Chapter 11

Transition matrices and Leslie matrices

Chapter objectives

- How do you construct a transition matrix from a transition diagram and vice versa?
- How do you construct a transition matrix to model the transitions in a population?
- How do you use a matrix recurrence relation, S_0 = initial state matrix, $S_{n+1} = TS_n$, to generate a sequence of state matrices?
- How do you informally identify the equilibrium state or steady-state matrix in the case of regular state matrices?
- How do you use a matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$ to model systems that include external additions or reductions at each step of the process?
- How do you use and interpret Leslie matrices to analyse population growth?

In this chapter we use matrices to model proportional change of the numbers in a particular state to itself and other states from one time to the next.

For example, in the first example of this chapter, there are two states which in this case are the towns in which rental cars finish up each day.

In sections A-E of this chapter we look at transition matrices which satisfy certain conditions and describe the proportional change.

In section F of the chapter Leslie matrices are used to model population change in a particular way.

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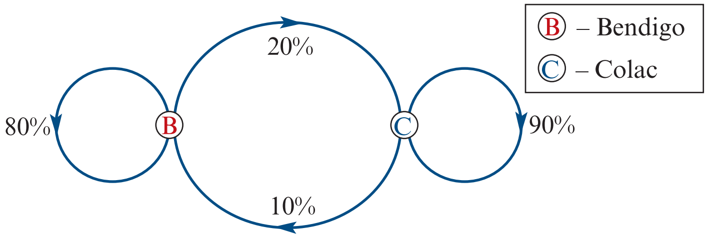
11A Transition matrices - setting up a transition matrix

Learning intentions

- ▶ To be able to set up a transition matrix from a diagram or written information.

A car rental firm has two branches: one in Bendigo and one in Colac. Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo each week are returned in Colac, and vice versa. The diagram below describes what happens on a weekly basis.



What does this diagram tell us?

From week to week:

- 0.8 (or 80%) of cars rented each week in Bendigo are returned to Bendigo
- 0.2 (or 20%) of cars rented each week in Bendigo are returned to Colac
- 0.1 (or 10%) of cars rented each week in Colac are returned to Bendigo
- 0.9 (or 90%) of cars rented each week in Colac are returned to Colac.

The percentages (written as proportions) are summarised in the form of the matrix below.

$$\begin{array}{cc} & \text{Rented in} \\ & \begin{array}{cc} \text{Bendigo} & \text{Colac} \end{array} \\ \text{Returned to} \begin{array}{c} \text{Bendigo} \\ \text{Colac} \end{array} & \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \end{array}$$

This matrix is an example of a **transition matrix (T)**. It describes the way in which transitions are made between two *states*:

- state 1: the rental car is based in Bendigo.
- state 2: the rental car is based in Colac.

Note: In this situation, where the total number of cars remains constant, the columns in a transitional matrix will always add to one (100%). For example, if 80% of cars are returned to Bendigo, then 20% must be returned to Colac.

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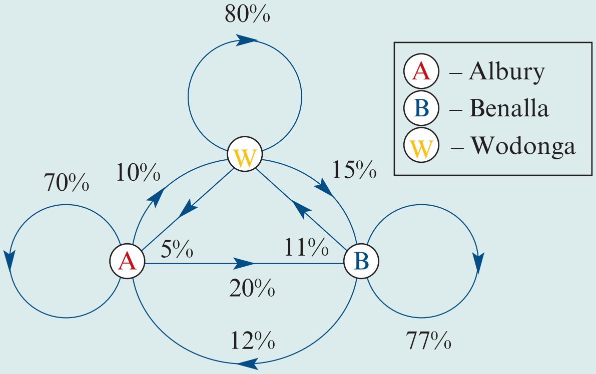
Example 1 Setting up a transition matrix

The diagram gives the weekly return rates of rental cars at three locations: Albury, Wodonga and Benalla. Construct a transition matrix that describes the week-by-week return rates at each of the three locations. Convert the percentages to proportions.



Explanation

1 There are three locations from which the cars can be rented and returned: Albury (A), Wodonga (W) and Benalla (B). To account for all the possibilities, a 3×3 matrix is needed. Construct a blank matrix labelling the rows and columns A, W and B, respectively. Column labels indicate where the car was rented. The row labels indicate where the cars were returned to.

Solution

$$\begin{array}{rc} & \text{Rented in} \\ & \begin{array}{ccc} A & \quad W \quad & B \end{array} \\ \begin{array}{r} A \\ \text{Returned to w} \\ B \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array}$$

2 Complete the matrix by writing each of the percentages (converted to proportions) into the appropriate locations. Start with column A and write in values for each row: 0.7 (70%), 0.1 (10%) and 0.2 (20%).

$$\begin{array}{rc} & \begin{array}{ccc} A & \quad W \quad & B \end{array} \\ \begin{array}{r} A \\ w \\ B \end{array} & \begin{bmatrix} 0.7 & \quad & \quad \\ 0.1 & & \\ 0.2 & & \end{bmatrix} \end{array}$$

3 Mentally check your answer by summing columns; they should sum to 1.

$$\begin{array}{rc} & \begin{array}{ccc} A & \quad W \quad & B \end{array} \\ \begin{array}{r} A \\ w \\ B \end{array} & \begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} \end{array}$$

Example 2 Setting up a transition matrix

A factory has a large number of machines. Machines can be in one of two states: operating or broken. Broken machines are repaired and come back into operation, and vice versa. On a given day:

- 85% of machines that are operational stay operating
- 15% of machines that are operating break down

- 5% of machines that are broken are repaired and start operating again
- 95% of machines that are broken stay broken.

Construct a transition matrix to describe this situation. Use the columns to define the situation at the ‘Start’ of the day and the rows to describe the situation at the ‘End’ of the day.

Explanation	Solution
1 There are two machine states: operating (O) or broken (B). To account for all the possibilities, a 2×2 transition matrix is needed. Construct a blank matrix, labelling the rows and columns O and B, respectively.	$\begin{array}{ccc} & & \text{Start} \\ & & \begin{array}{cc} O & \quad B \end{array} \\ \text{End} & \begin{array}{c} \text{O} \\ \text{B} \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$
2 Complete the matrix by writing each of the percentages (converted to proportions) into the appropriate locations. Start with column O and write in the values for each row: 0.85 (85%) and 0.15 (15%).	$\begin{array}{cc} & \begin{array}{cc} O & \quad B \end{array} \\ \begin{array}{c} \text{O} \\ \text{B} \end{array} & \begin{bmatrix} 0.85 & \quad \\ 0.15 & \quad \end{bmatrix} \end{array}$
3 Mentally check your answer by summing the columns; they should sum to 1.	$\begin{array}{cc} & \begin{array}{cc} O & \quad B \end{array} \\ \begin{array}{c} \text{O} \\ \text{B} \end{array} & \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \end{array}$

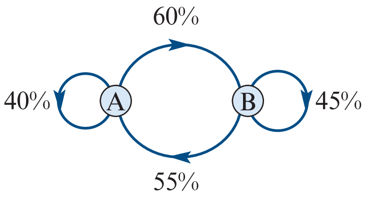
Exercise 11A

Setting up a transition matrix from a transition diagram

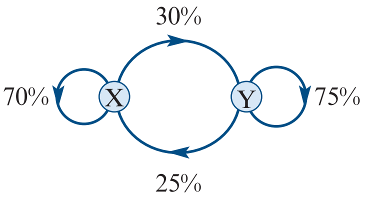
Example 1

1 The diagrams below describe a series of transitions between the states indicated. Construct a transition matrix that can be used to represent each of these diagrams. Use columns to define the starting points. Convert the percentages to proportions.

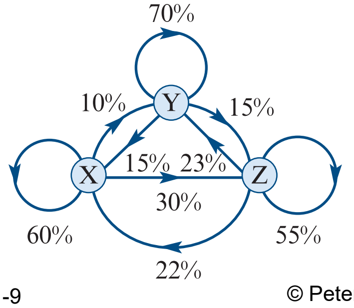
a



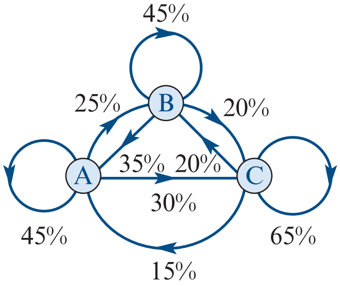
b



c



d



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Example 2

2 A factory has a large number of machines which can be in one of two states, *operating* (O) or *broken down* (B). It is known that that an operating machine breaks down by the end of the day on 4% of the days, and that 98% of machines which have broken down are repaired by the end of the day.
Complete the 2×2 transition matrix T to describe this.

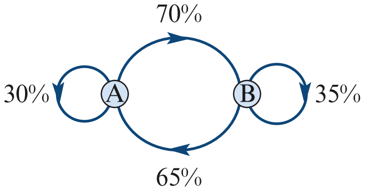
$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} O & B \end{array} \\ T = \begin{array}{c} O \\ B \end{array} & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \text{ Tomorrow} \end{array}$$

3 A large company has 1640 employees, 60% of whom currently work full-time (F) and 40% of whom currently work part-time (P). Every year 20% of full-time workers move to part-time work, and 14% of part-time workers move to full-time work.
Complete the 2×2 transition matrix T to describe this.

$$\begin{array}{cc} & \text{This year} \\ & \begin{array}{cc} F & P \end{array} \\ T = \begin{array}{c} F \\ P \end{array} & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \text{ Next year} \end{array}$$

Exam 1 style questions

4 In a particular newsagent, the two top-selling newspapers are the Argus and the Bastion. The transition diagram below shows the way shoppers at this newsagent change their newspaper choice from today to tomorrow.



A transition matrix that provides the same information as the transition diagram is

A
$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Tomorrow} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 65\% & 30\% \\ 70\% & 35\% \end{bmatrix} \end{array}$$

B
$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Tomorrow} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 30\% & 65\% \\ 70\% & 35\% \end{bmatrix} \end{array}$$

C
$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Tomorrow} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 30\% & 65\% \\ 35\% & 70\% \end{bmatrix} \end{array}$$

D
$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Tomorrow} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 30\% & 35\% \\ 65\% & 70\% \end{bmatrix} \end{array}$$

E
$$\begin{array}{cc} & \text{Today} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Tomorrow} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 65\% & 70\% \\ 35\% & 30\% \end{bmatrix} \end{array}$$

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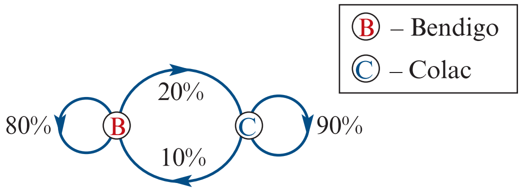
11B Interpreting transition matrices

Learning intentions

- ▶ To be able to interpret a transition matrix and a transition diagram.

Let us return to the car rental problem at the start of this section. As we saw then, the following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in Bendigo and Colac.

$$\begin{array}{cc} & \text{Rented in} \\ & \begin{array}{cc} B & C \end{array} \\ T = & \begin{bmatrix} 0.80 & 0.10 \\ 0.20 & 0.90 \end{bmatrix} \begin{array}{c} B \\ C \end{array} \text{ Returned to} \end{array}$$



Using this information alone, a number of predictions can be made.

