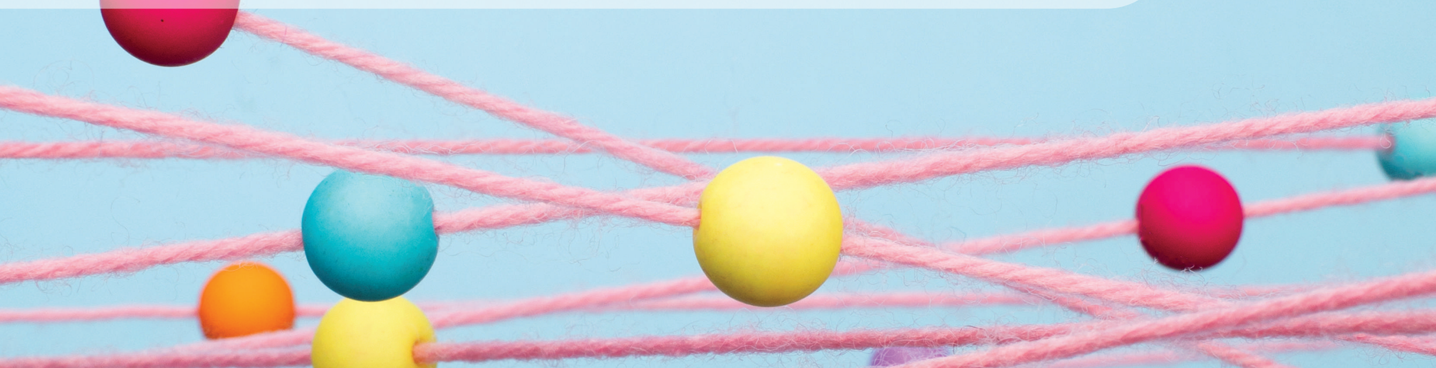
Chapter 13

Graphs, networks and trees: travelling and connecting problems



Chapter objectives

- ▶ What is a graph?
- ▶ How do we identify the features of a graph?
- ▶ How do we draw a graph?
- ▶ How do we apply graphs in practical situations?
- ▶ How do we construct an adjacency matrix from a graph?
- ▶ How do we define and draw a planar graph?
- ▶ How do we identify the type of walk on a graph?
- ▶ How do we find the shortest path between two vertices of a graph?
- ▶ How do we find the minimum distance required to connect all vertices of a graph?

In this chapter graphs and their use as networks representing connections between objects will be introduced, in addition to exploring their properties and applications.

Problems involving networks will be investigated and you will learn unique algorithms such as Djikstra's and Prim's to solve such problems.

ISBN 978-1-009-11041-9

Cambridge University Press

13A Graphs and networks

Learning intentions
- To be able to define and identify a graph, vertex, edge and loop.
- To be able to find the degree of a vertex and the sum of degrees.
- To be able to describe the features of a graph.
- To be able to define and identify a planar graph and its faces.
- To be able to apply Euler's formula and use it to verify if a graph is planar.

Representing connections with graphs

There are many situations in everyday life that involve connections between people or objects. Towns are connected by roads, computers are connected to the internet and people connect to each other through being friends on social media. A diagram that shows these connections is called a **graph**.

Edges and vertices

Six people – Anna, Brett, Cora, Dario, Ethan and Frances – have connections on a social media website. The graph shows these connections.

Anna is a friend of Brett, Ethan and Frances.

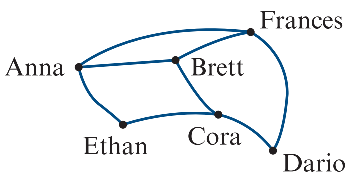
Brett is a friend of Anna, Cora and Frances.

Cora is a friend of Brett, Dario and Ethan.

Dario is a friend of Cora and Frances.

Ethan is a friend of Anna and Cora.

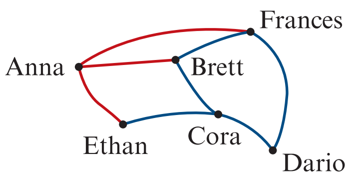
Frances is a friend of Anna, Brett and Dario.



The graph shows each of the people as a dot called a **vertex**. The *vertices* (plural of vertex) are joined together by a line that indicates the social media friendship between the people. The lines that join the vertices in the graph are called **edges**.

Degree of a vertex

Anna has three friends. The vertex representing Anna has three edges attached to it, connecting Anna to one of her friends. The number of edges attached to a vertex is called the **degree** of that vertex.



The degree of the vertex representing Anna is *odd*, because there is an odd number of edges connected to it. The degree of the vertex representing Dario is *even* because there is an even number of edges connected to it.

In symbolic form, we can let the letter A represent the vertex for Anna. The degree of this vertex can be written as $\deg(A)$. In this graph, $\deg(A) = 3$.

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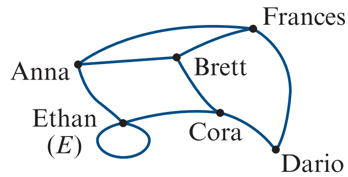
Loops

Imagine that Ethan is able to add himself as a friend on the social media website.

The edge representing this connection would connect the vertex representing Ethan, E, back to itself. This type of edge is called a **loop**.

A loop is attached twice to a vertex and so it will contribute two degrees. So $\deg(E) = 4$.



Edges, vertices and loops

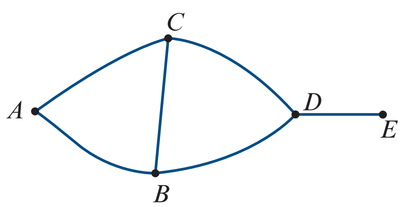
- A graph consists of **vertices** joined by **edges**.
- The number of edges attached to a vertex is called the degree of the vertex. The symbolic form for the degree of vertex A is $\deg(A)$.
- A loop connects a vertex to itself. Loops contribute two degrees to a vertex.

Describing graphs

Graphs that represent connections between objects can take different forms and have different features. This means that there is a variety of ways to describe these graphs.

Simple graphs

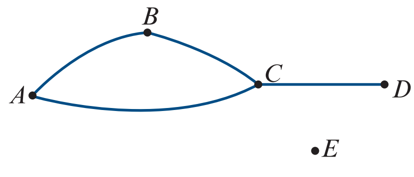
Simple graphs do not have any loops. There are no duplicate or **multiple edges** either.



Isolated vertex

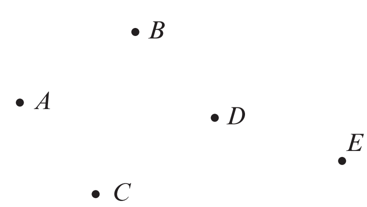
A graph has an **isolated vertex** if there is a vertex that is not connected to another vertex by an edge.

The isolated vertex in this graph is E, because it is not connected to any other vertex by an edge.



Degenerate graphs

Degenerate graphs have all vertices isolated. This means that there are no edges in the graph at all.



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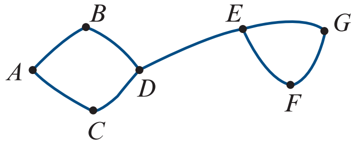
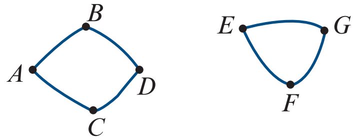
Connected graphs and bridges

A **connected graph** has every vertex connected to every other vertex, either directly or indirectly via other vertices. The graph on the right is connected. A **bridge** is an edge in a connected graph that, if removed, will cause the graph to be disconnected. The graph on the right has a bridge connecting vertex D to vertex E.

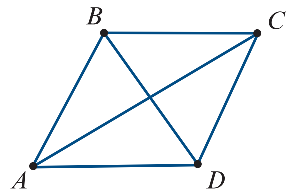


The graph on the right shows the bridge from vertex D to vertex E removed. There are now two separate sections of the graph that are not connected to each other.



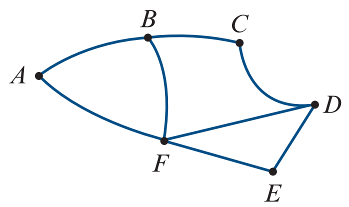
Complete graphs

If there is an edge between every pair of vertices, the graph is called a **complete graph**. Every vertex in the graph is connected directly by an edge to every other vertex in the graph.

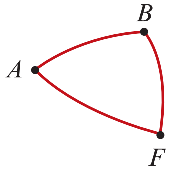


Subgraphs

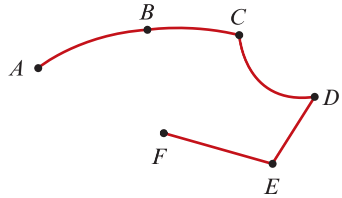
A **subgraph** is a part of a larger graph. All of the edges and vertices in the subgraph must exist in the original graph. If there are extra edges or vertices, the graph will not be a subgraph of the larger graph.



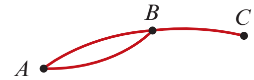
Graph 1



Graph 2



Graph 3



Graph 4

Graphs 2 and 3 above are subgraphs of graph 1. All of the vertices and edges in graphs 2 and 3 exist in graph 1.

Graph 4 above is not a subgraph of graph 1. There are two edges connecting vertex A to vertex B, but in graph 1 there is only one.

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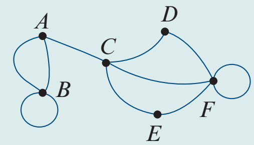
Example 1 Graphs

A connected graph is shown on the right.



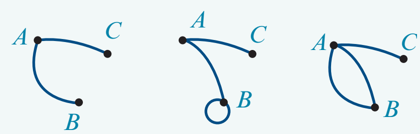
a What is the degree of vertex C?

b Which vertices have a loop?

c What is the degree of vertex F?

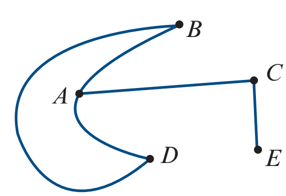
d A bridge exists between two vertices. Which vertices are they?

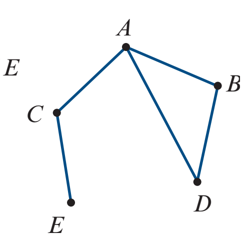
e Draw a subgraph of this graph that involves only vertices A, B and C.

Explanation	Solution
a Count the number of times an edge connects to vertex C. There are four connections.	The degree of vertex C is 4. deg(C) = 4.
b A vertex has a loop if an edge connects it to itself.	Vertex B and vertex F have loops.
c Count the number of times an edge connects to vertex F. Remember that a loop contributes two degrees.	The degree of vertex F is 5. deg(F) = 5
d Look for an edge that, if removed, would disconnect the graph.	A bridge exists between vertex A and vertex C.
e There are a few possible answers for this question. Some are shown on the right.	

Equivalent (Isomorphic) graphs

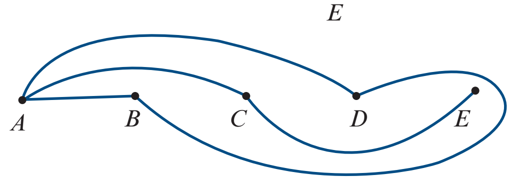
All of the graphs shown in the diagram below contain exactly the same information. For example, the edge between vertex E and C exists in all of them. The vertex A is connected to B, D and C as well.





The location of the vertices and edges in the diagram are unimportant. As long as the connections are all represented accurately, the graph can be drawn in any way that you prefer.

The first of the graphs has some curved edges and the second has all straight edges. The third has the vertices arranged in a straight line.



The style that they are drawn in and the position of the vertices relative to each other is unimportant. It *is* important that the information contained in the graph – the connections between the vertices – is correct.

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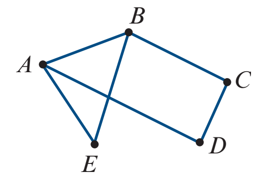
All of these graphs are considered to be *equivalent* to each other because they all contain identical information. Each has edges connecting the same vertices. Graphs that contain identical information like this are called **equivalent graphs** or **isomorphic graphs**.

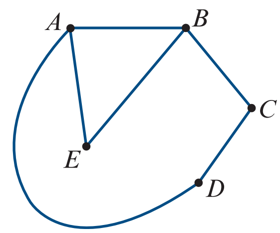
Planar graphs

The graph opposite has two edges that overlap. It is important to note that there is *no vertex at the point of overlap of the edges.*



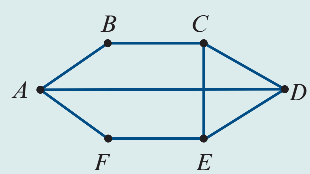
It can help to think of an edge as an insulated electrical wire. It is quite safe to cross two such electrical wires because the wires themselves never touch and never interfere with each other. The edges that cross over in this diagram are similar, in that they do not intersect and do not interfere with each other.

If a graph has edges that cross, it may be possible to redraw the graph so that the edges no longer cross. The edge between vertices A and D has been moved, but none of the information in the graph has changed. Graphs where this is possible are called **planar graphs**. If it is impossible to draw an equivalent graph without crossing edges, the graph is called a *non-planar graph*.



Example 2 **Redrawing a graph in planar form**

Show that this graph is planar by redrawing it so that no edges cross.



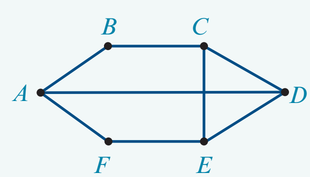
Explanation

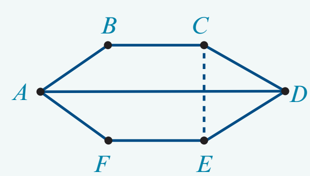
1 Choose one of the edges that crosses over another edge.

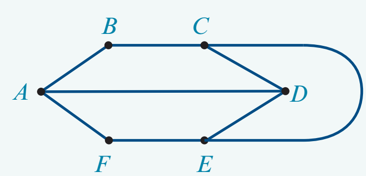
2 Remove it temporarily from the graph.

