite. Some of these roads are one-way only.



The matrix opposite indicates the direction that cars can travel along each of these roads.

In this matrix:

- the 1 in column A and row B indicates that cars can travel directly from A to B
- the 0 in column B and row A indicates that cars cannot travel directly from B to A (either it is a one-way road or no road exists).

From intersection

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \end{array}$$

To intersection

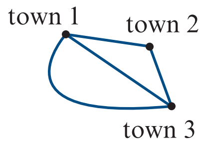
Cars can travel in both directions between intersections:

A A and D **B** B and C **C** C and D **D** D and E **E** C and E

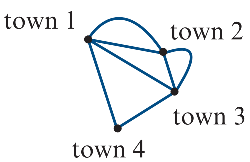
Written response questions

1 The following diagrams represent the road network joining several towns. Represent each road by a matrix.

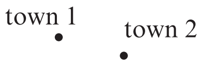
a



b



c



2 Heights in feet and inches can be converted into centimetres using matrix multiplication. The matrix $C = \begin{bmatrix} 30.45 \\ 2.54 \end{bmatrix}$ can be used as a conversion matrix (1 foot = 30.45 cm and 1 inch equals 2.54 cm).

a What is the order of matrix C?

Jodie tells us that her height is 5 feet 4 inches. We can write her height as a matrix $J = \begin{bmatrix} 5 & 4 \end{bmatrix}$.

b What is the order of matrix J?

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c Is the matrix product JC defined? Why?

d Evaluate the matrix product JC. Explain why it gives Jodie's height in centimetres.

e Matrix $H = \begin{bmatrix} 5 & 8 \\ 6 & 1 \end{bmatrix}$ gives the heights in feet and inches of two other people.
Use the conversion matrix C and matrix multiplication to generate a matrix that displays the heights of these two people in centimetres.

3 Books can be classified as *fiction* or *non-fiction* and come in either *hardback* or *paperback* form. The table shows the number of book titles carried by two bookshops in each of the categories.

	Bookshop 1		*Bookshop 2*	
Number of titles	*Hardback*	*Paperback*	*Hardback*	*Paperback*
Fiction	334	876	354	987
Non-fiction	213	456	314	586

a How many non-fiction paperback titles does bookshop 1 carry?

b The matrix $A = \begin{bmatrix} 334 & 876 \\ 213 & 456 \end{bmatrix}$ displays the number of book titles available at bookshop 1 in all categories. What is the order of this matrix?

c Write down a matrix equivalent to matrix A that displays the number of book titles available at bookshop 2. Call this matrix B.

d Construct a new matrix, $C = A + B$. What does this matrix represent?

e The average cost of books is \$45 for a hardback and \$18.50 for a paperback. These values are summarised in the matrix $E = \begin{bmatrix} 45.00 \\ 18.50 \end{bmatrix}$.

i What is the order of matrix E?

ii Construct the matrix product AE and evaluate.

iii What does the product AE represent?

f Bookshop 1 plans to double the number of titles it carries in every category. Write down a matrix expression that represents this situation and evaluate.

4 Mathematics and Physics are offered in a first year university science course.

The matrix $N = \begin{array}{c l} \begin{bmatrix} 600 \\ 320 \end{bmatrix} & \begin{array}{l} \textit{Mathematics} \\ \textit{Physics} \end{array} \end{array}$ lists the number of students enrolled in each subject.

The matrix $P = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0.15 & 0.225 & 0.275 & 0.25 & 0.10 \end{bmatrix} \end{array}$ lists the proportion of these students expected to be awarded an A, B, C, D or E grade in each subject.

a Write down the order of matrix P.

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b Let the matrix $R = NP$.

i Evaluate the matrix R.

ii Explain what the matrix element R_{13} represents.

c Students enrolled in Mathematics have to pay an extra fee of \$220, while students enrolled in Physics pay an extra fee of \$197.

i Write down a clearly labelled row matrix, called F, that lists these fees.

ii Show a matrix calculation that will give the total fees fees, L, paid in dollars by the students enrolled in Mathematics and Physics. Find this amount.

5 In a simplified game of darts, the possible scores are 25, 50 or 75.
G is a column matrix that lists the possible scores
In one game of 15 throws, Daniel achieved

$$G = \begin{bmatrix} 25 \\ 50 \\ 75 \end{bmatrix}$$

- Eight '25 scores'
- Six '50 scores'
- One '75 score'

a Write a row matrix, N, that shows the number of each score that Daniel had.

b Matrix P is found by multiplying matrix N with matrix G so that $P = N \times G$. Evaluate matrix P.

c In this context, what does the information in matrix P provide?

6 A mining company operates three mines, A, B and C. Each of the mines produces three types of minerals, p, q and r. Consider the following two matrices:

$$X = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} p & q & r \end{array} \\ \begin{bmatrix} 20 & 20 & 40 \\ 0 & 40 & 20 \\ 60 & 40 & 60 \end{bmatrix} \end{array} \quad \text{and} \quad Y = \begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 46\,000 \\ 34\,000 \\ 106\,000 \end{bmatrix}$$

Matrix X gives the number of tonnes of each of the minerals extracted per day from each of the mines, and matrix Y gives the total revenue (in dollars) from selling the minerals extracted from each of the mines on one day.

a Calculate the total number of tonnes of minerals produced by mine A.

b Calculate the total number of tonnes of mineral q produced.

c Calculate the total revenue of the three mines.

d In the matrix equation $XA = Y$

i What is the order of matrix A?

ii What do the elements of matrix A represent?

iii We know that $A = X^{-1}Y$. Find A.

Review

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