For example, if 50 cars are rented in Bendigo this week, the transition matrix predicts that:

- 80% or 40 of these cars will be returned to Bendigo next week ($0.80 \times 50 = 40$)
- 20% or 10 of these cars will be returned to Colac next week ($0.20 \times 50 = 10$).

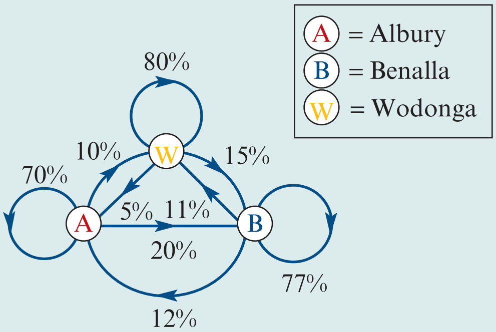
Further, if 40 cars are rented in Colac this week, the transition matrix predicts that:

- 10% or 4 of these cars will be returned to Bendigo next week ($0.10 \times 40 = 4$)
- 90% or 36 of these cars will be returned to Colac next week ($0.90 \times 40 = 36$).

Example 3 Interpreting a transition matrix

The following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in three locations: Albury, Wodonga and Benalla.

$$\begin{array}{cc} & \begin{array}{ccc} A & W & B \end{array} \\ T = & \begin{bmatrix} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{bmatrix} \begin{array}{c} A \\ W \\ B \end{array} \text{ Returned to} \end{array}$$



Use the transition matrix T and its transition diagram to answer the following questions.

a What percentage of cars rented in Wodonga each week are predicted to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

b Two hundred cars were rented in Albury this week. How many of these cars do we expect to be returned to:

i Albury? **ii** Benalla? **iii** Wodonga?

c What percentage of cars rented in Benalla each week are *not* expected to be returned to Benalla?

d One hundred and sixty cars were rented in Albury this week. How many of these cars are expected to be returned to either Benalla or Wodonga?

Solution

a **i** 0.5 or 5% **ii** 0.15 or 15% **iii** 0.80 or 80%

b **i** $0.70 \times 200 = 140$ cars **ii** $0.20 \times 200 = 40$ cars **iii** $0.10 \times 200 = 20$ cars

c $11 + 12 = 23\%$ or $100 - 77 = 23\%$

d 20% of $160 + 10\%$ of $160 = 48$ cars

Exercise 11B

Interpreting transition matrices

Example 3

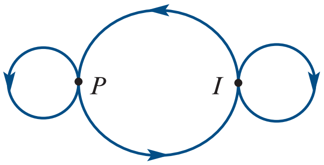
1 When people go to the movies they buy either a bag of popcorn (P) or an ice cream (I). Experience has shown that:

- 85% of people who buy popcorn this time will buy popcorn next time
- 15% of people who buy popcorn this time will buy an ice cream next time
- 75% of people who buy an ice cream this time will buy an ice cream next time
- 25% of people who buy ice cream this time will buy popcorn next time.

a Construct a transition matrix and transition diagram that can be used to describe this situation. Use the models below.

This time

$$T = \begin{array}{cc} \begin{array}{cc} P & I \end{array} & \\ \left[\begin{array}{cc} & \\ & \end{array}\right] & \begin{array}{c} P \\ I \end{array} \end{array} \text{ Next time}$$



b Eighty people are seen buying popcorn at the movies. How many of these are expected to buy popcorn next time they go to the movies?

c Sixty people are seen buying an ice cream at the movies. How many of these are expected to buy popcorn next time they go to the movies?

d On another occasion, 120 people are seen buying popcorn and 40 are seen buying an ice cream. How many of these are expected to buy an ice cream next time they attend the movies?

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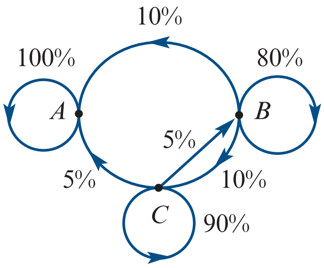
2 On Windy Island, sea birds are observed nesting at three sites: A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of these sea birds between these sites from year to year.

$$\begin{array}{c} \text{This year} \\ T = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array} \text{ Next year} \end{array}$$



a What percentage of sea birds nesting at site B this year were expected to nest at:

- **i** site A next year?
- **ii** site B next year?
- **iii** site C next year?

b This year, 850 sea birds were observed nesting at site B. How many of these are expected to:

- **i** still nest at site B next year?
- **ii** move to site A to nest next year?

c This year, 1150 sea birds were observed nesting at site A. How many of these birds are expected to nest at:

- **i** site A next year?
- **ii** site B next year?
- **iii** site C next year?

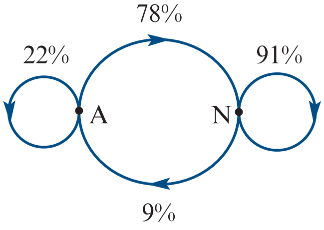
d What does the '1' in column A, row A of the transition matrix indicate?

3 A car insurance company finds that:

- 22% of car drivers involved in an accident this year (A) are also expected to be involved in an accident next year
- 9% of drivers who are *not* involved in an accident this year (N) are expected to be involved in an accident next year.

The transition diagram that can be used to describe this situation is shown below.

$$\begin{array}{c} \text{This year} \\ T = \begin{array}{c} \begin{array}{cc} A & N \end{array} \\ \begin{bmatrix} 0.22 & 0.09 \\ 0.78 & 0.91 \end{bmatrix} \end{array} \begin{array}{c} A \\ N \end{array} \text{ Next year} \end{array}$$



a In 2015, 84 000 drivers insured with the company were *not* involved in an accident.

- **i** How many of these drivers were *not* expected to be involved in an accident in 2016?
- **ii** How many of these drivers were expected to be involved in an accident in 2016?

b In 2015, 25 000 drivers insured with the company were involved in an accident.

- **i** How many of these drivers were expected to be involved in an accident in 2016?
- **ii** How many of these drivers were expected to be involved in an accident in 2017?
- **iii** How many of these drivers were expected to be involved in an accident in 2018?

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4 Fleas can move beteen three locations A, B and C. The way a flea moves after 5 seconds in a location can be exactly described by the transition matrix.

$$\begin{array}{ccl} & \begin{array}{ccc} & \text{Now} & \\ A & B & C \end{array} & \\ T = & \begin{bmatrix} 0.60 & 0.10 & 0.70 \\ 0.20 & 0.80 & 0.10 \\ 0.20 & 0.10 & 0.20 \end{bmatrix} & \begin{array}{ll} A & \\ B & \text{After 5 seconds} \\ C & \end{array} \end{array}$$

The move is not dependent on any previous move.

a If there are 30 fleas at location A at the beginning of the 5-second period, how many fleas would you expect to

i stay at A **ii** go to B **iii** go to C

at the end of the 5-second period?

b If there were 60 fleas at each of the locations how many fleas would you expect to have at

i A **ii** B **iii** C

after one 5-second period?

c Find the product $T\begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$ and comment.

d At the conclusion of the first 5-second period there are 30 fleas at C.

i How many of these go to A in the next 5-second period?

ii How many of these go to B in the next 5-second period?

iii How many of these go to C in the next 5-second period?

e Evaluate the product $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} T \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$ and comment.

Exam 1 style questions

5 There are 120 students in a Year 12 class. Each week every student participates in one of three activities: Sport (S), Outdoor Activities (O) or First Aid (F).

$$\begin{array}{ccl} & \begin{array}{ccc} & \text{This week} & \\ S & O & F \end{array} & \\ T = & \begin{bmatrix} 0.6 & 0.2 & 0.4 \\ 0.1 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.5 \end{bmatrix} & \begin{array}{ll} S & \\ O & \text{Next week} \\ F & \end{array} \end{array}$$

The activities that the children select each week change according to the transition matrix opposite.

From the transition matrix it can be concluded that:

A in the first week of the program, eighty students do Sport, twenty students children do Outdoor activities and twenty Students do First Aid.

B at least 50% of the students do not change their activities from the first week to the second week.

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C in the long term, all of the children will choose the same activity.
D Sport is the most popular activity in the first week
E 40% of the students will do First Aid each week.

6 Warren text messages a friend each week day of this week. His friends are Arthur (A), Belinda (B), Connie (C), Danielle (D) and Eleanor (E). On Monday, Warren will send a text message to Connie.
Based on the transition matrix, the order in which Warren will text message each of his friends for the next four days is:

$$\begin{array}{c} \text{Today} \\ T = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \text{ Tomorrow} \end{array}$$

A Arthur, Belinda, Connie, Danielle
B Danielle, Belinda , Arthur, Connie
C Danielle, Belinda, Arthur, Eleanor
D Eleanor, Arthur, Danielle, Belinda
E Eleanor, Danielle, Belinda, Arthur

11C Transition matrices – using recursion

Learning intentions

- To be able to use a matrix recurrence relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to generate a sequence of state matrices.
- To be able to informally identify the equilibrium state or steady-state matrix in the case of regular state matrices.

We return again to the car rental problem. The car rental firm now plans to buy 90 new cars. Fifty will be based in Bendigo and 40 in Colac.

Given this pattern of rental car returns, the first question the manager would like answered is:

> ‘If we start with 50 cars in Bendigo, and 40 cars in Colac, how many cars will be available for rent at both towns after 1 week, 2 weeks, etc?’

You have met this type of problem earlier when doing financial modelling (Chapter 8). For example, if we invest $1000 at an interest rate of 5% per annum, how much will we have after 1 year, 2 years, 3 years, etc?

We solved this type of problem by using a **recurrence relation** to model the growth in our investment year-by-year. We do the same with the car rental problem, the only difference being that we are now working with matrices.

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Constructing a matrix recurrence relation

A recurrence relation must have a *starting point*.

In this case it is the **initial state matrix**: $S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$

Generating S_1

To find out the number of cars in Bendigo and Colac after 1 week, we use the transition matrix $T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ to generate the next **state matrix** in the sequence, S_1, as follows:

$$\begin{aligned} S_1 &= T\,S_0 \\ &= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 0.8 \times 50 + 0.1 \times 40 \\ 0.2 \times 50 + 0.9 \times 40 \end{bmatrix} \\ \text{or } S_1 &= \begin{bmatrix} 44 \\ 46 \end{bmatrix} \end{aligned}$$

Thus, after 1 week we predict that there will be 44 cars in Bendigo and 46 in Colac.

Generating S_2

Following the same pattern, after 2 weeks;

$$S_2 = T\,S_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 44 \\ 46 \end{bmatrix} = \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$

Thus, after 2 weeks we predict that there will be 39.8 cars in Bendigo and 50.2 in Colac.

Generating S_3

After 3 weeks:

$$S_3 = T\,S_2 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix} = \begin{bmatrix} 36.9 \\ 53.1 \end{bmatrix}$$

Thus, after 3 weeks we predict that there will be 36.9 cars in Bendigo and 53.1 in Colac.