3 Redraw the edge between the same vertices but without crossing over another edge.

Solution







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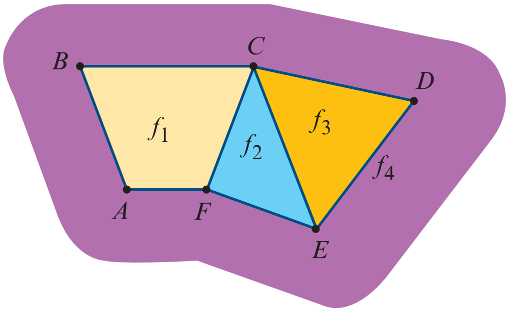
Euler's formula

Leonard Euler (pronounced 'oiler') was one of the most prolific mathematicians of all time. He contributed to many areas of mathematics and his proof of the rule named after him is considered to be the beginning of the branch of mathematics called topology.



Faces

A planar graph defines separate regions of the paper it is drawn on. These regions are enclosed spaces that you could colour in and these regions are called **faces**. An often-forgotten face of a graph is the space outside of the graph itself, covering the infinite space around it. This face is labelled f_4 in the graph above.

The number of faces for a graph can be counted. In the graph shown above, there are four faces, labelled f_1, f_2, f_3 and f_4.

Euler's formula

There is a relationship between the number of vertices, v, the number of edges, e, and the number of faces, f, in a connected planar graph.

In words: number of vertices + number of faces = number of edges +2

In symbols: $v + f = e + 2$

Euler's formula

For any planar graph:

$$v + f = e + 2$$

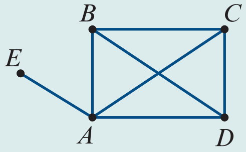
where v is the number of vertices, e is the number of edges and f is the number of faces in the graph.

Example 3 **Verifying Euler's formula**

For the graph shown on the right:

a redraw the graph into planar form

b verfiy Euler's formula for this graph.



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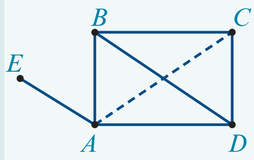
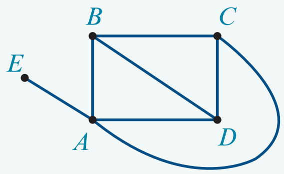
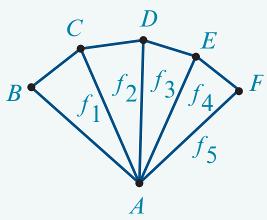
Explanation	Solution
a Temporarily remove an edge that crosses another edge and redraw it so that it does not cross another edge.	 
b Count the number of vertices, edges and faces.	In the planar graph there are five vertices, seven edges and four faces. $v + f = e + 2$ $5 + 4 = 7 + 2$ $9 = 9$ Euler's formula is verified.

Example 4 Using Euler's formula

A connected planar graph has six vertices and nine edges. How many faces does the graph have? Draw a connected planar graph with six vertices and nine edges.

Explanation	Solution
a Write down the known values.	$v = 6 \quad e = 9$
b Substitute into Euler's formula and solve for the unknown value.	$v + f = e + 2$ $6 + f = 9 + 2$ $6 + f = 11$ $f = 11 - 6$ $f = 5$ This graph has five faces, labelled f_1, f_2, f_3, f_4 and f_5.
c Sketch the graph. Note: There are other possible graphs.	

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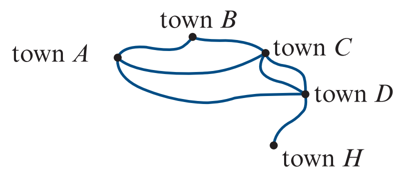
Skill-sheet

Exercise 13A

Drawing and describing graphs

Example 1

1 This section of a road map can be considered a graph, with towns as vertices and the roads connecting the towns as edges.



a Give the degree of:

i town A **ii** town B **iii** town H.

b What is the sum of the degrees of all the vertices of this graph?

c A bridge exists between two towns. Which towns are they?

d Draw a subgraph of this road map that contains only towns H, D and C.

2 Draw a graph that:

a has three vertices, two of which are odd

b has four vertices and five edges, one of which is a loop

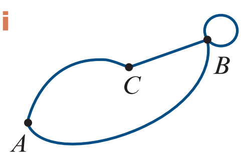
c has six vertices, eight edges and one bridge

d has six vertices, two of which are odd, and contains a subgraph that is a triangle.

Equivalent graphs

3 In each question below, three graphs are isomorphic and the fourth is not. Identify the graph which is not isomorphic to the others.

a **i**



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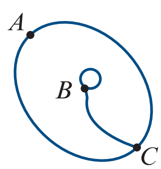

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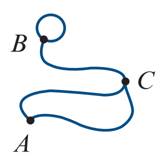

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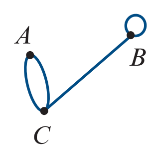
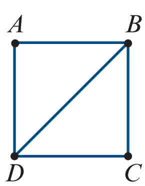
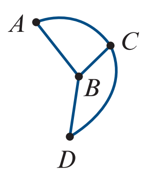


b **i**



ii



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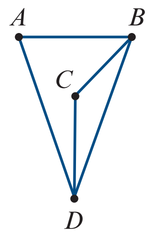

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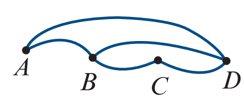
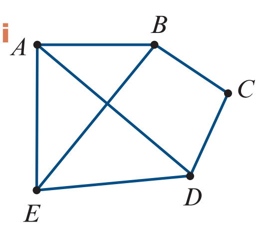
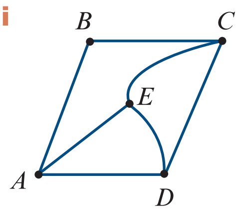


c **i**



ii



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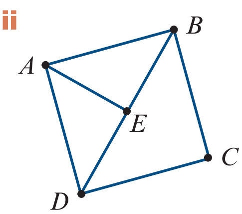

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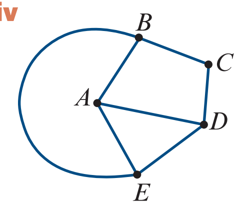

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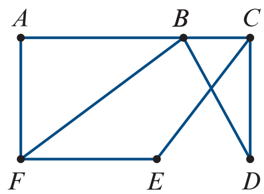
Drawing planar graphs

Example 2

4 Where possible, show that the following graphs are planar by redrawing them in a suitable planar form.

a


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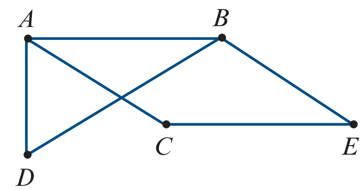
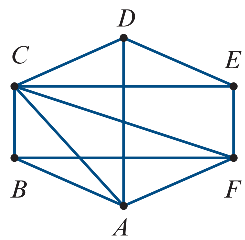
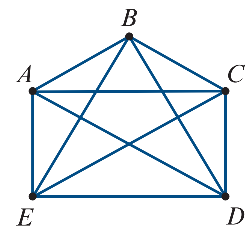


c


d


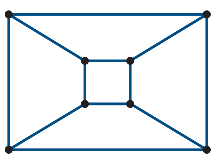
Euler's formula

Example 3

5 For each of the following graphs:

i state the values of v, e and f

ii verify Euler's formula.

a


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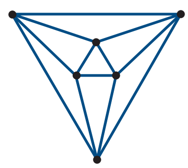

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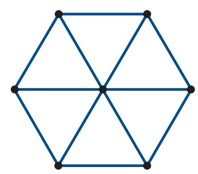

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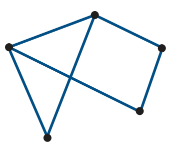

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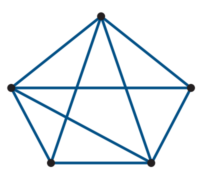

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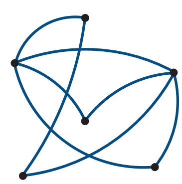

6 For a planar connected graph, find:

a f, if $v = 8$ and $e = 10$ **b** v, if $e = 14$ and $f = 4$ **c** e, if $v = 10$ and $f = 11$.

Properties of graphs

Example 4

7 A connected planar graph has eight vertices and thirteen edges. Find the number of faces of this graph.

8 A connected graph has five vertices and seven edges. Find the sum of the degrees of the vertices.

9 Find the number of edges needed to make a complete graph with six vertices.

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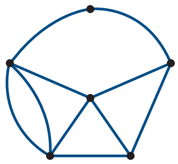
Exam 1 style questions

10 Consider the graph opposite.



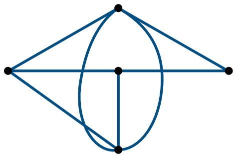
The number of vertices with a degree of 4 is

A 1 **B** 2 **C** 3
D 4 **E** 5

11 A planar graph has four faces. The graph could have

A Seven vertices and seven edges
B Seven vertices and four edges
C Seven vertices and five edges
D Four vertices and seven edges
E Five vertices and seven edges

Use the following information for questions 12 and 13.



12 The number of faces in the graph above is

A 3 **B** 4 **C** 5 **D** 6 **E** 7

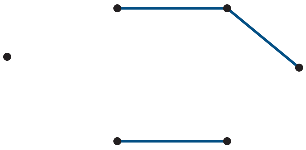
13 Consider the following five statements about the graph above:

- The graph is planar.
- The graph contains a bridge.
- It is a simple graph.
- The sum of degrees of the vertices is 16.
- It is a complete graph.

How many of these statements are true?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

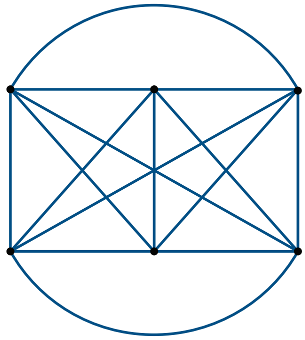
14 Consider the graph shown opposite.



The minimum number of edges that must be added to make this a complete graph is

A 1 **B** 2 **C** 6
D 8 **E** 12

15 The following graph with six vertices is a complete graph. Edges are removed so that the graph will have the minimum number of edges to remain connected. The number of edges that are removed is



A 6 **B** 8 **C** 10 **D** 12 **E** 14

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13B Adjacency matrices

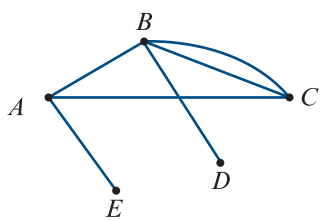
Learning intentions

- To be able to use an adjacency matrix to represent a graph.

Summarising the connections in a graph

A matrix can be used to summarise the information in a graph. A matrix that records the number of connections between vertices of a graph is called an **adjacency matrix**.

A graph and the adjacency matrix for that graph are shown here.



	A	B	C	D	E
A	0	1	1	0	1
B	1	0	2	1	0
C	1	2	0	0	0
D	0	1	0	0	0
E	1	0	0	0	0

The adjacency matrix has:

- five rows and five columns, one for each vertex in the graph
- row and column labels that match the vertices in the graph, A, B, C, D, E
- a '0' in the intersection of row A and column D because there is no edge connecting A to D
- a '0' in the intersection of row A and column A because there is no edge connecting A to itself; that is, there is no loop at vertex A
- a '1' in the intersection of row A column B because there is one edge connecting A to B
- a '2' in the intersection of row C and column B because there are two edges connecting C to B.

The number of edges between every other pair of vertices in the graph is recorded in the adjacency matrix in the same way.

Example 5 Drawing a graph from an adjacency matrix

Draw the graph that is represented by the following adjacency matrix.

	A	B	C	D	E
A	0	0	2	0	1
B	0	0	2	1	0
C	2	2	0	1	0
D	0	1	1	1	0
E	1	0	0	0	0

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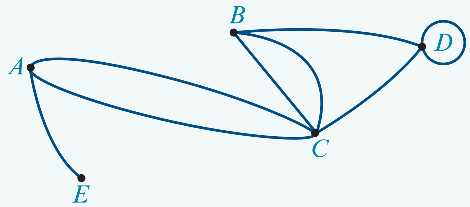
Explanation

1 Draw a dot for each vertex and label A to E.

2 There is a '2' in the intersection of row A and column C. This means there are two edges connecting vertex A and vertex C. Add these to the graph.

Solution



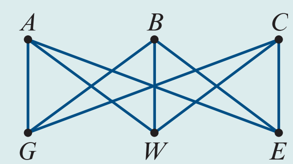
3 Note the '1' in the intersection of row D and column D. This shows that there is a loop at vertex D.

4 Look at every intersection of row and column and add edges to the graph, if they do not already exist.

Note: This graph is drawn as a planar graph, but this is not strictly necessary unless required by the question.

Example 6

Construct an adjacency matrix that can be used to represent the graph opposite. This graph represents the ways that three houses A, B and C are connected to three utility outlets, gas (G), water (W) and electricity (E).



Explanation

The convention used to enter the values is the same as discussed above.

Solution

	A	B	C	G	W	E
A	0	0	0	1	1	1
B	0	0	0	1	1	1
C	0	0	0	1	1	1
G	1	1	1	0	0	0
W	1	1	1	0	0	0
E	1	1	1	0	0	0

The graph in Example 6 is called a bipartite graph as the set of vertices is separated into two sets of objects Houses (A, B, C) and Utility outlets (G, W, E) with each edge connecting a vertex in each set. You will meet bipartite graphs again in Chapter 14 when studying allocation problems.

Adjacency matrices

The adjacency matrix A of a graph is an $n \times n$ matrix in which, for example, the entry in row C and column F is the number of edges joining vertices C and F.

A loop is a single edge connecting a vertex to itself.

Loops are counted as one edge.

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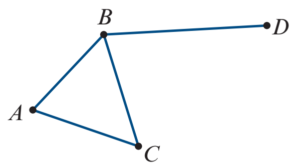
Exercise 13B

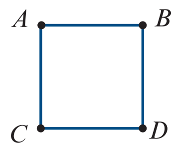
Writing adjacency matrices

1 For each of the following graphs, write down the adjacency matrix.

a

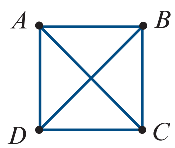


b

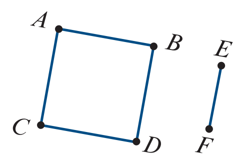


c

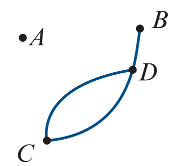
d



e



f



Drawing graphs from adjacency matrices

Example 5

2 Draw a graph from each of the following adjacency matrices.

a

	A	B	C
A	0	1	1
B	1	0	1
C	1	1	0

b

	A	B	C	D
A	1	0	0	0
B	0	0	1	1
C	0	1	1	0
D	0	1	0	0

c

	A	B	C	D
A	0	1	2	1
B	1	0	1	1
C	2	1	0	0
D	1	1	0	0

Properties of graphs

3 The adjacency matrix on the right has a row and column for vertex C that contains all zeros. What does this tell you about vertex C?

	A	B	C
A	0	1	0
B	1	0	0
C	0	0	0

4 Every vertex in a graph has one loop. What feature of the adjacency matrix would tell you this information?

5 A graph has five vertices: A, B, C, D and E. It has no duplicate edges and no loops. If this graph is complete, write down the adjacency matrix for the graph.

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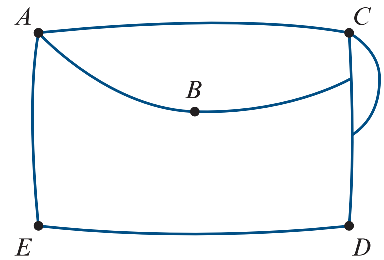
Exam 1 style questions

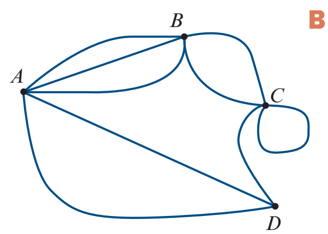
6 The map opposite shows the pathways between five buildings: A, B, C, D and E. An adjacency matrix for the graph that represents this map is formed. The number of zeros in this matrix is

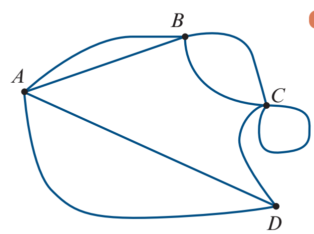


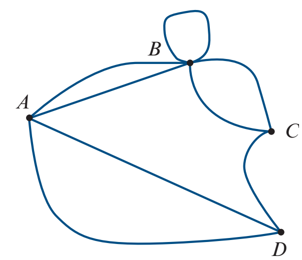
A 9 **B** 10 **C** 11
D 12 **E** 13

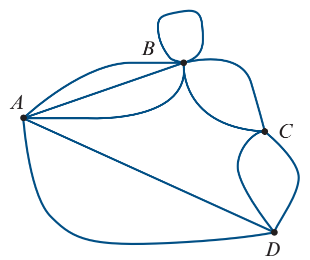
7 The adjacency matrix opposite shows the number of pathways between four points A, B, C and D. A graph that could be represented by the adjacency matrix is

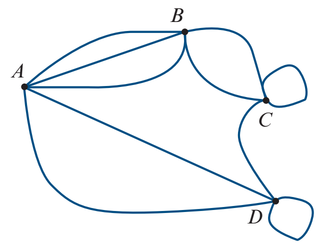
	A	B	C	D
A	0	3	0	2
B	3	0	2	0
C	0	2	1	1
D	2	0	1	0

A


B


C


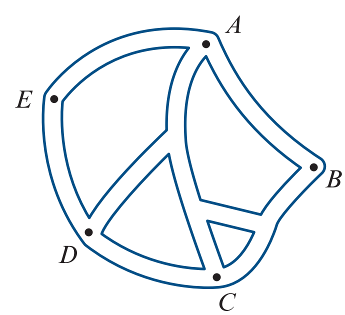
D


E


Use the following information to answer questions 8 and 9.

The map opposite shows the pathways between five schools: A, B, C, D and E.

An adjacency matrix for the graph that represents this map is formed.



8 Of the 25 elements in the adjacency matrix, the number '1' appears

A 7 times **B** 8 times **C** 9 times
D 10 times **E** 11 times

9 Of the 25 elements in the adjacency matrix, the numbers '2' or '3' appear

A 6 times **B** 7 times **C** 8 times **D** 9 times **E** 10 times

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Use the following information to answer questions 10 and 11.

10 A graph has four vertices A, B, C, and D. The adjacency matrix for this graph is shown opposite. Which one of the following statements about this graph is **not** true?

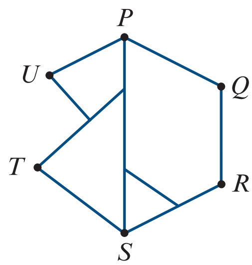
	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	1	0
D	1	1	0	0

A The graph is connected.
B The graph contains a loop.
C The graph contains multiple edges.
D The graph is planar.
E The graph contains a bridge

11 The number of faces of the graph represented by the adjacency matrix above is

A 2 **B** 3 **C** 4 **D** 5 **E** 6

12 The map opposite shows all the road connections between six towns, P, Q, R, S, T and U. The road connections could be represented by the adjacency matrix



A

	P	Q	R	S	T	U
P	0	1	1	1	1	1
Q	1	0	1	0	0	0
R	1	1	0	1	1	1
S	1	0	1	1	1	1
T	1	0	1	1	0	1
U	1	0	1	1	1	0

B

	P	Q	R	S	T	U
P	0	1	1	1	1	1
Q	0	1	1	1	1	1
R	0	1	1	1	1	1
S	1	0	1	1	2	2
T	1	0	1	2	0	1
U	1	0	1	2	1	0

C

	P	Q	R	S	T	U
P	0	1	1	2	1	2
Q	1	0	1	0	0	0
R	1	1	0	2	1	1
S	2	0	2	1	2	2
T	1	0	1	2	0	1
U	2	0	1	2	1	0

D

	P	Q	R	S	T	U
P	0	1	1	2	1	2
Q	1	0	1	0	0	0
R	1	1	0	2	1	1
S	2	0	2	0	3	2
T	1	0	1	3	0	1
U	2	0	1	2	0	0

E

	P	Q	R	S	T	U
P	0	1	1	2	1	2
Q	1	0	1	0	0	0
R	1	1	0	2	1	1
S	2	0	2	1	3	2
T	1	0	1	3	0	1
U	2	0	1	2	1	0

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13C Exploring and travelling

Learning intentions

- To be able to identify a walk as a trail, path, circuit or cycle.
- To be able to identify a walk as an Eulerian trail, Eulerian circuit, Hamiltonian path or Hamiltonian cycle.
- To be able to use the degrees of the vertices to identify if an Eulerian trail or circuit is possible.

Travelling

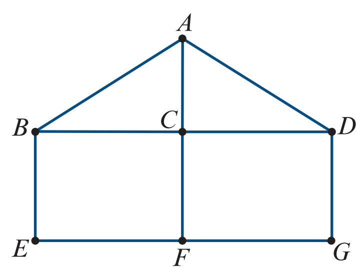
Graphs can be used to model and analyse problems involving exploring and **travelling**. These problems include minimising the distance travelled or time taken between different locations using different routes. For example, a courier driver would like to know the shortest route to use for deliveries, and a tour guide would like to know the quickest route that allows tourists to see a number of sights without retracing their steps.

To solve these types of problems, you will need to learn the language we use to describe the different ways of navigating through a graph, from one vertex to another.

Walks, trails, paths, circuits and cycles

The different ways of navigating through graphs, from one vertex to another, are described as *walks, trails, paths, circuits* and *cycles*.

The graph opposite will be used to explain and define each of these terms.

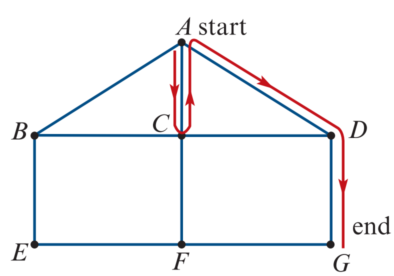


Walks

A **walk** is a sequence of edges, linking successive vertices in a graph.

A walk starts at one vertex and follows any route to finish at another vertex.

The red line in the graph opposite traces out a walk. This walk can be written down by listing the vertices in the order they are visited: A–C–A–D–G.



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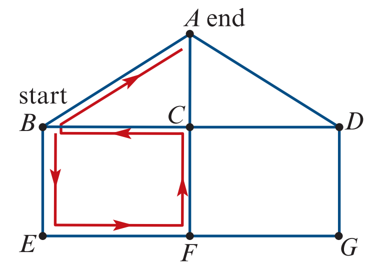
Trails

A **trail** is a walk with no repeated edges.

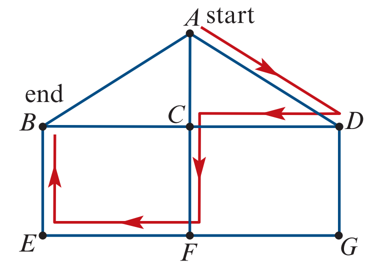


The red line in the graph opposite traces out a trail. This trail can be written down by listing the vertices in the order they are visited: $B–E–F–C–B–A$.

Note: There are *no repeated edges* in this trail, but one vertex (B) is repeated.

Paths

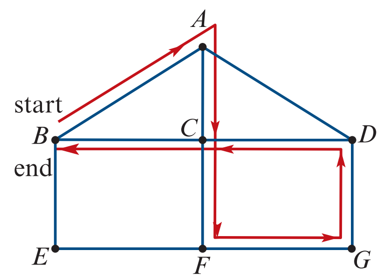
A **path** is a walk with no repeated edges and no repeated vertices.



The red line in the graph opposite traces out a path. This path can be written down by listing the vertices in the order they are visited: $A–D–C–F–E–B$.

Circuits

A **circuit** is a trail (no repeated edges) that starts and ends at the same vertex. Circuits are also called *closed trails*.

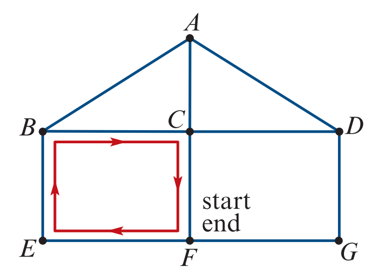


The red line in the graph opposite traces out a circuit. This circuit can be written down by listing the vertices in the order they are visited: $A–C–F–G–D–C–B–A$.

Note: There are *no repeated edges* in this circuit, but one vertex, C, is repeated. The start and end vertices are also repeated because of the definition of a circuit.

Cycles

A **cycle** is a path (no repeated edges, no repeated vertices) that starts and ends at the same vertex. The start and end vertex is an exception to repeated vertices. Cycles are also called *closed paths*.



The red line in the graph opposite traces out a cycle. This cycle can be written down by listing the vertices in the order they are visited: $F–E–B–C–F$.

Note: There are *no repeated edges* and *no repeated vertices* in this cycle, except for the start and end vertices.

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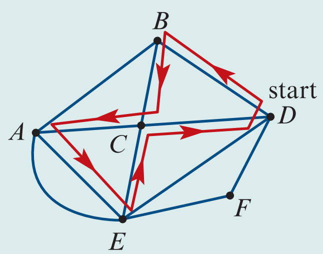
Walks, trails, paths, circuits and cycles

- A *walk* is a sequence of edges, linking successive vertices in a graph.
- A *trail* is a walk with no repeated edges.
- A *path* is a walk with no repeated edges and no repeated vertices.
- A *circuit* is a trail (no repeated edges) that starts and ends at the same vertex.
- A *cycle* is a path (no repeated edges and no repeated vertices) that starts and ends at the same vertex.

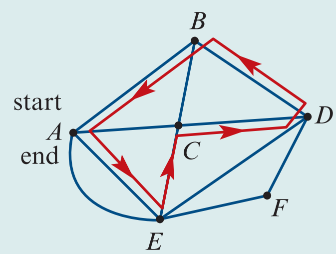
Example 7 Identifying types of walks

Identify the walk in each of graphs below as a trail, path, circuit, cycle or walk only.

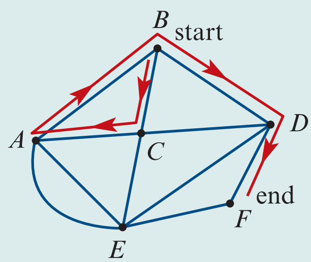
a



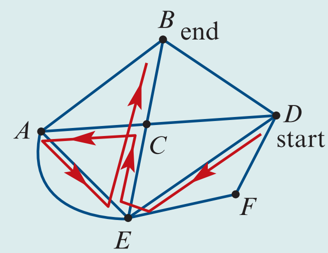
b



c



d



Solution

a This walk starts and ends at the same vertex so it is either a circuit or a cycle. The walk passes through vertex C twice without repeated edges, so it must be a circuit.

b This walk starts and ends at the same vertex so it is either a circuit or a cycle. The walk has no repeated vertex or edge so it is a cycle.

c This walk starts at one vertex and ends at a different vertex, so it is not a circuit or a cycle. There is one repeated vertex (B) and no repeated edge, so it must be a trail.

d This walk starts at one vertex and ends at a different vertex so it is not a circuit or a cycle. There are repeated vertices (C and E) and repeated edges (the edge between C and E), so it must be a walk only.

Eulerian trails and circuits

Trails and circuits that follow every edge, without duplicating any edge, of a graph are called **Eulerian trails** and **Eulerian circuits**. Eulerian trails and circuits are important for some real-life applications. If, for example, a graph shows towns as vertices and roads as edges, then being able to identify a route through the graph that follows every road can be important for mail delivery, or for checking the condition of the roads.

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Eulerian trails and circuits exist under easily identified conditions.

Eulerian trails and circuits

Eulerian trails

An Eulerian trail follows every edge of a graph.

An Eulerian trail will exist if the graph:

- is connected
- has exactly *zero* or *two* vertices that have an *odd degree*.
- If there are no odd vertices, the Eulerian trail can start at any vertex in the graph.
- if there are two odd vertices, the Eulerian trail will *start* at one of the odd vertices and *finish* at the other.

Eulerian circuits

An Eulerian circuit is an Eulerian trail (follows every edge) that starts and ends at the same vertex.

An Eulerian circuit will exist if the graph:

- is connected
- has vertices that *all* have an *even degree.*

An Eulerian circuit can start at *any* of the vertices.

Note: If a graph has more than two odd-degree vertices, neither an eulerian trail nor an eulerian circuit exists.

Hamiltonian paths and cycles

Paths and cycles that pass through every vertex of a graph only once are called **Hamiltonian paths** and **Hamiltonian cycles**, named after the mathematician William Rowan Hamilton.

Hamiltonian paths and cycles have real-life applications to situations where every vertex of a graph needs to be visited, but the route taken is not important. If, for example, the vertices of a graph represent people and the edges of the graph represent email connections between those people, a hamiltonian path would ensure that every person in the graph received a message intended for everyone.