A pattern is now emerging. So far we have seen that:

$$\begin{aligned} S_1 &= TS_0 \\ S_2 &= TS_1 \\ S_3 &= TS_2 \end{aligned}$$

If we continue this pattern we have:

$$\begin{aligned} S_4 &= TS_3 \\ S_5 &= TS_4 \end{aligned}$$

or, more generally, $S_{n+1} = TS_n$.

With this rule as a starting point, we now have a recurrence relation that will enable us to model and analyse the car rental problem on a step-by-step basis.

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Recurrence relation

S_0 = intial value, $S_{n+1} = TS_n$

Let us return to the factory problem in Example 2.

Example 4 Using a recursion relation to calculate state matrices step-by-step

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

At the start, 80 machines are operating and 20 are broken.

Use the recursion relation

$$S_0 = \text{intial value}, \ S_{n+1} = TS_n$$

where

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

to determine the number of operational and broken machines after 1 day and after 3 days.

Explanation

1 Write down a column matrix with S_0 representing the initial operational state of the machines, and the transition matrix.

Solution

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \quad T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

2 Use the rule $S_{n+1} = TS_n$ to determine the operational state of the machines after one day by forming the product $S_1 = TS_0$ and evaluate.

$$S_1 = TS_0 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 69 \\ 31 \end{bmatrix}$$

After 1 day, 69 machines are operational and 31 are broken.

3 To find the operational state of the machines after 3 days, we must first find the operating state of the machines after 2 days (S_2) and use this matrix to find S_3 using $S_3 = TS_2$.

$$S_2 = TS_1 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 69 \\ 31 \end{bmatrix} = \begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix}$$

$$S_3 = TS_2 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix} = \begin{bmatrix} 53.16 \\ 46.84 \end{bmatrix}$$

After 3 days, 53 machines are operating and 47 are broken.

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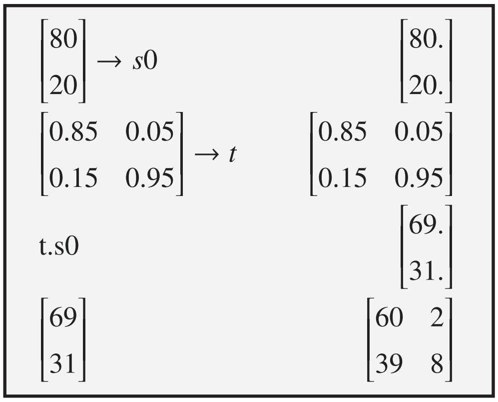
Calculator hint: In practice, generating matrices recursively is performed on your CAS calculator as shown opposite for the calculations performed in Example 11.



$\begin{bmatrix} 80 \\ 20 \end{bmatrix} \to s0$ $\qquad \begin{bmatrix} 80. \\ 20. \end{bmatrix}$

$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \to t$ $\qquad \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$

t.s0 $\qquad \begin{bmatrix} 69. \\ 31. \end{bmatrix}$

$\begin{bmatrix} 69 \\ 31 \end{bmatrix}$ $\qquad \begin{bmatrix} 60 & 2 \\ 39 & 8 \end{bmatrix}$

and so on.

A rule for determining the state matrix of a system after *n* steps

While we can use the recurrence relation:

$$S_0 = \text{intial value},\ S_{n+1} = TS_n$$

to generate state matrices step-by-step, there is a more efficient method when we need to determine the state matrix after a large number of steps.

If we follow through the process step-by-step we have:

$$S_1 = TS_0$$
$$S_2 = TS_1 = T(TS_0) = T^2S_0$$
$$S_3 = TS_2 = T(TS_1) = T^2S_1 = T^2(TS_0) = T^3S_0$$

Continuing the process

$$S_4 = T^4S_0$$
$$S_5 = T^5S_0$$

or more generally, $S_{\text{n}} = T^nS_0$.

We now have a simple rule for finding the value, S_n, of the state matrix after n steps.

A rule for finding the state matrix after *n* steps

If the recurrence rule for determining state matrices is

$$S_0 = \text{initial state matrix},\ S_{n+1} = TS_n,$$

the state matrix after n steps (or transitions) is given by $S_n = T^nS_0$.

Let us return to the factory problem we analysed in Example 2.

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Example 5 Determining the nth state of a system using the rule $S_n = T^n S_0$

The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa.

Initially, 80 machines are operating and 20 are broken, so:

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

Determine the number of operational and broken machines after 10 days.

Explanation

1 Write down the transition matrix, T, and initial state matrix, S_0. Enter the matrices into your calculator. Use T and S.

2 To find out how many machines are in operation and how many are broken after 10 days, write down the rule $S_n = T^n S_0$ and substitute $n = 10$ to give $S_{10} = T^{10} S_0$.

3 Enter the expression $T^{10}S$ into your calculator and evaluate.

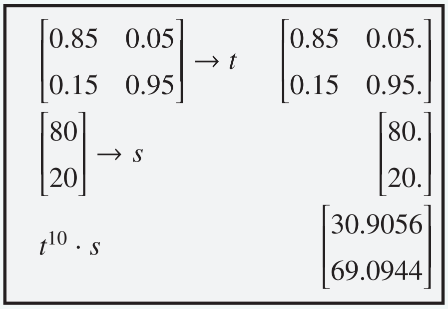
4 Write down your answer in matrix form and then in words.

Solution

$$T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \quad S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

$$S_n = T^n S_0$$

$$\therefore S_{10} = T^{10} S_0$$



$\begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \to t$ $\qquad \begin{bmatrix} 0.85 & 0.05. \\ 0.15 & 0.95. \end{bmatrix}$

$\begin{bmatrix} 80 \\ 20 \end{bmatrix} \to s$ $\qquad \begin{bmatrix} 80. \\ 20. \end{bmatrix}$

$t^{10} \cdot s$ $\qquad \begin{bmatrix} 30.9056 \\ 69.0944 \end{bmatrix}$

$$S_{10} = \begin{bmatrix} 30.9 \\ 69.1 \end{bmatrix}$$

After 10 days, 31 machines will be operational and 69 broken.

Using the inverse matrix of a transition matrix

In the above we have seen how to move from left to right in the sequence of state matrices by multiplying by the transition matrix.

$$S_0, S_1, S_2, \ldots S_n, S_{n+1} \ldots$$

We can move from right to left through the transition states by using the inverse of the transition matrix. In general, the inverse is not a transition matrix.

$$S_{n+1} = TS_n \qquad \text{and} \qquad S_n = T^{-1} S_{n+1}$$

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Example 6

We have a transition matrix $T = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

and we know that the state matrix $S_4 = \begin{bmatrix} 25\,587 \\ 34\,413 \end{bmatrix}$.

Determine S_3 and S_2.

Solution

We know that $S_4 = TS_3$. Hence $S_3 = T^{-1}S_4$. First $T^{-1} = \begin{bmatrix} \frac{7}{3} & -1 \\ -\frac{4}{3} & 2 \end{bmatrix}$.

You should hold this in your calculator and then

$$S_3 = T^{-1}S_4 = \begin{bmatrix} \frac{7}{3} & -1 \\ -\frac{4}{3} & 2 \end{bmatrix}\begin{bmatrix} 25\,587 \\ 34\,413 \end{bmatrix} = \begin{bmatrix} 25\,290 \\ 34\,710 \end{bmatrix}$$

and

$$S_2 = T^{-1}S_3 = \begin{bmatrix} \frac{7}{3} & -1 \\ -\frac{4}{3} & 2 \end{bmatrix}\begin{bmatrix} 25\,290 \\ 34\,710 \end{bmatrix} = \begin{bmatrix} 24\,300 \\ 35\,700 \end{bmatrix}$$

Note: To calculate S_2 given S_3 we could have used:

$$S_2 = \left(T^{-1}\right)^2 S_4$$

The steady-state solution

A second question a manager might like answered about the car rental is as follows.

> 'Will the number of rental cars available from each location vary from week to week or will they settle down to some fixed value?'

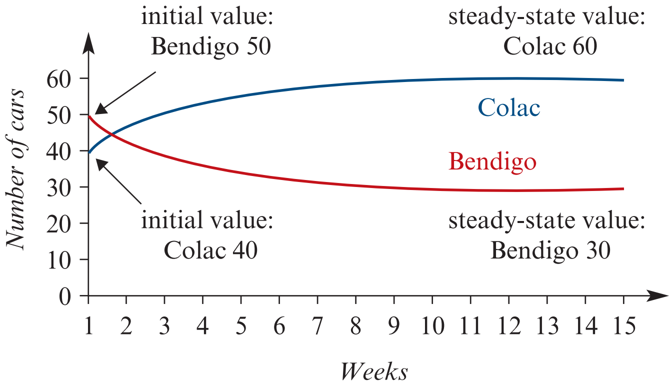
To investigate this question, we start by listing the state matrices from week 0 to week 15.

Week	0	1	2	3	4–11	12	13	14	15
State matrix	$\begin{bmatrix} 50 \\ 40 \end{bmatrix}$	$\begin{bmatrix} 44 \\ 46 \end{bmatrix}$	$\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$	$\begin{bmatrix} 36.9 \\ 53.1 \end{bmatrix}$	…	$\begin{bmatrix} 30.3 \\ 59.7 \end{bmatrix}$	$\begin{bmatrix} 30.2 \\ 59.8 \end{bmatrix}$	$\begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$	$\begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$

What you should notice is that, as the weeks go by, the number of cars at each of the locations starts to settle down. We call this the *steady- or equilibrium- state solution*.

For the rental car problem, the *steady-state solution* is 30.1 (in practice, 30) cars at the Bendigo branch and 59.9 (in practice, 60) cars at the Colac branch, which means the numbers of cars at each location will *not* change from then on.

This can be seen more clearly in the graph below (the points have been joined to guide the eye).



In summary, even though the number of cars returned to each location varied from day to day, the numbers at each location eventually settled down to an equilibrium or steady-state solution. In the steady state, the number of cars at each location remained the same.

Important

1. In the *steady state*, cars are still moving between Bendigo and Colac, but the number of cars rented in Bendigo and returned to Colac is balanced by the number of cars rented in Colac and returned to Bendigo. Because of this balance, the steady state is also called the *equilibrium state*.
2. For a system to have a steady state, the transition matrix must be *regular* and the columns must add up to 1. A *regular matrix* is one whose powers never contain any zero elements. In practical terms, this means that every state represented in the transition matrix is accessible, either directly or indirectly from every other state.

A strategy for estimating the steady-state solution

In the car rental problem we found that, even though the number of cars returned to each location initially varied from day to day, it eventually settled down so the number of cars at each location remained the same.

Although we arrived at this conclusion by repeated calculations, we can arrive at the solution much faster by using the rule $S_n = T^n S_0$ to find the nth state.

Estimating the steady state solution

If S_0 is the initial state matrix, then the **steady-state matrix**, S, is given by

$$S = T^n S_0$$

as n tends to infinity (∞).

Note: While in practice we cannot evaluate T^n for $n = \infty$, we find that, depending on the circumstances, large values of n can often give a very close approximation to the steady-state solution.