Unlike Eulerian trails and circuits, Hamiltonian paths and cycles do not have a convenient rule or feature that identifies them. Inspection is the only way to identify them.

Hamiltonian paths and cycles

Hamiltonian paths

A Hamiltonian path visits every vertex of a graph.

Hamiltonian cycles

A Hamiltonian cycle is a Hamiltonian path (every vertex) that starts and ends at the same vertex.

Note: Inspection is the only way to identify Hamilton paths and cycles.

Remember: Eulerian trails and circuits do not repeat edges. Hamiltonian paths and cycles do not repeat vertices.

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Hint: To remember the difference between eulerian and hamiltonian travels, remember that eulerian refers to edges, and both start with ‘e’.

Example 8 Eulerian and Hamiltonian travel

A map showing the towns of St Andrews, Kinglake, Yarra Glen, Toolangi and Healesville is shown on the right. Consider only the yellow routes in your answer. St Andrews and Yarra Glen are considered connected, although this route is not fully shown in the image.

a Draw a graph with a vertex representing each of these towns and edges representing the direct road connections between the towns.

b Explain why an Eulerian trail, but not an Eulerian circuit, is possible through this graph.

c Write down an Eulerian trail that begins at Toolangi.

d Write down a Hamiltonian cycle that begins at Healesville.

Explanation

a A road connection exists between:

- St Andrews and Kinglake
- St Andrews and Yarra Glen
- Kinglake and Yarra Glen
- Kinglake and Toolangi
- Yarra Glen and Toolangi
- Yarra Glen and Healesville
- Healesville and Toolangi.

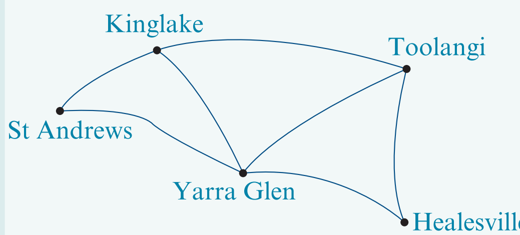
b The graph has two odd-degree vertices (Toolangi and Kinglake).

c There are a few different answers to this question. One of these is shown.

d There are two different answers to this question. One of these is shown.

Solution

a



b There are exactly two odd-degree vertices in this graph. An Eulerian trail will exist, but an Eulerian circuit does not.

c An Eulerian trail, starting at Toolangi is: Toolangi–Healesville–Yarra Glen–Toolangi–Kinglake–Yarra Glen–St Andrews–Kinglake

d A Hamiltonian cycle that begins at Healesville is: Healesville–Yarra Glen–St Andrews–Kinglake–Toolangi–Healesville

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Skill-sheet

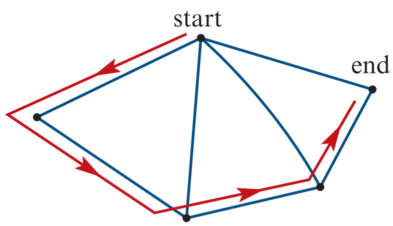
Exercise 13C

Describing travels

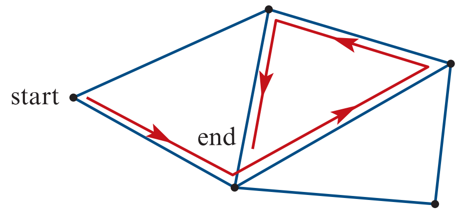
Example 7

1 Identify the walk in each of the graphs below as a trail, path, circuit or walk only.

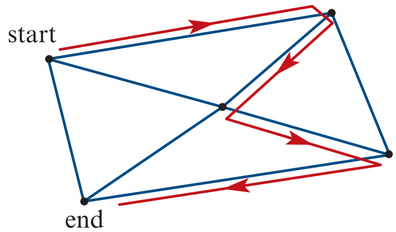
a



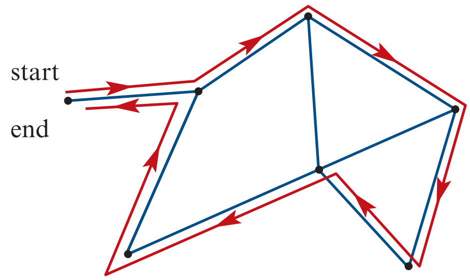
b



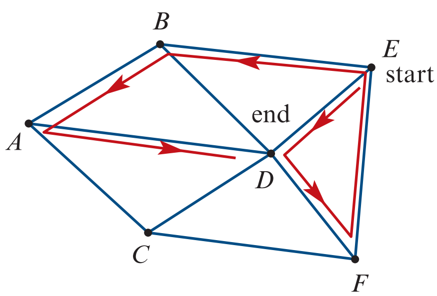
c



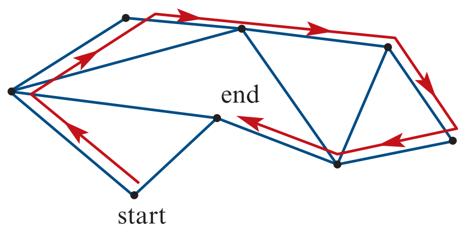
d



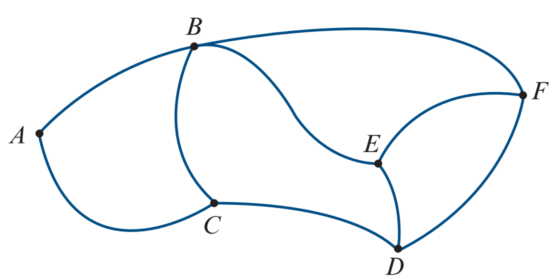
e



f



2 Using the graph opposite, identify the walks below as a trail, path, circuit, cycle or walk only.



a $A-B-E-B-F$

b $B-C-D-E-B$

c $C-D-E-F-B-A$

d $A-B-E-F-B-E-D$

e $E-F-D-C-B$

f $C-B-E-F-D-E-B-C-A$

g $F-E-B-C-A-B-F$

h $A-C-D-E-B-A$

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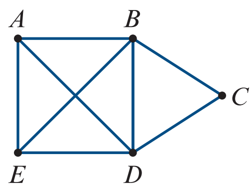
Eulerian trails and circuits

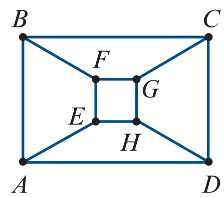
3 **i** Identify whether each graph below has an Eulerian circuit, an Eulerian trail, both or neither.

ii Name the Eulerian circuits or trails found.

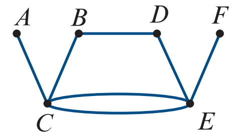
a



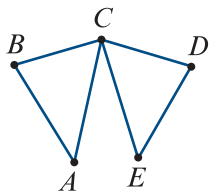
b



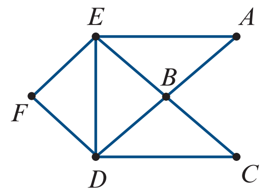
c



d



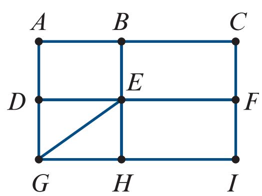
e



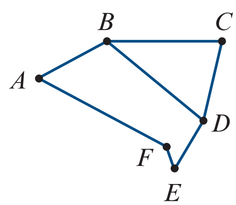
Hamiltonian paths and cycles

4 List a Hamiltonian cycle for each of the following.

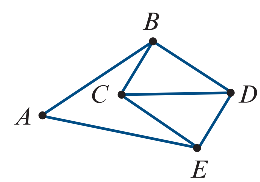
a



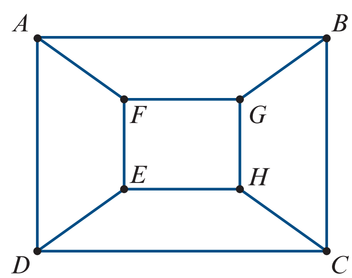
b



c



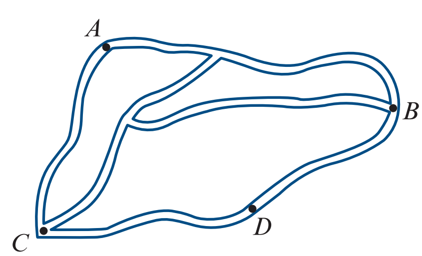
5 List a Hamiltonian path for this graph, starting at F and finishing at G.



Applications

Example 8

6 Four children each live in a different town. The diagram opposite is a map of the roads that link the four towns, A, B, C and D.



a How many different ways can a vehicle travel between town A and town B without visiting any other town?

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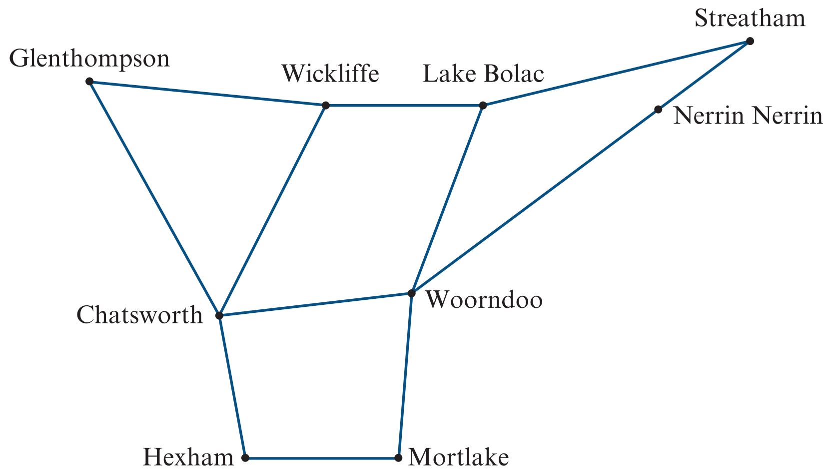
b How many different trails are there from town A to town D?

c Draw this map as a graph by representing towns as vertices and each different route between two towns as an edge.

d Explain why a vehicle at A could not follow an Eulerian circuit through this graph.

7 The following graph shows the roads linking nine Victorian country towns.



a Verify Euler's formula for this graph.

Cycling enthusiasts from the nine towns are planning a race that uses the roads linking the towns.

b The organisers who live in Lake Bolac want to visit each town to gain support for the race. They plan to visit each town on the same day but not pass through any town more than once. They will start and finish at Lake Bolac.

- **i** What name is given for the route they plan to take?
- **ii** Identify the two routes they can follow.

c In planning the race route, the organisers would like the cyclists to travel along each road linking the towns, but only once, and start and finish at Lake Bolac.

- **i** What name is given for the type of route they would like the cyclists to take?
- **ii** Explain why this cannot be done.

d The race planning in part **c** above can start and finish at Lake Bolac with only one road being travelled along twice.

- **i** Which road is this?
- **ii** Identify one possible route the race can follow starting and finishing at Lake Bolac, with only one road being travelled along twice.

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Properties of graphs

8 A graph has six vertices, A, B, C, D, E and F. The adjacency matrix for this graph is shown opposite.

	A	B	C	D	E	F
A	0	0	1	1	0	0
B	0	0	0	0	1	1
C	1	0	0	2	1	0
D	1	0	2	0	0	1
E	0	1	1	0	0	0
F	0	1	0	1	0	1

a Is the graph connected?

b Is the graph planar?

c Does the graph contain a bridge?

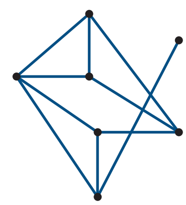
d Does the graph contain an Eulerian trail?

e Does the graph contain an Eulerian circuit?

f Does the graph contain a Hamiltonian path?

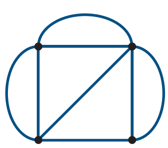
g Does the graph contain a Hamiltonian cycle?

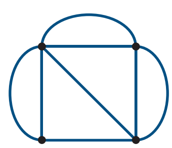
9 Consider the graph opposite.
What is the minimum number of edges that must be added for an Eulerian circuit to exist?

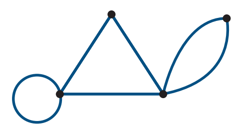


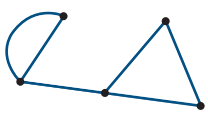
Exam 1 style questions

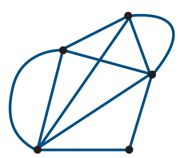
10 Which one of the following graphs has an Eulerian circuit?

A


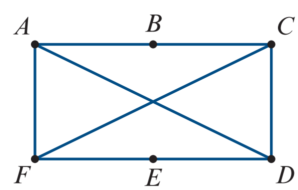
B


C


D


E


11 Consider the graph opposite. Which one of the following is **not** a Hamiltonian cycle for this graph?



A $ABCFEDA$

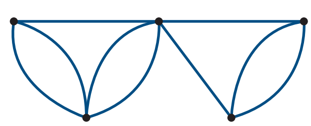
B $BADEFCB$

C $CDEFABC$

D $DEFACBD$

E $EFCBADE$

12 An Eulerian trail for the graph opposite will be possible if only one edge is removed. In how many different ways could this be done?



A 1 **B** 2 **C** 3 **D** 4 **E** 5

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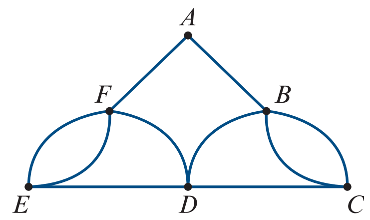
Use the following graph to answer questions 13 and 14.



13 Which one of the following is a Hamiltonian path for the graph above?

A $AFEDBA$ **B** $ABCDEF$ **C** $AFEDBCBA$
D $ABDFEDCBA$ **E** $FDEFABC$

14 The graph above will have a Eulerian circuit if an edge could be added between the vertices

A E and C **B** A and B **C** A and F
D A and D **E** F and B

13D Weighted graphs and networks

Learning intentions

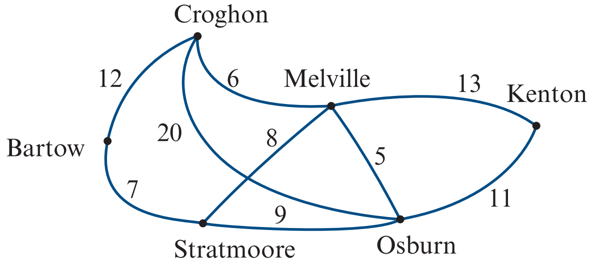
- To be able to analyse a weighted graph.
- To be able to find the shortest path between two vertices for a network.