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Example 7 Estimating the steady-state solution

For the car rental problem:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Estimate the steady-state solution by calculating S_n for $n = 10, 15, 17$ and 18.

Explanation

1 Write down the transition matrix T and initial state matrix S_0. Enter the matrices into your calculator. Use T and S.

2 Use the rule $S_n = T^n S_0$ to write down the expression for the nth state for $n = 10$.

3 Enter the expression $T^{10}S$ into your calculator and evaluate.

4 Repeat the process for $n = 15, 17$ and 18.

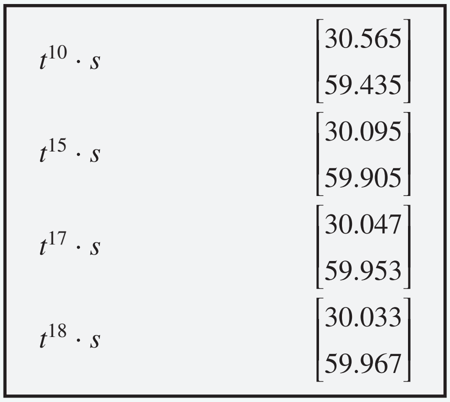
5 Write down your answer in matrix form and then in words. This result agrees with the graphical result arrived at earlier.

Solution

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}, \; S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

$$S_n = T^n S_0$$

$$\therefore S_{10} = T^{10} S_0 = \begin{bmatrix} 30.6 \\ 59.4 \end{bmatrix}$$



$$t^{10} \cdot s \quad \begin{bmatrix} 30.565 \\ 59.435 \end{bmatrix}$$

$$t^{15} \cdot s \quad \begin{bmatrix} 30.095 \\ 59.905 \end{bmatrix}$$

$$t^{17} \cdot s \quad \begin{bmatrix} 30.047 \\ 59.953 \end{bmatrix}$$

$$t^{18} \cdot s \quad \begin{bmatrix} 30.033 \\ 59.967 \end{bmatrix}$$

$$S_{15} = \begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}, \quad S_{17} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}, \quad S_{18} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}$$

The estimated steady-state solution is 30 cars based in Bendigo and 60 cars based in Colac.

Note: To establish a steady state to a given degree of accuracy, in this case one decimal place, at least two successive state matrices must agree to this degree of accuracy.

Skill-sheet

Exercise 11C

Calculating state matrices step-by-step and by rule

Example 4

1 For the initial state matrix $S_0 = \begin{bmatrix}200\\400\end{bmatrix}$ and the transition matrix $T = \begin{bmatrix}0.7 & 0.6\\0.3 & 0.4\end{bmatrix}$: use the recursion relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to determine:

a S_1 **b** S_2 **c** S_3

Example 5

2 For the initial state matrix $S_0 = \begin{bmatrix}200\\400\end{bmatrix}$ and the transition matrix $T = \begin{bmatrix}0.7 & 0.6\\0.3 & 0.4\end{bmatrix}$: use the recursion relation: S_0 = initial state matrix, $S_{n+1} = T^nS_0$, to determine:

a S_5 **b** S_7 **c** S_{12}

Example 6

3 We have a transition matrix $T = \begin{bmatrix}0.65 & 0.4\\0.35 & 0.6\end{bmatrix}$

and we know that the state matrix $S_5 = \begin{bmatrix}5461\\4779\end{bmatrix}$.

Determine S_4 and S_3.

Example 7

4 For the initial state matrix $S_0 = \begin{bmatrix}100\\200\end{bmatrix}$ and the transition matrix $T = \begin{bmatrix}0.9 & 0.2\\0.1 & 0.8\end{bmatrix}$:

a use the recursion relation: S_0 = initial state matrix, $S_{n+1} = TS_n$, to determine:

i S_1 **ii** S_2 **iii** S_3

b determine the value of T^5

c use the rule $S_n = T^nS_0$ to determine:

i S_2 **ii** S_3 **iii** S_7

d by calculating $S_n = T^nS_0$ for $n = 10, 15, 21$ and 22, show that the steady-state matrix is close to $\begin{bmatrix}200\\100\end{bmatrix}$.

5 For the initial state matrix $S_0 = \begin{bmatrix}100\\200\\300\end{bmatrix}$ and the transition matrix $T = \begin{bmatrix}0.7 & 0.4 & 0.1\\0.2 & 0.1 & 0.3\\0.1 & 0.5 & 0.6\end{bmatrix}$:

a use a recurrence relation to determine:

i S_1 **ii** S_2 **iii** S_3

b use the relationship $S_n = T^nS_0$ to determine:

i S_2 **ii** S_3 **iii** S_7

c by calculating $S_n = T^nS_0$ for $n = 10, 15, 17$ and 18, show that the steady-state matrix is close to $\begin{bmatrix}247.1\\129.4\\223.5\end{bmatrix}$.

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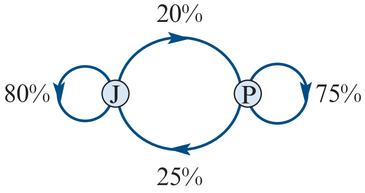
Practical applications of transition matrices

6 Two fast-food outlets, Jill's and Pete's, are located in a small town.

In a given week:

- 80% of people who go to Jill's return the next week
- 20% of people who go to Jill's go to Pete's the next week
- 25% of people who go to Pete's go to Jill's the next week
- 75% of people who go to Pete's return the next week.



a Construct a transition matrix to describe this situation. Call the matrix T.

b Initially, 400 people eat at Jill's and 400 eat at Pete's. Write down a column matrix S_0 that describes this situation.

c How many of these people do we expect to go to Jill's the next week? How many to Pete's?

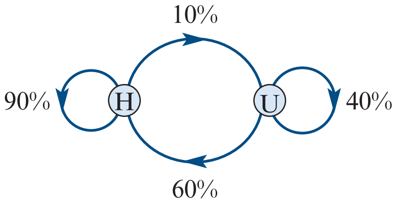
d How many do we expect to go to Jill's after 5 weeks? How many to Pete's?

e In the long term, how many do we expect to eat at Jill's each week? How many of these people do we expect to eat at Pete's?

7 Imagine that we live in a world in which people are either 'happy' or 'unhappy', but the way people feel can change from day to day.

In this world:

- 90% of people who are happy today will be happy tomorrow
- 10% of people who are happy today will be unhappy tomorrow
- 40% of people who are unhappy today will be unhappy tomorrow
- 60% of people who are unhappy today will be happy tomorrow.



a Construct a transition matrix to describe this situation. Call the matrix T.

b On a given day, out of 2000 people, 1500 are happy and 500 are unhappy. Write down a column matrix, S_0, that describes this situation.

c The next day, how many of these people do we expect to be 'happy' and how many 'unhappy'?

d After 4 days, how many of these people do we expect to be 'happy' and how many 'unhappy'?

e In the long term, how many people do we expect to be 'happy' and how many 'unhappy'?

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8 In another model of this world, people can be 'happy', 'neither happy nor sad', or 'sad', but the way people feel can change from day to day.

The transition matrix opposite shows how people's feelings may vary from day to day in this world, and the proportions of people involved.

$$T = \begin{array}{c} \\ H \\ N \\ S \end{array}\begin{array}{c} \begin{array}{ccc} H & N & S \end{array} \\ \begin{bmatrix} 0.80 & 0.40 & 0.35 \\ 0.15 & 0.30 & 0.40 \\ 0.05 & 0.30 & 0.25 \end{bmatrix} \end{array}$$

In the transition matrix, the columns define the situation today and the rows define the situation tomorrow.

a On a given day, out of 2000 people, 1200 are 'happy', 600 are 'neither happy nor sad' and 200 are 'sad'. Write down a matrix, S_0, that describes this situation.

b The next day, how many people do we expect to be happy?

c After 5 days, how many people do we expect to be happy?

d In the long term, how many of the 2000 people do we expect to be happy?

Exam 1 style questions

9 Students at a boarding school have a choices of two breakfast cereals, Crispies (C) and Krunchies (K). The change in the percentage of students who have each cereal on consecutive days is descibed by the transition matrix T shown below.

$$\begin{array}{c} \text{Today} \\ T = \begin{array}{c|cc} & C & K \\ \hline C & 0.42 & 0.56 \\ K & 0.58 & 0.44 \end{array} \quad \text{Tomorrow} \end{array}$$

On Monday 25% of the students ate Crispies. What percentage of the students ate Krunchies on Tuesday?

A 47.5% **B** 48% **C** 50.25% **D** 52.5% **E** 62%

10 A factory employs the same number of workers each day. The workers are allocated to work with either machine A or machine B. The workers may be allocated to work on a different machine from day to day, as shown in the transition matrix below.

$$T = \begin{array}{c|cc} & A & B \\ \hline A & 0.32 & 0.16 \\ B & 0.68 & 0.84 \end{array}$$

Machine A has 72 workers each day on it. Each day, the number of workers machine B will be

A 24 **B** 36 **C** 72 **D** 288 **E** 306

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11 Consider the matrix recurrence relation below.

$$S_0 = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, S_{n+1} = TS_n \text{ where } T = \begin{bmatrix} x & 0.4 & y \\ 0.6 & z & 0.4 \\ 0.1 & 0.2 & w \end{bmatrix}$$

Matrix T is a regular transition matrix. Given that $S_1 = \begin{bmatrix} 14 \\ 26 \\ 20 \end{bmatrix}$ which of the following is true?

A $x = 0.3, y = 0.1, z = 0.4, w = 0.5$

B $x = 0.3, y = 0.3, z = 0.4, w = 0.7$

C $x = 0.2, y = 0.7, z = 0.3, w = 0.3$

D $x = 0.2, y = 0.8, z = 0.3, w = 0.2$

E $x = 0.3, y = 0.6, z = 0.4, w = 0.4$

11D Transition matrices – using the rule $S_{n+1} = TS_n + B$

Learning intentions

- To be able to use a matrix recurrence relation $S_0 =$ initial state matrix, $S_{n+1} = TS_n + B$ to model systems that include external additions or reductions at each step of the process.

To date, we have only considered matrix recurrence models of the form

$$S_0 = \text{ initial state matrix, } S_{n+1} = TS_n$$

This recurrence model can be used to model situations where the total number of objects in the system, like cars, machines, people or birds, remains unchanged. For example, in the car rental problem 90 cars are available for rental. But what happens if management wants to increase the total number of cars available for rent by adding, say, an extra car at each location each week?

To allow for this situation we need to use the matrix recurrence relation:

$$S_0 = \text{initial state matrix, } S_{n+1} = TS_n + B$$

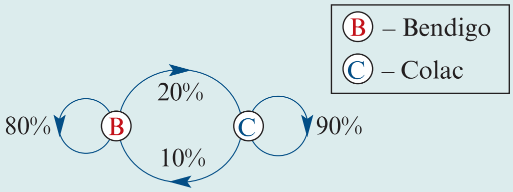
where B is a column matrix.

The next example applies this model to the rental car problem.

Example 8 Determining the nth state of a system using the rule $S_n = T^nS_0 + B$

A rental starts with 90 cars, 50 located at Bendigo and 40 located at Colac.