Weighted graphs

The edges of graphs represent connections between the vertices. Sometimes there is more information known about that connection. If the edge of a graph represents a road between two towns, we might also know the length of this road, or the time it takes to travel this road.

Extra numerical information about the edge that connects vertices can be added to a graph by writing the number next to the edge. Graphs that have a number associated with each edge are called **weighted graphs**.

The weighted graph in the diagram on the right shows towns, represented by vertices, and the roads between those towns, represented by edges. The numbers, or *weights*, on the edges are the distances along the roads.



Weighted graphs in which the weights are physical quantities, for example distance, time or cost, are called **networks**.

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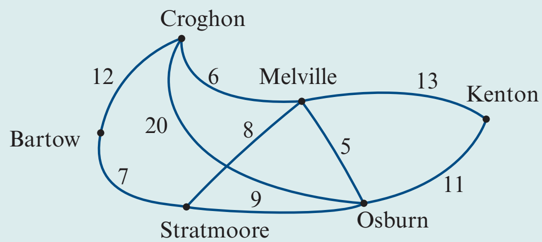
Shortest path problems

When we know numerical information about the connections, travelling through a graph will have extra considerations. If the weights of a network represent time, we can choose a route that will allow us to travel in the shortest time. If the weights represent distance, we can determine a route that will allow us to travel the shortest distance.

These types of problems involve finding the **shortest path** from one vertex to another. In networks that have only a few vertices, it is often easy to find the shortest path between two vertices by inspection. All of the possible route options should be listed, but it is sometimes obvious that certain routes are going to be much longer than others.

Example 9 Finding the shortest path from one vertex to another

Find the shortest path from Bartow to Kenton in the network shown on the right.



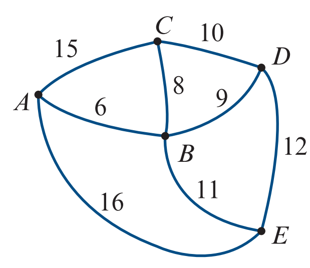
Explanation	Solution
1 List options for travelling from Bartow to Kenton.	B–S–M–O–K B–S–M–K B–S–O–K
2 Add the weights for each route.	$B–S–M–O–K$ $7 + 8 + 5 + 11 = 31$ km $B–S–M–K$ $7 + 8 + 13 = 28$ km $B–S–O–K$ $7 + 9 + 11 = 27$ km
3 Write your answer.	The shortest path from Bartow to Kenton is 27 km with route $B–S–O–K$.

Skill-sheet

Exercise 13D

Weighted graphs and networks

1 The network on the right shows towns A, B, C, D and E represented by vertices. The edges represent road connections between the towns. The weights on the edges are the average times, in minutes, it takes to travel along each road.



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a Which two towns are 12 minutes apart by road?

b How long will it take to drive from C to D via B?

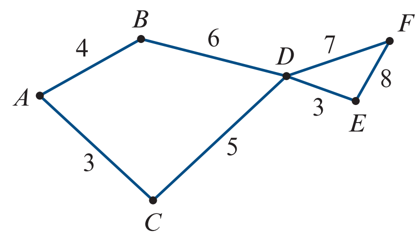
c A motorist intends to drive from D to E via B. How much time will they save if they travel directly from D to E?

d Find the shortest time it would take to start at A, finish at E and visit every town exactly once.

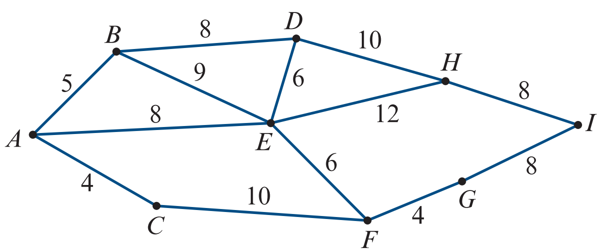
Shortest path by inspection

Example 9

2 By inspection, find the length of the shortest path from A to E.



3 The network on the right shows the distance, in kilometres, along walkways that connect landmarks A, B, C, D, E, F, G, H and I in a national park.



a What distance is travelled on the path A–B–E–H–I?

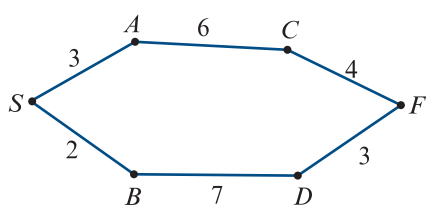
b What distance is travelled on the circuit F–E–D–H–E–A–C–F?

c What is the distance travelled on the shortest cycle starting and finishing at E?

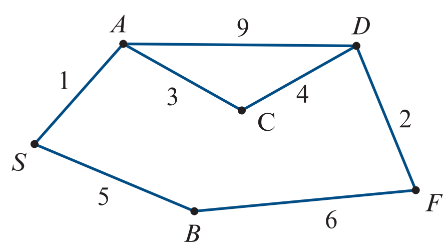
d Find the shortest path from A to I.

4 Determine the shortest path from S to F in the following weighted graphs. Write down the length of the shortest path.

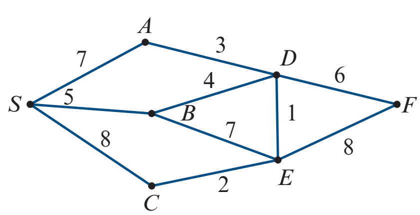
a



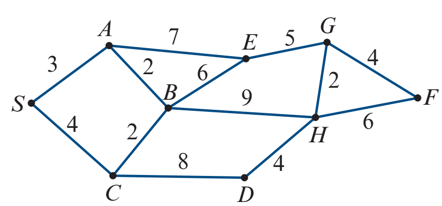
b



c



d



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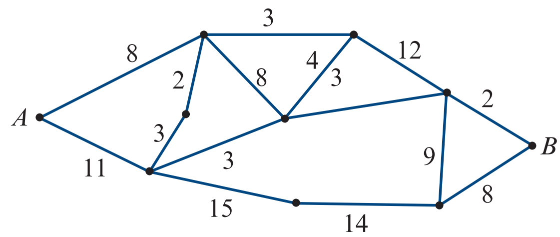
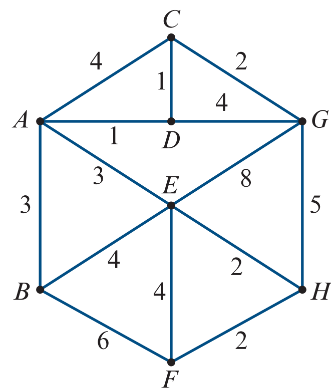
5 In the network opposite, the vertices represent small towns and the edges represent roads. The numbers on each edge indicate the distances (in kilometres) between towns.
Determine the length of the shortest path between the towns labelled A and B.



Exam 1 style questions

Use the following information to answer questions 6 and 7.



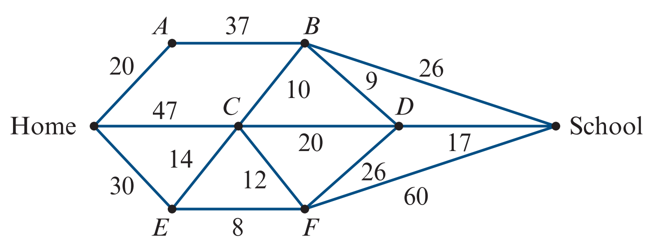
6 What is the shortest path between C and E?

A $C - A - E$ **B** $C - D - A - E$ **C** $C - D - G - E$
D $C - G - E$ **E** $C - G - H - E$

7 What is the shortest path between B and G?

A $B - E - G$ **B** $B - E - H - G$ **C** $B - A - C - G$
D $B - A - D - G$ **E** $B - A - D - C - G$

Use the following information to answer questions 8 and 9.



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8 Victoria rides her bike to school each day. The edges of the network on the previous page represent the roads that Victoria can use to ride to school from her home. The numbers on the edges give the time taken, in minutes, to travel along each road.
The fastest Victoria can ride from home to school is

A 80 minutes **B** 81 minutes **C** 83 minutes
D 84 minutes **E** 98 minutes

9 Which of the following represent the fastest route for Victoria's journey from home to school.

A *Home – A – B – School* **B** *Home – A – B – D – School*
C *Home – C – D – School* **D** *Home – C – B – D – School*
E *Home – E – C – B – D – School*

13E Dijkstra's algorithm

Finding the shortest path from one vertex of a graph to another is easy to determine if the graph is small and does not have too many vertices and edges. When there are many vertices and many edges, a systematic method, called an **algorithm**, can be used to find the shortest path.

Dutch computer scientist, Edsger Wybe Dijkstra (pronounced 'Dyke-stra') developed an algorithm for determining the shortest path through a graph. This algorithm, and others like it, have important applications to computerised routing and scheduling programs, such as GPS navigation devices.

Note: An alternative version is available online. Either method can be used. There is no curriculum requirement for you to know both methods.

The algorithm

You may choose to read through the example first to see a detailed implementation. Here we write the algorithm to emphasise its repetitive aspect.

Step 1: Assign the starting vertex a label of value zero and circle the vertex and the zero together.

Once a vertex and its label have been circled it cannot be changed

Step 2: Consider the vertex which has been most recently circled. Suppose this vertex to be X and the label of value d assigned to it. Then, in turn, consider each vertex directly joined to X but not yet permanently circled. For each such vertex, Y say, temporarily assign it with the value d + (the weight of edge XY) if Y does not have a temporary value or if it does, assign the lesser of d + (the weight of edge XY) and the existing temporary value.

Step 3: Choose the least of all temporary value labels on the network. Make this value label permanent by circling it.

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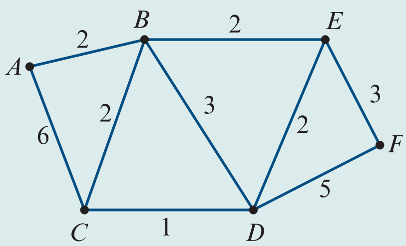
Step 4: Repeat Steps 2 and 3 until the destination node has a permanent label.

Step 5: Go backwards through the network, retracing the path of shortest length from the destination vertex to the starting vertex by

- starting at destination and go to the circled vertex with value = destination value − edge value.
- continuing to move back to the start vertex folowing this procedure.

Example 10 Using Dijkstra's algorithm to find the shortest path in a network: graphical method

Find the shortest path from A to F in the weighted graph shown on the right.

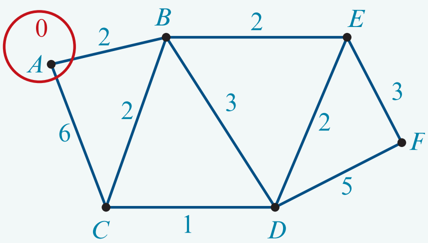


Explanation

Step 1

- Assign the starting vertex a zero and circle the vertex and its new value of zero.

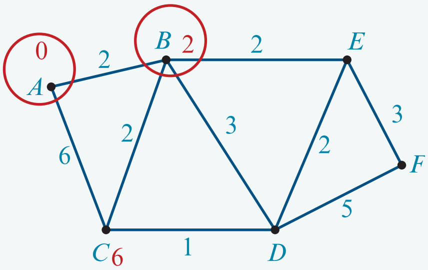
Solution



- A is the starting vertex, it is assigned zero and it is circled.

Step 2

- Assign a value to each vertex connected to the starting vertex. The value assigned is the length of the edge connecting it to the starting vertex.
- Circle the vertex with the lowest assigned value.



- The starting vertex A is connected to vertices B and C.
- The vertex B is assigned 2 and the vertex C is assigned 6.
- Vertex B is circled because it has the lowest value.

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Explanation

Step 3

- From the newly circled vertex, assign a value to each vertex connected to it by adding the value of each connecting edge to the newly circled vertex's value.
- If a connecting vertex already has a value assigned and the new value is less than it, replace it with the new value.
- If a vertex is circled, it cannot have its value changed.
- Consider all uncircled vertices and circle the one with the lowest value.

Step 4

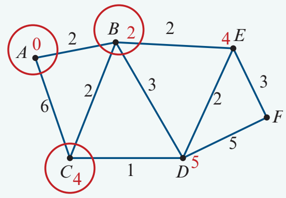
- Repeat **Step 3** until the destination vertex and its assigned value are circled.
- The length of the shortest path will be the assigned value of the destination vertex.

Step 5

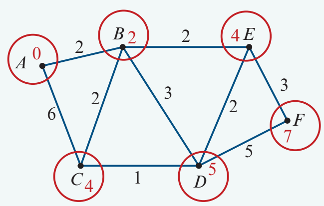
- The shortest path is found by backtracking. Starting at the destination vertex, move to the circled vertex whose value is equal to the destination vertex's assigned value, subtract the connecting edge value. Continue to subtract the connecting edge value from one circled vertex to the next until you reach the starting vertex.

Note: Once a vertex is assigned a value, it cannot be assigned a larger value, even if it has not been circled yet. You do not need to circle all vertices. Stop when the destination vertex is circled.

Solution



- The newly circled vertex B is connected to three vertices; C, D and E. Starting with vertex B's value of 2, E is assigned 4 (adding 2 from the connecting edge) and D is assigned 5 (adding 3 from the connecting edge).
- The vertex C will be re-assigned 4 (adding 2 from the connecting edge) because it is lower than 6.
- Now there are two uncircled vertices with the lowest assigned value 4, vertices C and E; it **does not** matter which one is circled. C is circled.



- Vertex F is the destination vertex, assigned a value of 7. Therefore the shortest path from A to F has a length of 7.
- To find the shortest path, start at F and consider the two connecting edges to it. The edge of length 3 is correct, because 7 minus 3 equals 4, the value of vertex E.
- Likewise from E, subtract the connecting edge of 2 to vertex B to equal 2, then subtract the connecting edge of 2 to A to equal zero.

Therefore the shortest path from A to F is:
$A - B - E - F$ with a length of 7.

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Skill-sheet

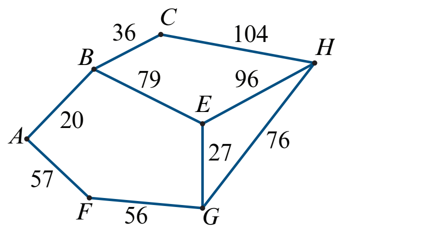
Exercise 13E

Calculations within Dijkstra's algorithm

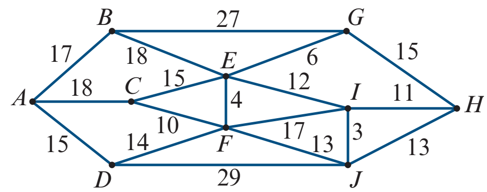
Example 10

1 Using Dijkstra's algorithm, determine the shortest path and write down the length of the shortest path from A to H in the following weighted graphs.

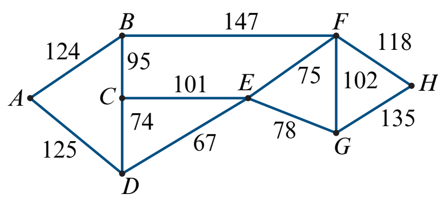
a



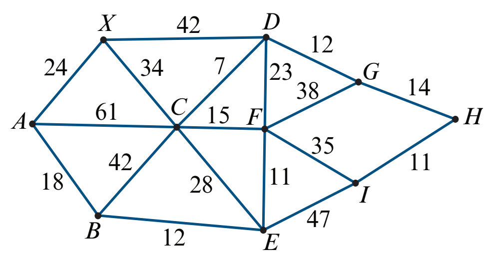
b



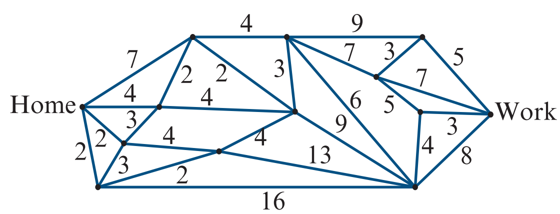
c



d



2 Renee drives to work each day. The edges of the network opposite represent the roads that Renee can use to drive to work. The numbers on the edges give the time, in minutes, to travel along each road. What is the shortest time that Renee can drive between home and work?



Note: See Chapter review, written-response question 1 for more problems using Dijkstra's algorithm.

13F Trees and minimum connector problems

Learning intentions

- ▶ To be able to identify a tree.
- ▶ To be able to find a spanning tree for a graph.
- ▶ To be able to find the minimum spanning tree for a network using Prim's algorithm.

In the previous applications of networks, the weights on the edges of the graph were used to determine a minimum pathway through the graph. In other applications, it is more important to minimise the number and weights of the edges in order to keep all vertices connected to the graph. For example, a number of towns might need to be connected to a water supply. The cost of connecting the towns can be minimised by connecting each town into a network of water pipes only once, rather than connecting each town to every other town.

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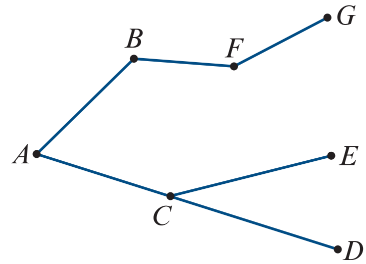
Problems of this type are called connector problems. In order to solve connector problems, you need to learn the language of networks that have as few edges as possible.

Trees

A **tree** is a connected graph that has no loops, multiple edges or cycles.

This tree has seven vertices and six edges.

The number of edges is always one less than the number of vertices.



Spanning trees

Every connected graph will have at least one subgraph that is a tree. A subgraph is a tree, and if that tree connects all of the vertices in the graph, then it is called a **spanning tree**.

Trees

A *tree* has no loops, multiple edges or cycles.

If a tree has n vertices, it will have $n - 1$ edges.

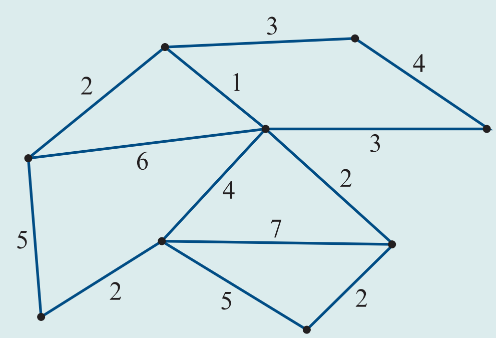
A *spanning tree* is a tree that connects all of the vertices of a graph.

There can be more than one spanning tree for any connected graph. The *total weight* of a spanning tree is the total of all the weights on the edges that make up the tree.

Example 11 **Finding the weight of a spanning tree**

a Draw one spanning tree for the graph shown.

b Calculate the weight of this spanning tree.



Explanation	Solution
a **1** Count the number of vertices and edges in the graph.	There are 9 vertices and 13 edges.
2 Calculate the number of edges in the spanning tree.	The spanning tree will have 8 edges.

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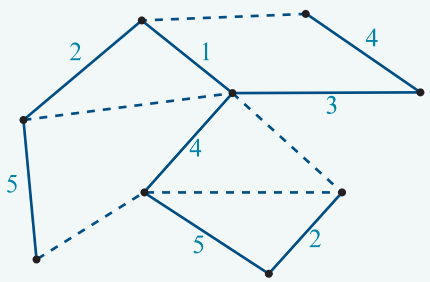
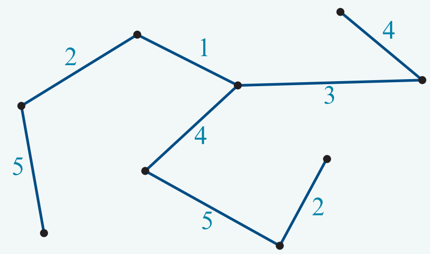
3 Calculate how many edges must be removed.

Remove 13 − 8 = 5 edges.

4 Choose edges to remove.



b Add the weights of the remaining edges.



Weight = 5 + 2 + 1 + 4 + 5 + 2 + 3 + 4
= 26

Minimum spanning trees

One of the spanning trees from a particular connected graph will have the *smallest* total weight. This tree is called the **minimum spanning tree**. Minimum spanning trees can be found using an algorithm called **Prim's algorithm**.

Prim's algorithm for finding a minimum spanning tree

- Choose a starting vertex (any will do).
- Inspect the edges starting from the starting vertex and choose the one with the smallest weight. (If there are two edges that have the same weight, it does not matter which one you choose). The starting vertex, the edge and the vertex it connects to form the beginning of the minimum spanning tree.
- Now inspect all of the edges starting from both of the vertices you have in the tree so far. Choose the edge with the smallest weight, ignoring edges that would connect the tree back to itself. The vertices and edges you already have, plus the extra edge and vertex it connects form the minimum spanning tree so far.
- Keep repeating this process until all of the vertices are connected.

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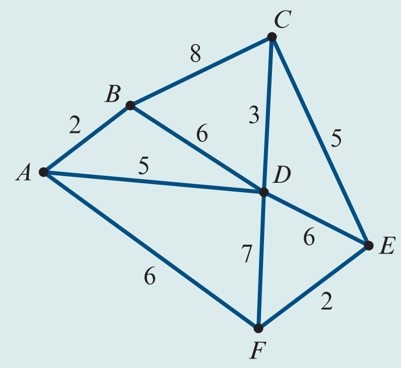
Example 12 Finding the minimum spanning tree

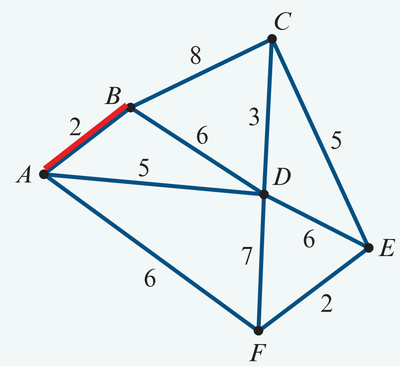
Apply Prim's algorithm to find the minimum spanning tree for the graph shown on the right. Write down the total weight of the minimum spanning tree.



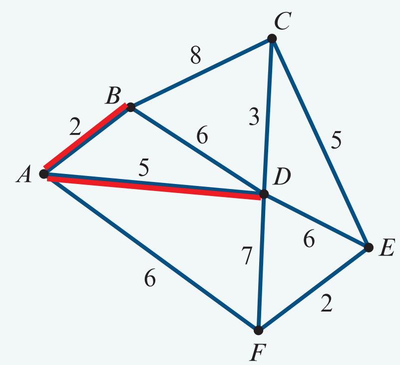
Explanation

Solution

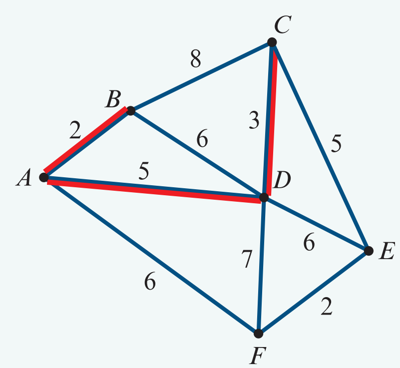
Start with vertex A.
The smallest weighted edge from vertex A is to B with weight 2.



Look at vertices A and B. The smallest weighted edge from either vertex A or vertex B is from A to D with weight 5.



Look at vertices A, B and D. The smallest weighted edge from vertex A, B or D is from D to C with weight 3.



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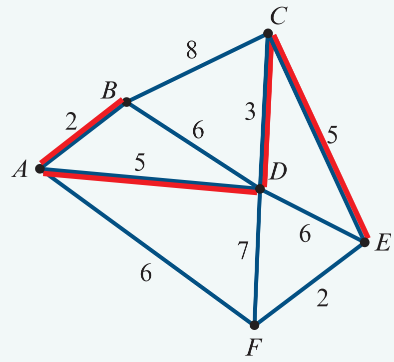
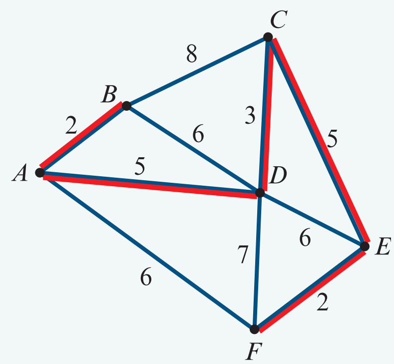
Look at vertices A, B, D and C. The smallest weighted edge from vertex A, B, D or C is from C to E with weight 5.



Look at vertices A, B, D, C and E. The smallest weighted edge from vertex A, B, D, C or E is from E to F with weight 2.
All vertices have been included in the graph. This is the minimum spanning tree.



Add the weights to find the total weight of the minimum spanning tree.

The total weight of the minimum spanning tree is $2 + 5 + 3 + 5 + 2 = 17$.

Minimum Connector problems

Minimum spanning trees represent the least weight required to keep all of the vertices connected in the graph. If the edges of a graph represent the cost of connecting towns to a gas pipeline, then the total weight of the minimum spanning tree would represent the minimum cost of connecting the towns to the gas. This is an example of a *connector problem*, where the cost of keeping towns or other objects connected together is important to make as low as possible.

Example 13 Solving a connector problem

Water is to be piped from a water tank to seven outlets on a property. The distances (in metres) of the outlets from the tank and from each other are shown in the network below.

Starting at the tank, the aim is to find the minimum length of pipe, in metres, which will be needed to have water piped to all outlets in the property.

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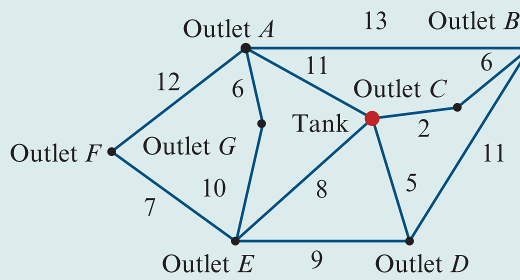


a On the diagram, show where the water pipes will be placed in order to minimise the length required.

b Calculate the total length, in metres, of the water pipe that is required to obtain this minimum length.

Explanation

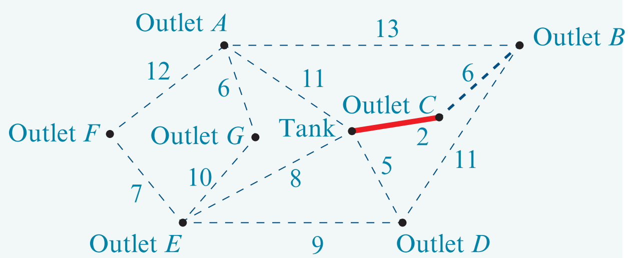
a **1** The water pipes will be a minimum length if they are placed on the edges of the minimum spanning tree for the network.

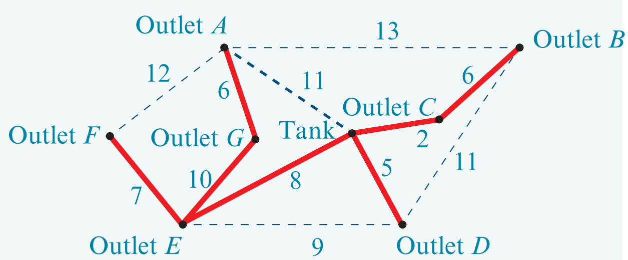
A starting point for Prim's algorithm is the vertex that is connected to the tank by the edge with the smallest weight. The starting vertex (Tank), the edge and the vertex it connects to form the beginning of the minimum spanning tree.

2 Follow Prim's algorithm to find the minimum spanning tree.

b Add the weights of the minimum spanning tree. Write your answer.

Solution





The length of water pipe required is 2 + 6 + 5 + 8 + 7 + 10 + 6 = 44 metres

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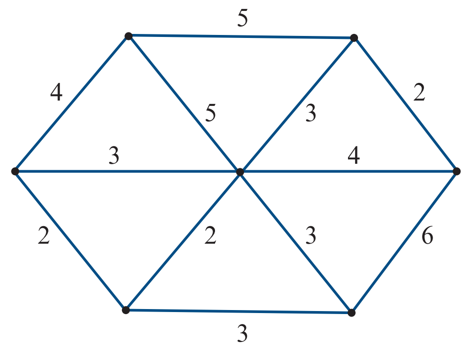
Skill-sheet

Exercise 13F

Spanning trees

Example 11

1 A weighted graph is shown on the right.



a How many edges must be removed in order to leave a spanning tree?

b Remove some edges to form three different trees.