Cars are usually rented and returned in the same town. However, a small percentage of cars rented in Bendigo are returned in Colac and vice versa. The transition diagram opposite gives these percentages.



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To increase the number of cars, two extra cars are added to the rental fleet at each location each week. The recurrence relation that can be used to model this situation is:

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix},\ S_{n+1} = TS_n + B \quad \text{where} \quad T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Determine the number of cars at Bendigo and Colac after:

a 1 week **b** 2 weeks.

Explanation	**Solution**
a Use the rule $S_1 = TS_o + B$ to determine the state matrix after 1 week and write your conclusion.	$S_1 = TS_0 + B$ $= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 50 \\ 40 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $= \begin{bmatrix} 44 \\ 46 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 46 \\ 48 \end{bmatrix}$ Thus, we predict that there will be 46 cars in Bendigo and 48 cars in Colac.
b Use the rule $S_2 = TS_1 + B$ to determine the state matrix after 2 weeks and write your conclusion.	$S_2 = TS_1 + B$ $= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} 46 \\ 48 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $= \begin{bmatrix} 41.6 \\ 52.4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 43.6 \\ 54.4 \end{bmatrix}$ Thus, we predict that there will be 43.6 cars in Bendigo and 54.4 cars in Colac.

Unfortunately, the recurrence rule $S_{n+1} = TS_n + B$ does not lead to a simple rule for the state matrix after n steps, so we need to work our way through this sort of problem step-by-step.

Using the inverse matrix of a transition matrix

In the above we have seen how to move from left to right in the sequence of state matrices by applying $S_{n+1} = TS_n + B$

$$S_0, S_1, S_2, \ldots S_n, S_{n+1} \ldots$$

We can move from right to left through the transition states by the observation

$$S_{n+1} = TS_n + B \qquad \text{and} \qquad S_n = T^{-1}(S_{n+1} - B)$$

Question 4 in the Exercise can be completed in this way.

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Skill-sheet

Exercise 11D

Using a recurrence rule to calculate state matrices

Example 7

1 For the transition matrix $T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ and the state matrix $S_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$:

a use the recurrence rule $S_{n+1} = TS_n$ to determine:

i S_1 **ii** S_3

b use the recurrence rule $S_{n+1} = TS_n + R$, where $R = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, to determine:

i S_1 **ii** S_2

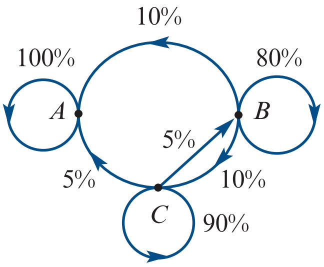
c use the recurrence rule $S_{n+1} = TS_n - B$, where $B = \begin{bmatrix} -20 \\ 20 \end{bmatrix}$, to determine:

i S_1 **ii** S_2

Practical application

2 On Windy Island, sea birds are observed nesting at three sites: A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of sea birds between these sites from year to year.

$$T = \begin{array}{cl} \text{This year} & \\ \begin{array}{ccc} A & B & C \end{array} & \\ \left[\begin{array}{ccc} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{array}\right] & \begin{array}{c} A \\ B \\ C \end{array} \text{ Next year} \end{array}$$



Initially, 10 000 sea birds were observed nesting at each site, so $S_0 = \begin{bmatrix} 10\,000 \\ 10\,000 \\ 10\,000 \end{bmatrix}$.

a Use the recurrence rule $S_{n+1} = TS_n$ to:

i determine S_1, the state matrix after 1 year

ii predict the number of sea birds nesting at site B after 2 years.

b Without calculation, write down the number of sea birds predicted to nest at each of the three sites in the long term. Explain why this can be done without calculation.

c To help solve the problem of having all the birds eventually nesting at site A, the ranger suggests that 2000 sea birds could be removed from site A each year and relocated in equal numbers to sites B and C.

The state matrix, S_2, is now given by

$$S_2 = TS_1 + N$$

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where $S_1 = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \end{bmatrix}$, $T = \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix}$ and $N = \begin{bmatrix} -2000 \\ 1000 \\ 1000 \end{bmatrix}$.

Evaluate:

i S_2 **ii** S_3 (assuming that $S_3 = TS_2 + N$) **iii** S_4 (assuming that $S_4 = TS_3 + N$).

Exam 1 style questions

3 The matrix S_{n+1} is determined from the matrix S_n using the recurrence relation $S_{n+1} = T \times S_n - C$, where

$$T = \begin{bmatrix} 0.6 & 0.75 & 0.1 \\ 0.2 & 0.2 & 0.1 \\ 0.2 & 0.05 & 0.8 \end{bmatrix}, S_0 = \begin{bmatrix} 2000 \\ 1000 \\ 1000 \end{bmatrix}, S_1 = \begin{bmatrix} 1975 \\ 650 \\ 1125 \end{bmatrix}.$$

and C is a column matrix. Matrix $S_2 =$

A $\begin{bmatrix} 1860 \\ 687.5 \\ 1452.5 \end{bmatrix}$ **B** $\begin{bmatrix} 1700 \\ 560.5 \\ 1022.5 \end{bmatrix}$ **C** $\begin{bmatrix} 1710 \\ 587.5 \\ 1202.5 \end{bmatrix}$ **D** $\begin{bmatrix} 1600 \\ 1725.5 \\ 1200.5 \end{bmatrix}$ **E** $\begin{bmatrix} 1650 \\ 550.5 \\ 1032.5 \end{bmatrix}$

4 Supporters of a football team attend home games. There are 3 areas, bays A, B and C, where they sit. There is considerable moving of position from game to game and the numbers attending the home games gradually decline as the year progresses. Let X_n be the state matrix that shows the number of supporters in each bay n weeks into the the season. The number of supporters in each location can be determined by the matrix recurrence relation

$$X_{n+1} = TX_n - D$$

where

$$T = \begin{array}{c} \text{This game} \\ \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.3 & 0.7 & 0.2 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix} \ \text{Next game} \qquad \text{and } D = \begin{bmatrix} 70 \\ 70 \\ 70 \end{bmatrix}$$

If $X_3 = \begin{bmatrix} 9830 \\ 11\,130 \\ 7830 \end{bmatrix}$ then $X_2 =$

A $\begin{bmatrix} 12370 \\ 9510 \\ 4650 \end{bmatrix}$ **B** $\begin{bmatrix} 4000 \\ 10\,000 \\ 15\,000 \end{bmatrix}$ **C** $\begin{bmatrix} 12\,000 \\ 11\,000 \\ 5000 \end{bmatrix}$ **D** $\begin{bmatrix} 7500 \\ 8600 \\ 12\,000 \end{bmatrix}$ **E** $\begin{bmatrix} 79300 \\ 245232 \\ 231011 \end{bmatrix}$

11E Leslie matrices

Learning intentions

- To use and interpret Leslie matrices to analyse changes in population over time.

Leslie matrices are used to construct discrete models of population growth. In particular, they are used to model changes in the sizes of different age groups within a population.

The general setting

Leslie matrices were developed by Patrick Holt Leslie (1900–1972) while he was working in the Bureau of Animal Population at the University of Oxford. They are used by biologists and ecologists to model changes over time in various animal populations.

Age groups First the population is divided into age groups. The time period for each age group is the same length. Together they cover the life span of the population. For example, in a study of a human population, we could use a time period of 10 years and consider eleven age groups as follows:

Age group(i)	1	2	3	4	⋯	10	11
Age range (years)	0–10	10–20	20–30	30–40	⋯	90–100	100–110

Note: Only the females of the species are counted in the population, as they are the ones who give birth to the new members of the population.

A **Leslie matrix** is a transition matrix that can be used to describe the way population changes over time. It takes into account two factors for the females in each age group: the *birth rate*, b_i, and *survival rate*, s_i, where i is the number of the age group.

Birth rates We ignore migration, and so the population growth is entirely due to new female births. The birth rate, b_i, for age group i is the average number of female offspring from a mother in age group i during one time period. For example, average birth rate of women in age group 4 $(20 - 30$ years) might be 1.7 female children for the 10 year period.

Survival rates The survival rate, s_i, for age group i is the proportion of the population in age group i that progress to age group $i + 1$. Note that $0 \leq s_i \leq 1$.

For example, the survival rate for age group 2 might be 0.95, that is 95% of females in this $10 - 20$ year age group would survive to progress to age group 3, $20 - 30$ years.

Note: The survival rate of the last age group $(100 - 110)$ is taken to be 0.

A simple example

We start with a simple example where the life span of the species is 9 years. We will divide the population into three age groups. This means we use a time period of 3 years.

Age group(i)	1	2	3
Age range (years)	0–3	3–6	6–9

A Leslie matrix for three age groups is a 3 × 3 matrix of the form

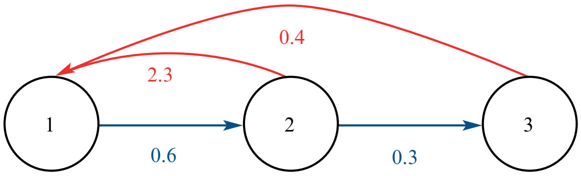
$$L = \begin{bmatrix} b_1 & b_2 & b_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}$$

Suppose that the survival rates are $s_1 = 0.6$ and $s_2 = 0.3$, and that the birth rates are $b_1 = 0$, $b_2 = 2.3$ and $b_3 = 0.4$. Then the Leslie matrix is

$$\begin{array}{c} \text{From age group } i \\ \begin{array}{ccccc} & 1 & 2 & 3 & \\ L = & \left[\begin{array}{c} 0 \\ 0.6 \\ 0 \end{array}\right. & \begin{array}{c} 2.3 \\ 0 \\ 0.3 \end{array} & \left.\begin{array}{c} 0.4 \\ 0 \\ 0 \end{array}\right] & \begin{array}{l} 1 \text{ birth rate} \\ 2 \text{ survival rate} \\ 3 \text{ survival rate} \end{array} \end{array} \end{array} \quad \text{To age group } i+1$$

Life cycle transition diagram

The above Leslie matrix can be represented by a diagram which we will refer to as a **Life cycle transition diagram**.



The population state matrix

The **population state matrix** is a column matrix that lists the number in each age group at a given time.

The **initial population state matrix** is denoted by S_0. Suppose that for our example, initially the population has 400 females in each age group. We represent the initial population state matrix S_0, as a 3 × 1 column matrix as shown below.

$$\begin{array}{ccc} & & \text{Age group} \\ S_0 = & \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

We can now use the Leslie matrix, L, in combination with the initial state matrix S_0 to generate the state matrix after one time period, S_1, to find the size of each age group after one time period (3 years) as follows:

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$$S_1 = LS_0 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix}$$

Thus after one time period, there are 1080 females in age group 1, 240 in age group 2 and 120 in age group 3 and the total population size has increased from 1200 (= 400 + 400 + 400) to 1440 (= 1080 + 240 + 120). Similarly, to find the number in each age group after two time periods we calculate S_2 from S_1 as follows:

$$S_2 = LS_1 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix} = \begin{bmatrix} 600 \\ 648 \\ 72 \end{bmatrix}$$

Thus, after two time periods, there are 600 females in age group 1648 in age group 2 and 72 in age group 3 and the over-all population size has decreased to 1320.

Finding the population matrix S_n after n time periods.

To speed up the process we can make use of the explicit formula for the state matrix S_n after n time periods. Notice that there is a pattern when calculating the population state matrices:

$$\begin{aligned} S_1 &= LS_0 \\ S_2 &= LS_1 = L^2S_0 \\ S_3 &= LS_2 = L^3S_0 \\ &\vdots \\ S_{n+1} &= LS_n = L^nS_0 \end{aligned}$$

In general, we can find the population matrix S_n using the rule

$$S_n = L^nS_0$$

Using this rule, to find S_3, we have

$$S_3 = L^3S_0 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}^3 \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1519.2 \\ 360 \\ 194.4 \end{bmatrix}$$

Continuing in this way, we can see the change over time in the total population and in the distribution of the age groups.

Change in the population over time

Time period	0	1	2	3	4	5
Age 0–3 years	400	1080	600	1519.2	905.76	2139.70
Age 3–6 years	400	240	648	360.0	911.52	543.46
Age 6–9 years	400	120	72	194.4	108.00	273.46
Total	1200	1440	1320	2073.6	1925.28	2956.61

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Leslie matrices

An $m \times m$ **Leslie matrix** has the form

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

where:

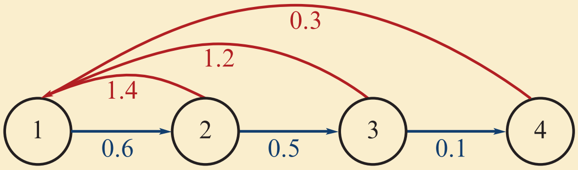
- m is the number of age groups being considered
- s_i, the survival rate, is the proportion of the population in age group i that progress to age group $i + 1$
- b_i, the birth rate, is the average number of female offspring from a mother in age group i during one time period.

Leslie matrix and its interpretation

From age group

$$L = \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 0 & 1.4 & 1.2 & 0.3 & 1 \\ 0.6 & 0 & 0 & 0 & 2 \\ 0 & 0.5 & 0 & 0 & 3 \\ 0 & 0 & 0.1 & 0 & 4 \end{array} \quad \text{To age group}$$

This is a Leslie matrix with 4 age groups. The corresponding life-cycle transition diagram is shown here.



Recursive rules

The population matrix S_n is an $m \times 1$ matrix representing the size of each age group after n time periods. This is calculated using a recursive formula

S_0 is the initial state matrix, $S_{n+1} = LS_n$

or the explicit rule

$S_n = L^n S_0$

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Example 9 Determining state matrices and life cycle diagrams

Use the Leslie matrix and initial state matrix below to answer the following questions.

From age group

$$L = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0 & 1.8 & 2.6 & 0.1 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \text{To age group} \qquad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

a Write down

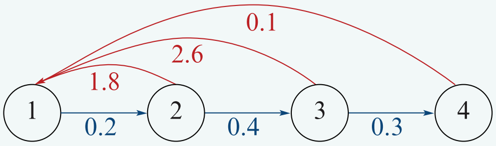
i the birth rate for age group 2 **ii** the survival rate for age group 3

b Complete a life cycle diagram for this Leslie matrix.

c Evaluate the following population state matrices.

S_1, S_5 and S_{20}

d Given that $S_{16} = \begin{bmatrix} 9.53 \\ 2.42 \\ 1.22 \\ 0.16 \end{bmatrix}$, determine S_{17}

Explanation	**Solution**
a i The birth rate for age group 2 is given in the matrix position, row 1, column 2.	Birth rate for age group 2 = 1.8
ii The survival rate for age group 3 is given in the matrix position, row 4, column 3.	Survival rate for age group 3 = 0.3
b **Survival rates** $s_1 = 0.2, s_2 = 0.4, s_3 = 0.3$ **Birth rates** $b_2 = 1.8, b_3 = 2.6, b_4 = 0.1$	
c $S_1 = LS_0$ $S_5 = L^5 S_0$ $S_{20} = L^{20} S_0$	Using a calculator. $S_1 = \begin{bmatrix} 0 \\ 200 \\ 0 \\ 0 \end{bmatrix}, S_5 = \begin{bmatrix} 149.76 \\ 26.4 \\ 16.64 \\ 8.64 \end{bmatrix}, S_{20} = \begin{bmatrix} 3.84 \\ 0.97 \\ 0.49 \\ 0.19 \end{bmatrix}$

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d $S_{17} = LS_{16}$

(Further investigation would reveal that the population continues to decrease over time.)

$$S_{17} = \begin{bmatrix} 7.54 \\ 1.91 \\ 0.97 \\ 0.37 \end{bmatrix}$$

Example 10 Entering information in a Leslie matrix and state matrix

Information about a population of female goats is given in the following table.

Age group (years)	0 – 1	1 – 2	2 – 3	3 – 4	4 – 5
Initial population	10	25	40	20	15
Birth rates	0	0.2	0.9	0	0
Survival rates	0.6	0.7	0.5	0.2	0

a Write down the initial population state matrix, S_0.

b Using the information in the table above, write down a Leslie matrix to describe the change in population of female goats.

c Construct a life cycle transition diagram for this Leslie matrix.

d Determine the number of 3 – 4 year old female goats in the population after 3 years. Round your answer to the nearest whole number.

Explanation

a Enter the initial population numbers into a 5 × 1 matrix.

Solution

$$S_0 = \begin{bmatrix} 10 \\ 25 \\ 40 \\ 20 \\ 15 \end{bmatrix}$$

b Enter the birthrates and survival rates into a 5 × 5 matrix.

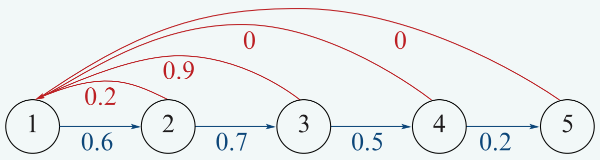
$$L = \begin{bmatrix} 0 & 0.2 & 0.9 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

c **Survival rates**

$s_1 = 0.6, s_2 = 0.7, s_3 = 0.5,$
$s_4 = 0.2$

Birth rates

$b_2 = 0, 2, b_3 = 0.9, b_4 = 0,$
$b_5 = 0$



d $S_3 = L^3 S_0$

$$\begin{bmatrix} 0 & 0.2 & 0.9 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}^3 \begin{bmatrix} 10 \\ 25 \\ 40 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 8.7 \\ 10.17 \\ 17.22 \\ 2.1 \\ 1.75 \end{bmatrix}$$

There are two goats in the 3-4 year old age group in this population.

Long term (limiting) behaviour of population numbers

The following examples demonstrate numerical techniques for modelling the use of Leslie matrices.

Example 11 Limiting behaviour for Leslie matrices

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 4 & 4 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

a Find

i S_5 **ii** S_{10} **iii** S_{50}

Premultiply each of these state matrices by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ to calculate the total populations at each of these stages and comment.

b Determine S_{25} and S_{26}. Divide each age group population for S_{26} by the corresponding age group population for S_{25} and show that $S_{26} \approx 1.1915 S_{25}$ and comment.

Solution

a

i $S_5 = \begin{bmatrix} 1000 \\ 250 \\ 62.5 \end{bmatrix}$ **ii** $S_{10} = \begin{bmatrix} 2500 \\ 531.25 \\ 218.75 \end{bmatrix}$ **iii** $S_{50} = \begin{bmatrix} 2\,777\,063 \\ 582\,688.05 \\ 244521.18 \end{bmatrix}$

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Finding the total populations according to this model

i $\begin{bmatrix}1 & 1 & 1\end{bmatrix} S_5 = [1312.5]$. The population is approximately 1312 after 5 years.

ii $\begin{bmatrix}1 & 1 & 1\end{bmatrix} S_{10} = [3250]$. The population is approximately 3250 after 10 years.

iii $\begin{bmatrix}1 & 1 & 1\end{bmatrix} S_{50} = [3\,604\,272.2]$. The population is approximately 3 604 272 after 50 years.

A numerical investigation reveals that the population increases without bound.

b We calculate S_{25} and S_{26}:

$$S_{25} = L^{25} S_0 = \begin{bmatrix} 34\,781.25 \\ 7297.85 \\ 3062.50 \end{bmatrix}, \quad S_{26} = L^{26} S_0 = \begin{bmatrix} 41\,441.41 \\ 8695.31 \\ 3648.93 \end{bmatrix}$$

Then we can find the rate of increase in each age group during the 26th time period:

$$\frac{41\,441.41}{34\,781.25} \approx \frac{8695.31}{7297.85} \approx \frac{3648.93}{3062.50} \approx 1.1915$$

This suggests that the age-group proportions have stabilised after the 25 time periods.

The long-term growth rate is approximately 1.19. (That is, after a certain stage, the population is increasing by 19% each time period.)

Note: Try different entries in S_0 to see if you get the same behaviour. The long-term growth rate is largely dependent on the Leslie matrix L.

Limiting behaviour of Leslie matrices

Often we will find that, after a long enough time, the proportion of the population in each age group does not change from one time period to the next. This happens if we can find a real number k such that $LS_{n+1} = kS_n$ for some sufficiently large n. This does not happen with every Leslie matrix as we see in Example 13.

Example 12 A Leslie matrix and state matrix with constant rate of increase

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1600 \\ 800 \\ 200 \end{bmatrix}.$$

a Determine

i S_{10} **ii** S_{14} **iii** S_{15}

b The rate of increase of the population is a constant and each of the age group populations increase in the same way. Find this rate by comparing S_{14} and S_{15}.

c Confirm the ratio of the age group populations stays constant for S_0, S_1 and S_{10} at 8 : 4 : 1

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Solution

a

i $S_{10} = \begin{bmatrix} 9906.78 \\ 4953.39 \\ 1238.35 \end{bmatrix}$ **ii** $S_{14} = \begin{bmatrix} 20542.70 \\ 10271.35 \\ 2567.84 \end{bmatrix}$ **iii** $S_{15} = \begin{bmatrix} 24651.24 \\ 12325.62 \\ 3081.4043 \end{bmatrix}$

By comparing S_{14} and S_{15},

$$\frac{24651.24}{20542.70} = \frac{12325.62}{10271.35} = \frac{3081.4043}{2567.84} \approx 1.2$$

we find that the growth rate is 1.2.

b $8 : 4 : 1 = 1600 : 800 : 200 \approx 9906.78 : 4953.39 : 1238.35$

Example 13 Periodic, increasing and decreasing rates of change

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 0 & b_3 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

Investigate the long-term behaviour of the population if:

a $b_3 = 8$ **b** $b_3 = 4$ **c** $b_3 = 10$

Solution

a Let $b_3 = 8$. Use your calculator to store the matrices L and S_0. Then compute:

$$S_1 = LS_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad S_2 = L^2S_0 = \begin{bmatrix} 0 \\ 0 \\ 125 \end{bmatrix}, \quad S_3 = L^3S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

The population will continue to cycle through these three states; this is because $L^3 = I$.

b Let $b_3 = 4$. Then a numerical investigation suggests that the population decreases over the long term:

$$S_1 = LS_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad S_5 = L^5S_0 = \begin{bmatrix} 0 \\ 0 \\ 62.5 \end{bmatrix}, \quad S_{50} = L^{50}S_0 = \begin{bmatrix} 0 \\ 0 \\ 0.0019 \end{bmatrix}$$

c Let $b_3 = 10$. Then a numerical investigation suggests that the population increases over the long term:

$$S_1 = LS_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad S_5 = L^5S_0 = \begin{bmatrix} 0 \\ 0 \\ 156.25 \end{bmatrix}, \quad S_{50} = L^{50}S_0 = \begin{bmatrix} 0 \\ 0 \\ 4440.89 \end{bmatrix}$$

Note: A population can increase, decrease, become constant or oscillate.