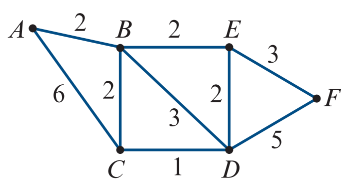
c For each tree in part **b**, find the total weight.

Minimum spanning trees and Prim's algorithm

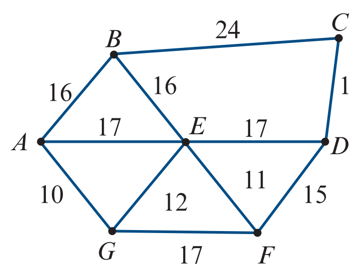
Example 12

2 Find a minimum spanning tree for each of the following graphs and give the total weight.

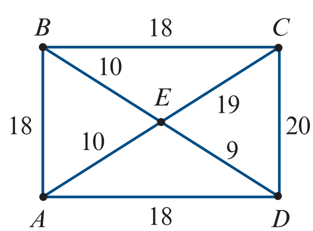
a



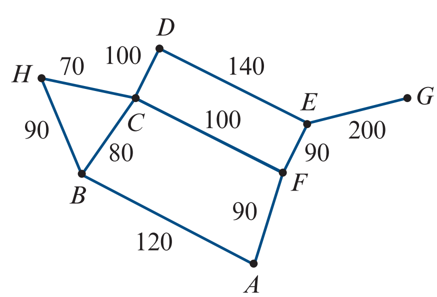
b



c



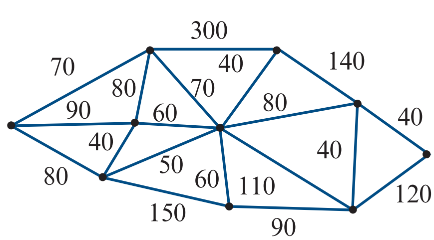
d



Connector problems

Example 13

3 In the network opposite, the vertices represent water tanks on a large property and the edges represent pipes used to move water between these tanks. The numbers on each edge indicate the lengths of pipes (in m) connecting different tanks.



Determine the shortest length of pipe needed to connect all water storages.

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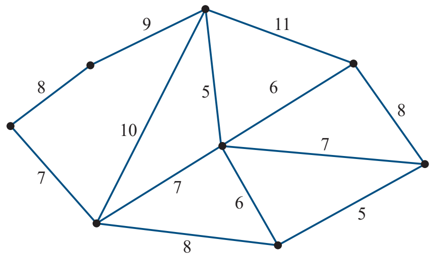
Exam 1 style questions

4 For the graph opposite, the length of the minimum spanning tree is

A 44 **B** 45 **C** 46

D 47 **E** 48



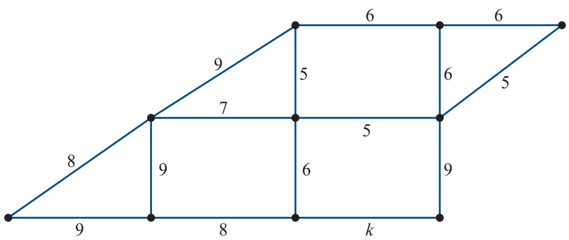
5 The minimum spanning tree for the graph below includes the edge with weight labelled k.

The total weight of all edges for the minimum spanning tree is 58.

The value of k is

A 6 **B** 7 **C** 8

D 9 **E** 10



6 The minimum spanning tree for the graph opposite includes two edges with weights a and b. The length of the minimum spanning tree is 124.

The values of a and b could be

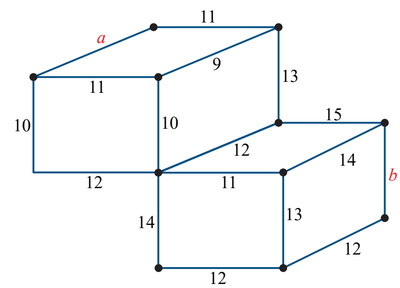
A $a = 6$ and $b = 17$

B $a = 12$ and $b = 12$

C $a = 10$ and $b = 14$

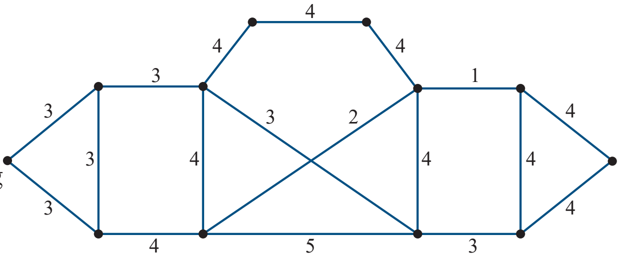
D $a = 10$ and $b = 15$

E $a = 13$ and $b = 12$



7 A minimum spanning tree is to be drawn for the weighted graph opposite. How many edges with weight 4 will **not** be included in any particular minimum spanning tree?

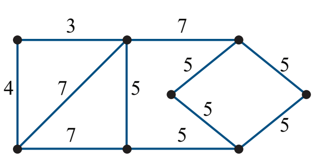
A 1 **B** 2 **C** 3 **D** 4 **E** 5



8 Consider the weighted graph opposite.

How many different minimum spanning trees are possible?

A 1 **B** 2 **C** 3 **D** 4 **E** 5



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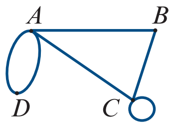
Review

Key ideas and chapter summary

Assignment

Graph	A **graph** is a diagram that consists of a set of points called **vertices** and a set of lines called **edges**. Each edge joins two vertices.
Edge	In the graph above, the lines joining A, B, C and D are edges.
Vertex	In the graph above, the points A, B, C, D are vertices.
Loop	A **loop** is an edge that connects a vertex to itself. In the graph above there is a loop at vertex C.
Degree of a vertex	The **degree of a vertex** is the number times edges attach to a vertex. The degree of vertex A is written as $\deg(A)$. In the graph above, $\deg(A) = 4$, $\deg(B) = 2$ and $\deg(D) = 2$. A loop has degree of 2. In the graph above, $\deg(C) = 4$.
Multiple edge	Sometimes a graph has two or more identical edges. These are called **multiple edges**. In the graph above, there are multiple edges between vertex A and vertex D.
Simple graph	**Simple graphs** are graphs that do not have loops and do not have multiple edges.
Isolated vertex	An **isolated vertex** is one that is not connected to any other vertex. Isolated vertices have degree of zero.
Degenerate graph	A **degenerate graph** has no edges. All of the vertices are isolated.
Connected graph	A **connected graph** has no isolated vertex. There is a path between each pair of vertices.
Bridge	A **bridge** is a single edge in a connected graph that, if it were removed, leaves the graph disconnected.
Complete graph	A **complete graph** has every vertex connected to every other vertex by an edge.
Subgraph	A **subgraph** is a graph that is part of a larger graph and has some of the same vertices and edges as that larger graph. A subgraph does not have any extra vertices or edges that do not appear in the larger graph.
Equivalent graph (isomorphic graph)	Graphs that contain identical information (connections between vertices) to each other are **equivalent graphs** or **isomorphic graphs**.



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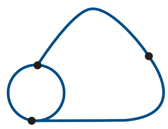
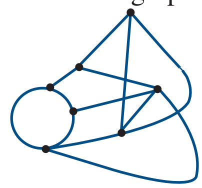
Face An area in a graph or network that can only be reached by crossing an edge. One such area is always the area surrounding a graph.

Planar graph A **planar graph** can be drawn so that no two edges overlap or intersect, except at the vertices.

This graph is planar.



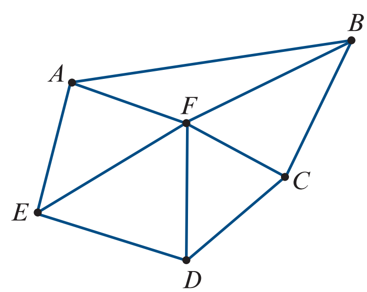
This graph is **not** planar.



Euler's formula **Euler's formula** applies to planar graphs.
If v = the number of vertices, e = the number of edges and f = the number of faces then $v + f = e + 2$

Adjacency matrix An **adjacency matrix** is a square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

Travelling Movement through a graph from one vertex to another along the edges is called **travelling** through the graph.



Walk A **walk** is a sequence of edges, linking successive vertices, in a graph.
In the graph above, $E–A–F–D–C–F–E–A$ is an example of a walk.

Trail A **trail** is a walk with no repeated edges.
In the graph above, $A–F–D–E–F–C$ is an example of a trail.

Path A **path** is a walk with no repeated vertices and no repeated edges.
In the graph above, $F–A–B–C–D$ is an example of a path.

Circuit A **circuit** is a trail (no repeated edges) that starts and ends at the same vertex.
In the graph above, $A–F–D–E–F–B–A$ is an example of a circuit.

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Cycle A **cycle** is a path (no repeated edges nor vertices) that starts and ends at the same vertex. The start and end vertex is an exception to repeated vertices.
In the graph above, $B–F–D–C–B$ is an example of a cycle.

Eulerian trail An **Eulerian trail** is a trail (no repeated edges) that includes all of the edges of a graph. Eulerian trails exist if the graph is connected and has exactly zero or two vertices of odd degree. The remaining vertices have even degree.

Eulerian circuit An **Eulerian circuit** is a trail (no repeated edges) that includes all of the edges of a graph and that starts and ends at the same vertex. Eulerian circuits exist if the graph is connected and has all of the vertices with an even degree.

Hamiltonian path A **Hamiltonian path** is a path (no repeated edges or vertices) that includes all of the vertices of a graph.

Hamiltonian cycle A **Hamiltonian cycle** is a path (no repeated edges or vertices) that starts and ends at the same vertex. The starting vertex is an exception to repeated vertices.

Weighted graph A **weighted graph** has numbers, called weights, associated with the edges of a graph. The weights often represent physical quantities as additional information to the edge, such as time, distance or cost.

Network A **network** is a weighted graph where the weights represent physical quantities such as time, distance or cost.

Shortest path The **shortest path** through a network is the path along edges so that the total of the weights of that path is the minimum for that network. Shortest path problems involve finding minimum distances, costs or times through a network. Shortest paths can be determined by inspection or by using Dijkstra's algorithm.

Dijkstra's algorithm **Dijkstra's algorithm** is an algorithm for determining the shortest path through a network from one vertex to another.

Tree A **tree** is a connected graph that contains no cycles, multiple edges or loops.
A tree with n vertices has $n - 1$ edges.

Spanning tree A **spanning tree** is a tree that connects every vertex of a graph.
A spanning tree is found by counting the number of vertices (n) and removing enough edges so that there are $n - 1$ edges left that connect all vertices.

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Minimum spanning tree A **minimum spanning tree** is a spanning tree for which the sum of the weights of the edges is as small as possible.

Prim's algorithm **Prim's algorithm** is an algorithm for determining the minimum spanning tree of a network.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ✓

13A **1** I can identify edges, vertices and loops in a graph. ☐

See Example 1, and Exercise 13A Question 1

13A **2** I can determine the degree of a vertex in a graph. ☐

See Example 1, and Exercise 13A Question 1

13A **3** I can define and identify simple graphs, isolated vertices, degenerate graphs, connected graphs, bridges and subgraphs. ☐

See Example 1, and Exercise 13A Question 1

13A **4** I can recognise isomorphic graphs. ☐

See Exercise 13A Question 3

13A **5** I can define planar graphs. ☐

See Example 2, and Exercise 13A Question 5

13A **6** I can redraw graphs in planar form. ☐

See Example 3, and Exercise 13A Question 4

13A **7** I can use Euler's formula. ☐

See Example 4, and Exercise 13A Question 6

13B **8** I can write an adjacency matrix from a graph. ☐

See Exercise 13B Question 1

13B **9** I can construct a graph from an adjacency matrix. ☐

See Example 5, and Exercise 13B Question 2

13C **10** I can define walks, trails, paths, circuits and cycles through a graph. ☐

See Example 6, and Exercise 13C Question 1

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Review

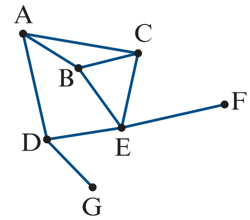
13C	11	I can identify Eulerian trails and circuits through graphs.	☐
		See Example 7, and Exercise 13C Question 3	
13C	12	I can determine whether an Eulerian trail or circuit exists in a graph.	☐
		See Example 7, and Exercise 13C Question 3	
13C	13	I can identify Hamiltonian paths and cycles through graphs.	☐
		See Example 7, and Exercise 13C Question 4	
13D	14	I can define a weighted graph.	☐
		See Exercise 13D Question 1	
13D	15	I can calculate the shortest path from one vertex to another by inspection.	☐
		See Example 8, and Exercise 13D Question 3	
13D	16	I can calculate the shortest path from one vertex to another using Dijkstra's algorithm.	☐
		See Example 9, and Exercise 13D Question 9	
13E	17	I can define tree, spanning tree, minimum spanning tree.	☐
		See Example 11, and Exercise 13E Question 1	
13E	18	I can draw a minimum spanning tree using Prim's algorithm.	☐
		See Example 12, and Exercise 13E Question 2	

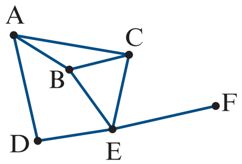
Multiple-choice questions

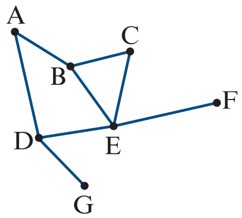
1 The minimum number of edges for a graph with seven vertices to be connected is:

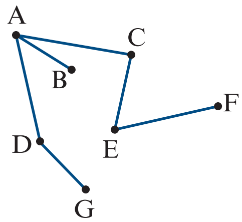
A 4 **B** 5 **C** 6 **D** 7 **E** 21

2 Which of the following graphs is a spanning tree for the network shown?



A


B


C


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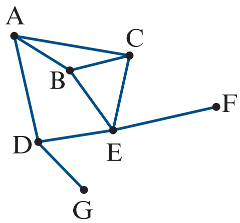
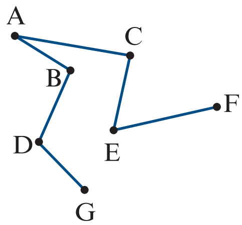
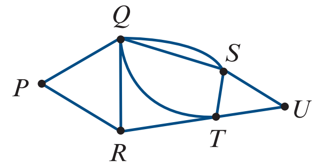
D 

E 

3 For the graph shown, which vertex has degree 5?

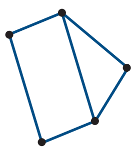
A *Q* **B** *T* **C** *S* **D** *R* **E** *U*

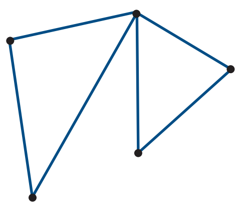


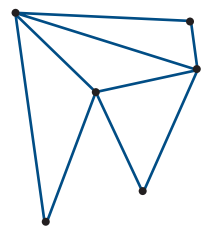
4 A connected graph with 15 vertices divides the plane into 12 regions. The number of edges connecting the vertices in this graph will be:

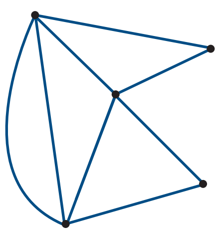
A 15 **B** 23 **C** 24 **D** 25 **E** 27

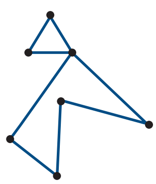
5 Which of the following graphs does *not* have an Eulerian circuit?