Skill-sheet

Exercise 11E

Determining state matrices and life cycle diagrams

Example 8

1 Use the Leslie matrix and initial state matrix below to answer the following questions.

From age group

$$L = \begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \left[\begin{array}{cccc} 0 & 1.9 & 2.1 & 1.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{array}\right. & & & \left.\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \text{ To age group} \qquad S_0 = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

a Write down

i The birth rate for age group 2 **ii** The survival rate for age group 3

b Complete the life cycle diagram for this Leslie matrix.

c Evaluate the following population state matrices.

i S_1 **ii** S_3 **iii** S_{20}

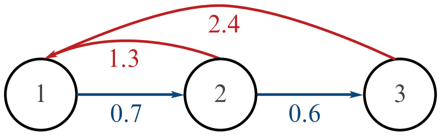
d Given that $S_7 = \begin{bmatrix} 2613 \\ 1200 \\ 485 \\ 168 \end{bmatrix}$ determine S_8. Give your values correct to the nearest whole number.

2 Complete the life cycle diagram corresponding to each of the following Leslie matrices:

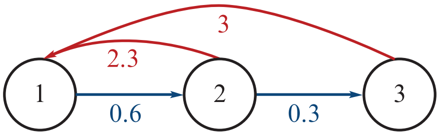
a $\begin{bmatrix} 0 & 2.9 & 3.1 & 2.1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0.42 \\ 0.6 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 0 & 3 & 8 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \end{bmatrix}$

3 Construct the Leslie matrix corresponding to each life cycle diagram.

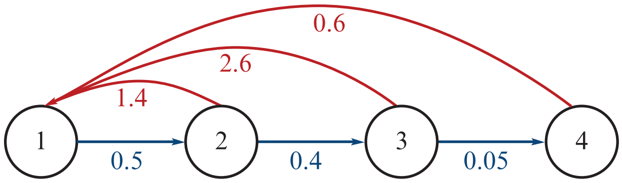
a



b



c



Entering information in a Leslie matrix and state matrix

Example 9

4 Information about a population of female kangaroos in a particular area is given in the following table.

Age group (years)	0 – 4	4 – 8	8 – 12	12 – 16	4 – 5
Initial population	15	20	30	15	10
Birth rates	0	0.2	0.9	1.1	0
Survival rates	0.8	0.9	0.7	0.8	0

a Write down the initial population state matrix, S_0.

b Write down the Leslie matrix.

c Complete the life cycle diagram for this Leslie matrix.

d Determine the population state matrix after

  i one year, (S_1) **ii** after 5 years, (S_5).

e Determine the number of 4 – 8 year old female kangaroos in the population after 5 years.

f State the initial total population.

g Use multiplication of state matrices by the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ to find the total population after

  i one year, **ii** after 5 years **iii** after 10 years.

h It is conjectured that the population is increasing by about 10% per annum. Calculate each of the following and comment

  i 1.1×90 **ii** $1.1^5 \times 90$ **iii** $1.1^{10} \times 90$.

5 Information about a population of female locusts is given in the following table.

Stage	Eggs	Nymphs	Adults
Initial population	0	0	50
Birth rates	0	0	1000
Survival rates	0.02	0.05	0

a Write down the initial population state matrix, S_0.

b Write down the Leslie matrix.

c Construct a time life-cycle transition diagram for this Leslie matrix.

d Determine the population state matrix after

  i one year, (S_1) **ii** after 3 years, (S_3) **iii** after 4 years, (S_4).

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e If the Initial population is now:

Stage	Eggs	Nymphs	Adults
Initial population	50	100	50

find the populations of each after

i one year (S_1) **ii** after 3 years (S_3). **iii** after 4 years (S_4).

Limiting behaviour for Leslie matrices

Example 10

6 Consider the following Leslie matrix L and initial population state matrix S_0:

$$L = \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix}$$

a Find

i S_5 **ii** S_{10} **iii** S_{20}

Premultiply each of these state matrices by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ to calculate the total populations at each of these stages and comment.

b Determine S_{20} and S_{21}. Divide each age group population for S_{21} by the corresponding age group population for S_{20} and show that $S_{21} \approx 1.057S_{20}$ and comment.

Example 11

7 A Leslie matrix that models a certain population of female animals is

$$\mathbf{L} = \begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

where the animals have a maximum life span of 9 years, and the population has been divided into three age groups of 3 years each.

a Assume that each age group initially consists of 400 females. What is the number of females in each age group after:

i 3 years **ii** 6 years **iii** 9 years?

b Now assume that the initial population is 1200 and $S_0 = \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix}$. Find S_n after

i 3 years **ii** 6 years **iii** 9 years.

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c Calculate

i $1.27S_0$ **ii** 1.27^2S_0 **iii** 1.27^3S_0

Compare these answers to the answers of part **b**

Example 12 **8** Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1200 \\ 0 \\ 0 \end{bmatrix}$$

a Find:

i LS_0 **ii** L^2S_0 **iii** L^3S_0

b Comment on these results in terms of the population behaviour. Try using a different initial population matrix S_0.

c Now investigate for each of the following Leslie matrices. Comment on population increase or decrease.

i $L = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$ **ii** $L = \begin{bmatrix} 0 & 0 & 15 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$

9 A Leslie matrix that models a certain population of female insects is

$$\mathbf{L} = \begin{bmatrix} 0 & 3 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

where the insects have a maximum life span of 4 months, and the population has been divided into four age groups of 1 month each.

Assume that each age group initially consists of 400 females. What is the number of females in each age group after:

a 1 month **b** 2 months **c** 3 months?

10 For a certain species of fish, we consider three age groups each of one year in length. These fish reproduce only during their third year and then die. Assume that 20% of fish survive their first year and that 50% of these survivors make it to reproduction age. The initial population consists of 1000 newborns.

a Investigate what happens for each of the following values of b_3:

i $b_3 = 10$ **ii** $b_3 = 15$ **iii** $b_3 = 6$

b For $b_3 = 20$, determine the long-term growth rate and the proportion of fish in each age group.

Exam 1 style questions

11 The Leslie matrix for a certain endangered species is:

$$L = \begin{bmatrix} 0.9 & 2.5 & 0.4 \\ 0.3 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$$

Some of the species were moved into a sanctuary. The initial female population in the sanctuary is given by

$$S_0 = \begin{bmatrix} 130 \\ 40 \\ 20 \end{bmatrix}$$

The best estimate of the total female population after 7 years is

A 1000 **B** 1500 **C** 2000 **D** 2500 **E** 3000

12 The Leslie matrix $L = \begin{bmatrix} 0 & 2 & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$ satisfies the matrix equation

$$L \begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix}$$

The values of b, c and d are

A $b = 2, d = \frac{1}{4}, c = \frac{1}{2}$ **B** $b = 2, d = \frac{1}{2}, c = \frac{1}{4}$ **C** $b = 4, d = \frac{1}{4}, c = \frac{1}{2}$

D $b = 4, d = \frac{1}{2}, c = \frac{1}{4}$ **E** $b = 2, d = \frac{1}{4}, c = \frac{1}{2}$

13 A population of birds is modelled by using the Leslie matrix

$$L = \begin{bmatrix} 0 & 2 & 1.5 \\ 0.44 & 0 & 0 \\ 0 & 0.55 & 0 \end{bmatrix}$$

The growth has reached the point where the rates of growth of the different age groups of the population are constant and the state matrix at this point is $S_k = \begin{bmatrix} 1000 \\ 400 \\ 200 \end{bmatrix}$. The rate of growth per time period is

A 10% **B** 11% **C** 12%

D 13% **E** 14%

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Key ideas and chapter summary

Assign-ment

State matrix Transition matrixes

A **state matrix** S_n is a column matrix whose elements represent the nth state of a dynamic system defined by a recurrence relation of the form: S_0 = initial state, $S_{n+1} = TS_n$. Here T is a square matrix called a **transition matrix**.

Steady-state matrix

The **steady-state matrix**, S, represents the equilibrium state of a system. For regular matrices, this equilibrium state of a system can be estimated by calculating T^nS_0 for a large value of n.

Leslie matrices

- An $m \times m$ **Leslie matrix** has the form

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

  where:
  - m is the number of age groups being considered
  - s_i, the survival rate, is the proportion of the population in age group i that progress to age group $i + 1$
  - b_i, the birth rate, is the average number of female offspring from a mother in age group i during one time period.
- The population matrix S_n is an $m \times 1$ matrix representing the size of each age group after n time periods. This is calculated using a recursive formula

$$S_0 \text{ is the initial state matrix}, S_{n+1} = LS_n$$

  or the explicit rule

$$S_n = L^nS_0$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

11A **1** I can set up a transition matrix from a diagram. ☐

See Example 1, and Exercise 11A Question 1

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11A **2** I can set up a transition matrix from a written information. ☐

See Example 2, and Exercise 11A Question 2

11B **3** I can interpret a transition matrix and a transition diagram. ☐

See Example 3, and Exercise 11B Question 1

11C **4** I can use a recurrence relation to calculate state matrices step by step ☐

See Example 4, and Exercise 11C Question 1

11C **5** I can use a recurrence relation $S_{n+1} = T^n S_0$ to determine the nth state. ☐

See Example 5, and Exercise 11C Question 2

11C **6** I can use the inverse of a transition matrix. ☐

See Example 6, and Exercise 11C Question 4

11C **7** I can estimate steady state solution for suitable transition matrices. ☐

See Example 7, and Exercise 11C Question 4

11D **8** I can use the matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$. ☐

See Example 8, and Exercise 11D Question 1

11E **9** I can determine state matrices and construct life cycle diagrams in situations modelled by Leslie matrices. ☐

See Example 9, and Exercise 11E Question 1

11E **10** I can enter information into a Leslie matrix from written information. ☐

See Example 10, and Exercise 11E Question 2

11E **11** I can use numerical techniques to consider the limiting behaviour of Leslie matrices. ☐

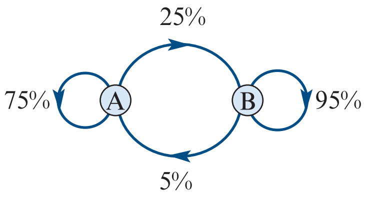
See Example 12, and Exercise 11E Question 3

11E **12** I can identify the properties of a Leslie matrix and the state matrices when there is a constant rate. ☐