A 

B 

C 

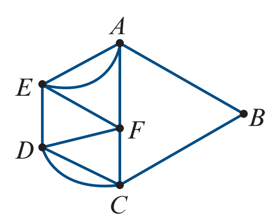
D 

E 

6 A connected planar graph divides the plane into a number of regions. If the graph has eight vertices and these are linked by 13 edges, then the number of regions is:

A 5 **B** 6 **C** 7 **D** 8 **E** 10

7 For the graph shown, which of the following paths is a Hamiltonian cycle?



A *A–B–C–D–C–F–D–E–F–A–E–A*

B *A–E–F–D–C–B–A*

C *A–F–C–D–E–A–B–A*

D *A–B–C–D–E–A*

E *A–E–D–C–B–A–F*

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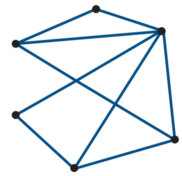
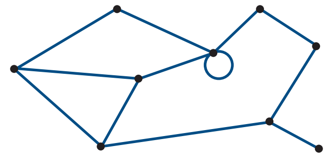
Review

8 The graph opposite has:



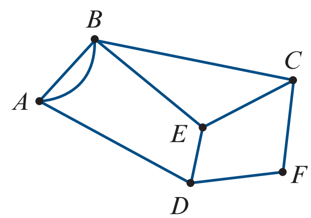
A four faces **B** five faces
C six faces **D** seven faces
E eight faces

9 The sum of the degrees of the vertices on the graph shown here is:



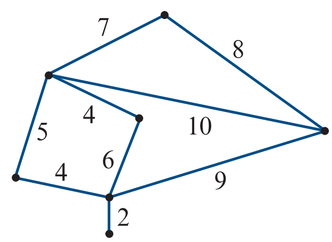
A 20 **B** 21 **C** 22
D 23 **E** 24

10 For the graph shown, which additional edge could be added to the network so that the graph formed would contain an Eulerian trail?



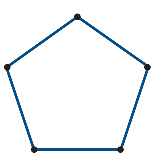
A A–F **B** D–E **C** A–B
D C–F **E** B–F

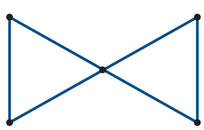
11 For the graph shown here, a minimum spanning tree has length:

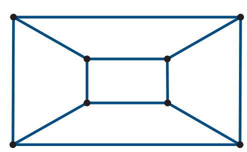


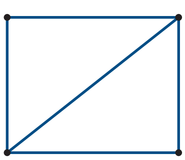
A 30 **B** 31 **C** 33
D 34 **E** 26

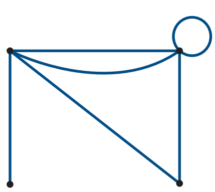
12 Of the following graphs, which one has both Eulerian circuit and Hamiltonian cycles?

A


B


C


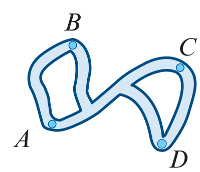
D


E


13 A graph with six vertices is connected with the minimum number of edges. The minimum number of extra edges needed to make this a complete graph is

A 5 **B** 6 **C** 10 **D** 14 **E** 16

14 Four towns, A, B, C and D, are linked by roads as shown. Which of the following graphs could be used to represent the network of roads? Each edge represents a route between two towns.



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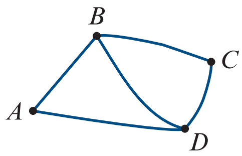
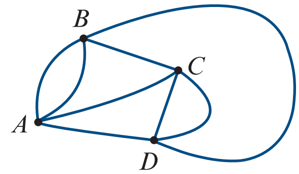
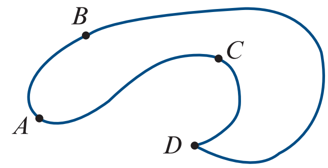
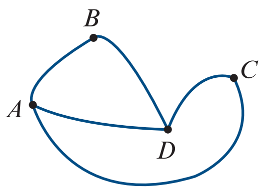
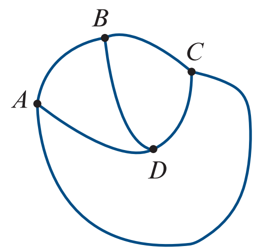
A


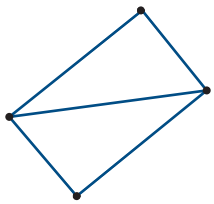
B


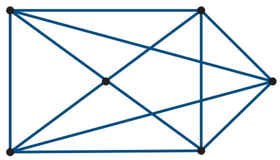
C


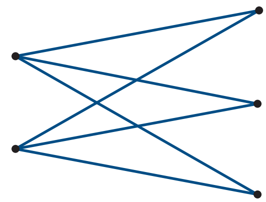
D


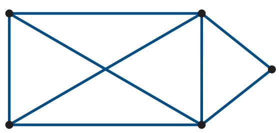
E


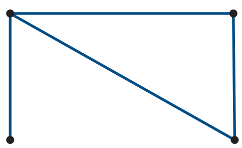
15 Which *one* of the following graphs has an Eulerian circuit?

A


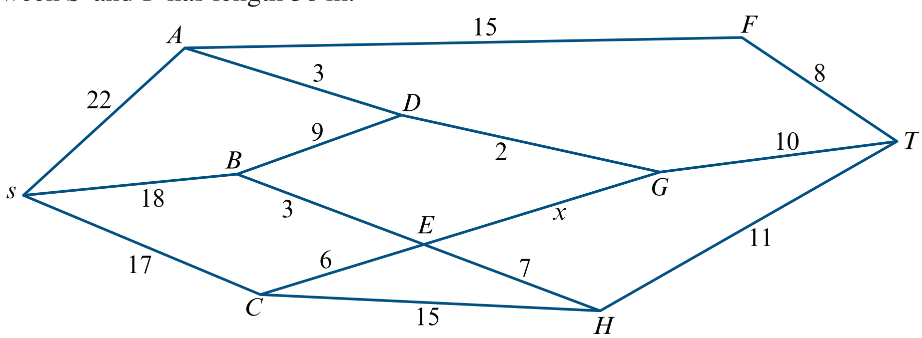
B


C


D


E


16 The network below shows the distance, in metres, between points. The shortest path between S and T has length 36 m.



The value of x is.

A 4 **B** 5 **C** 6 **D** 7 **E** 8

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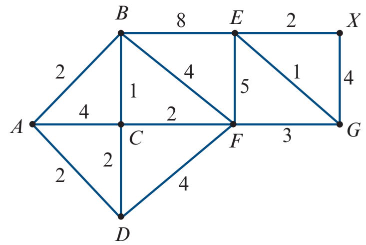
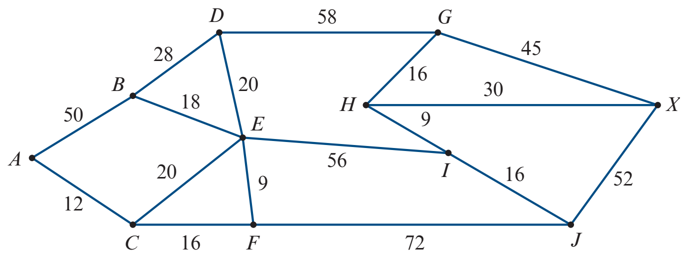
Written response questions

1 Determine the shortest path and write down the length of the shortest path from vertex A to vertex X in the following weighted graphs.

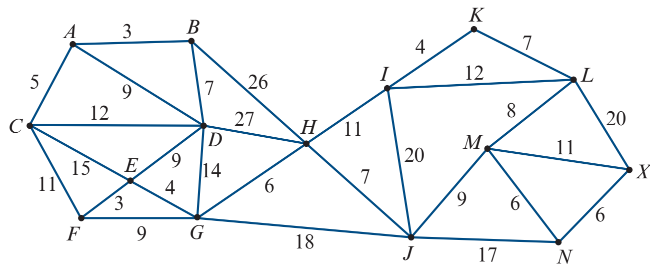
a



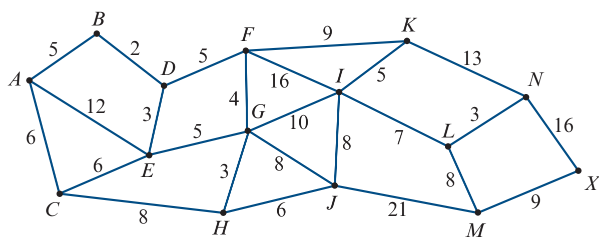
b



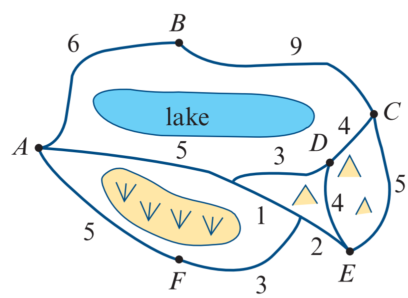
c



d



2 The map shows six campsites, A, B, C, D, E and F, which are joined by tracks. The numbers by the paths show lengths, in kilometres, of that section of track.



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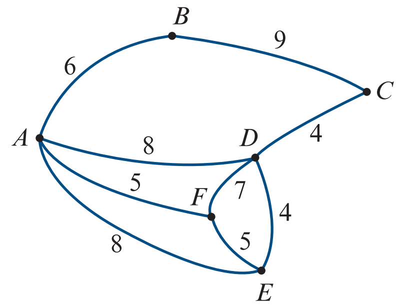
a **i** Complete the graph opposite, which shows the shortest direct distances between campsites. (The campsites are represented by vertices and tracks are represented by edges.)



ii A telephone cable is to be laid to enable each campsite to phone each other campsite. For environmental reasons, cables can only be laid along the tracks and cables can only connect to one another at the campsites. What is the minimum length of cable necessary to complete this task?

iii Fill in the missing entries for the adjacency matrix shown for the completed graph formed above.

	A	B	C	D	E	F
A	0	1	0	1	1	1
B	1	0	1	0	0	0
C	0	1	0	1	1	0
D	1	0	–	–	–	–
E	1	0	–	–	–	–
F	1	0	–	–	–	–

b A walker follows the route $A–B–A–F–E–D–C–E–F–A$.

i How far does this person walk?

ii Why is the route *not* a Hamiltonian cycle?

iii Write down a route that a walker could follow that is a Hamiltonian cycle.

iv Find the distance walked in following this Hamiltonian cycle.

c It is impossible to start at A and return to A by going along each track exactly once. An extra track joining two campsites can be constructed so that this is possible. Which two campsites need to be joined by a track to make this possible? (Two tracks are only considered the same if they share their entire length. For example, A–E and F–E are considered different tracks even though they share a common stretch near E.)

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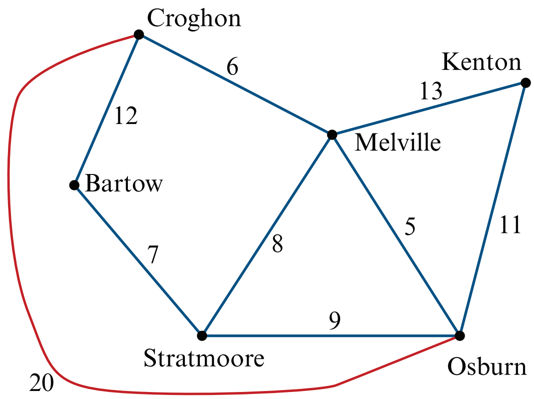
Review

3 The network on the right shows six villages represented as vertices of the graph. The edges represent the roads connecting the villages. The weights on the edges are the distances, in kilometres, alog each of the roads.



a What is the degree of the vertex representing Melville?

b Determine the sum of the degrees of the vertices of this graph.

c Verify Euler's formula for this graph.

A salesperson might need to travel to every village in this network to conduct business.

d If the salesperson follows the path Stratmoore – Melville – Kenton– Osburn – Melville– Croghon – Bartow – Stratmoore, has the salesperson followed a Hamiltonian cycle? Give a reason to justify your answer.

e If the salesperson follows the path Croghorn– Bartow – Stratmoore – Melville– Kenton – Osburn, what is the mathematical term for this path?

It would make sense for the salesperson to avoid visiting a certain village more than once, and it would also make sense for them to return 'home' after travelling the shortest distance possible.

f If the salesperson starts and ends in Bartow, find the shortest route and state the shortest distance the salesperson would have to travel.

g If the salesperson can start and end at any village in the network, what is the shortest route possible?

A road inspector must travel along every road connecting the six villages.

h Explain why the inspector could not follow an Eulerian circuit through this road network.

i The inspector may start and end their route at different villages, but would like to travel along each road once only. Which villages can the inspector start their route from? Write down a path the inspector could take to complete their work.

j The speed limit for each of these roads is 60 km/hr. If the inspector must complete their work by 5 p.m, what is the latest time that the inspector can begin their work?

New electrical cables connecting the villages are required. They will be installed along some of the roads listed in the graph above. These cables will form a connected graph and the shortest total length of cable will be used.

k Give a mathematical term to describe a graph that represents these cables.

l Draw the graph that represents these cables and find the total length of cable required.

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