See Example 13, and Exercise 11E Question 4

Multiple choice questions

1 The transition matrix that can be used to represent the information in the diagram shown is:



A
$$\text{To}\ \begin{array}{cc} & \text{From} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.05 & 0.95 \end{bmatrix} \end{array}$$

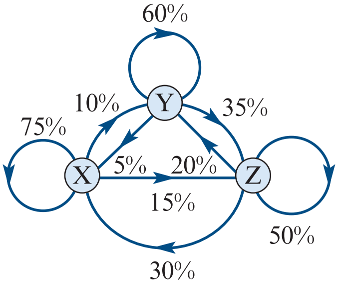
B
$$\text{To}\ \begin{array}{cc} & \text{From} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{bmatrix} \end{array}$$

C
$$\text{To}\ \begin{array}{cc} & \text{From} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.95 & 0.05 \end{bmatrix} \end{array}$$

D
$$\text{To}\ \begin{array}{cc} & \text{From} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.95 \\ 0.25 & 0.05 \end{bmatrix} \end{array}$$

E
$$\text{To}\ \begin{array}{cc} & \text{From} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.25 & 0.05 \\ 0.75 & 0.95 \end{bmatrix} \end{array}$$

2 The transition matrix that can be used to represent the information in the diagram shown is:



A
$$\text{To}\ \begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.30 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$$

B
$$\text{To}\ \begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.10 & 0.15 \\ 0.60 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20 \end{bmatrix} \end{array}$$

C
$$\text{To}\ \begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.10 & 0.15 \\ 0.10 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20 \end{bmatrix} \end{array}$$

D
$$\text{To}\ \begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.15 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$$

E
$$\text{To}\ \begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.15 \\ 0.15 & 0.35 & 0.50 \\ 0.10 & 0.60 & 0.20 \end{bmatrix} \end{array}$$

The following information is needed for Questions 3 to 8.

3 A system is defined by a transition matrix $T = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ with $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

For this system, $S_1 =$

A $\begin{bmatrix} 60 \\ 200 \end{bmatrix}$ **B** $\begin{bmatrix} 140 \\ 160 \end{bmatrix}$ **C** $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ **D** $\begin{bmatrix} 166 \\ 144 \end{bmatrix}$ **E** $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$

4 For this system, T^2 is:

A $\begin{bmatrix} 0.36 & 0.25 \\ 0.16 & 0.25 \end{bmatrix}$ **B** $\begin{bmatrix} 0.56 & 0.55 \\ 0.44 & 0.45 \end{bmatrix}$ **C** $\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ **D** $\begin{bmatrix} 1.2 & 1.0 \\ 0.8 & 1.0 \end{bmatrix}$

E not defined

5 For this system, S_3 is closest to:

A $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ **B** $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ **C** $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ **D** $\begin{bmatrix} 640 \\ 560 \end{bmatrix}$ **E** $\begin{bmatrix} 400 \\ 800 \end{bmatrix}$

6 For this system, the steady-state matrix is closest to:

A $\begin{bmatrix} 166.5 \\ 133.5 \end{bmatrix}$ **B** $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ **C** $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ **D** $\begin{bmatrix} 166.8 \\ 133.2 \end{bmatrix}$ **E** $\begin{bmatrix} 166.9 \\ 133.1 \end{bmatrix}$

7 If $L_1 = TS_0 + B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then L_1 equals:

A $\begin{bmatrix} 70 \\ 220 \end{bmatrix}$ **B** $\begin{bmatrix} 150 \\ 180 \end{bmatrix}$ **C** $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ **D** $\begin{bmatrix} 176 \\ 164 \end{bmatrix}$ **E** $\begin{bmatrix} 210 \\ 120 \end{bmatrix}$

8 If $P_1 = TS_0 - 2B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then P_1 equals:

A $\begin{bmatrix} 140 \\ 100 \end{bmatrix}$ **B** $\begin{bmatrix} 170 \\ 100 \end{bmatrix}$ **C** $\begin{bmatrix} 180 \\ 100 \end{bmatrix}$ **D** $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ **E** $\begin{bmatrix} 180 \\ 180 \end{bmatrix}$

9 A system of state matrices S_n is defined by the matrix equation $S_{n+1} = GS_n$ where $G = \begin{bmatrix} 0 & -0.5 \\ 1.5 & 0.5 \end{bmatrix}$.

If $S_1 = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then S_2 equals:

A $\begin{bmatrix} -12.5 \\ -2.5 \end{bmatrix}$ **B** $\begin{bmatrix} -10 \\ 25 \end{bmatrix}$ **C** $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ **D** $\begin{bmatrix} 10 \\ 25 \end{bmatrix}$ **E** $\begin{bmatrix} 15 \\ 30 \end{bmatrix}$

10 $T = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}$ is a transition matrix. $S_5 = \begin{bmatrix} 22 \\ 18 \end{bmatrix}$ is a state matrix.
If $S_5 = TS_4$, then S_4 equals:

A $\begin{bmatrix} 18 \\ 22 \end{bmatrix}$ **B** $\begin{bmatrix} 20 \\ 20 \end{bmatrix}$ **C** $\begin{bmatrix} 21.8 \\ 18.2 \end{bmatrix}$ **D** $\begin{bmatrix} 22 \\ 18 \end{bmatrix}$ **E** $\begin{bmatrix} 18.2 \\ 21.2 \end{bmatrix}$

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11 A large population of birds lives on a remote island. Every night each bird settles at either location A or location B.

On the first night the number of birds at each location was the same. On each subsequent night, a percentage of birds changed the location at which they settled.

The movement of birds between the two locations is described by the transition matrix T shown opposite. Assume this pattern of movement continues.

In the long term, the number of birds that settle at location A will:

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0 \\ 0.2 & 1 \end{bmatrix} \end{array}$$

A not change **B** gradually decrease to zero **C** gradually increase

D eventually settle at around 20% of the island's bird population

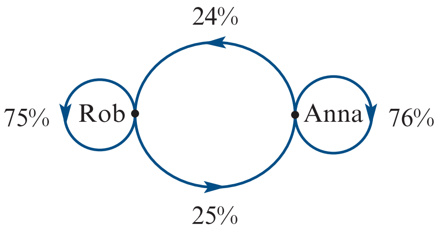
E eventually settle at around 80% of the island's bird population

Use the following information to answer Questions 12 and 13.

Two politicians, Rob and Anna, are the only candidates for a forthcoming election. At the beginning of the election campaign, people were asked for whom they planned to vote. The numbers were as per the table.

Candidate	*Number of people planing to vote for candidate*
Rob	5692
Anna	3450

During the election campaign, it is expected that people may change the candidate that they plan to vote for each week according to the transition diagram shown.



12 The total number of people who are expected to change the candidate that they plan to vote for 1 week after the election campaign begins is:

A 828 **B** 1423 **C** 2251 **D** 4269 **E** 6891

13 The election campaign will run for 10 weeks. If people continue to follow this pattern of changing the candidate they plan to vote for, the expected winner after 10 weeks will be:

A Rob by about 50 votes **B** Rob by about 100 votes

C Rob by fewer than 10 votes **D** Anna by about 100 votes

E Anna by about 200 votes

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Written response questions

1 The Diisco (D) and the Spin (S) are two large music venues in the same city. They both open on the same Saturday night and will open on every Saturday night.

The matrix A_1 opposite is the attendance matrix for the first Saturday. This matrix shows the number of people who attended the first Diisco and the number of people who attended the Spin.

$$A_1 = \begin{array}{c}\begin{bmatrix} 500 \\ 240 \end{bmatrix}\end{array}\begin{array}{c} D \\ S \end{array}$$

The number of people expected to attend the second Saturday for each venue can be determined using the matrix equation

$$A_2 = GA_1$$

$$\text{where } G \text{ is the matrix } G = \begin{array}{cc} & \text{This Saturday} \\ & \begin{array}{cc} D & S \end{array} \\ & \begin{bmatrix} 1.2 & -0.4 \\ 0.2 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ D \\ S \end{array} \quad \text{Next Saturday}$$

a **i** Determine A_2, the attendance matrix for the second Saturday.

ii What was the total attendance on the second Saturday.

Assume that the attendance matrices for successive Saturdays can be determined as follows:

$$A_3 = GA_2, \quad A_4 = GA_3, \text{ and so on such that } A_{n+1} = GA_n$$

b Determine the attendance matrix (with the elements written correct to the nearest whole number) for the eighth Saturday.

c Describe the way in which the number of people attending the Diiscoo is expected to change over the next 80 or so Saturdays.

Suppose instead that 500 people attend the first Diiscoo, and 490 people attend the Spin.

d Describe the way in which the attendance at both venues changes if attendance follows this prediction.

2 Suppose that the trees in a forest are classified into three age groups: young trees (0–15 years), middle-aged trees (16–30 years) and old trees (more than 30 years). A time period is 15 years, and it is assumed that in each time period:

- 10% of young trees, 20% of middle-aged trees and 40% of old trees die
- surviving trees enter into the next age group; old trees remain old
- dead trees are replaced by young trees.

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Complete the 3 × 3 transition matrix T to describe this.

$$\begin{array}{c} \\ Y \\ M \\ O \end{array} \begin{array}{c} \begin{array}{ccc} Y & M & O \end{array} \\ \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \end{array}$$

3 The following table represents a study of a particular population of marsupials, which has been divided into eight age groups. The table gives the initial population, birth rate and survival rate for each age group.

Age group	1	2	3	4	5	6	7	8
Initial population	0	100	100	50	0	0	0	0
Birth rate	0	0.1	0.9	0.2	0	0	0	0
Survival rate	0.98	0.95	0.95	0.9	0.7	0.5	0.1	0

a Write down the Leslie matrix for this population.

b Calculate S_2 and S_3.

c Estimate the long-term growth rate of the population.

4 The growth of algae in a particular lake is being studied to protect the ecology from a disastrous algal bloom. The algae can live for up to four days. So the population is divided into four age groups of one day each. The fertility rates and survival rates are being monitored so that the population can be modelled using a Leslie matrix.

At the beginning of the study in late winter (day 0), it was observed that the algae concentration in the lake was 3200 cells per millilitre of water, with equal numbers in each age group. The fertility rates on the four days of life were 0.2, 0.5, 0.6 and 0.4 respectively. The survival rate for each of the first three days of life was 0.7.

a Write down a Leslie matrix to represent this particular model.

b Find the population matrix for cells per millilitre of water on day 20, correct to three significant figures.

c Find the population matrix for cells per millilitre of water on day 21. Hence find the rate of change in the algae concentration per day at this stage.

d With the coming of spring on day 21, the fertility rates increased to 0.3, 0.6, 0.7 and 0.5; the survival rate remained unchanged. Find the population matrix after a further three weeks (i.e. on day 42).

e With the arrival of warmer weather on day 42, the fertility rates increased to 0.3, 0.7, 0.8 and 0.5; the survival rate increased to 0.85. Suppose that an algal bloom is declared if the concentration of algae reaches 100 000 cells per millilitre of water. Using trial and error, find the day of the study on which an algal bloom was declared.