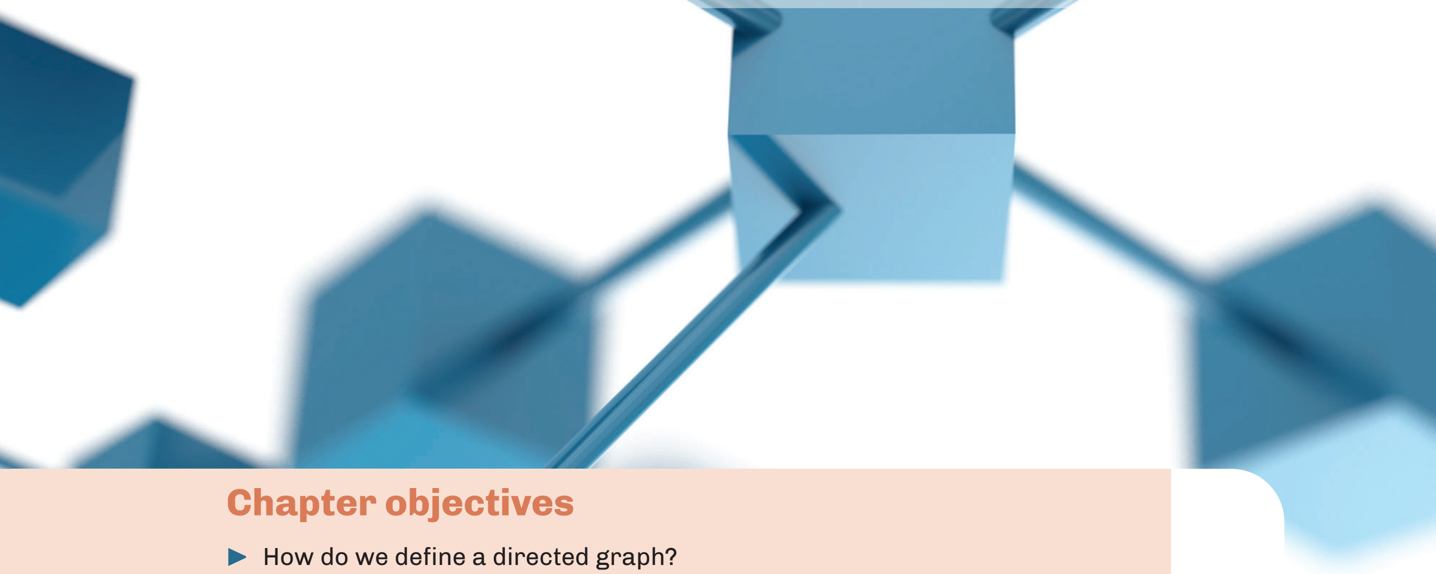
Chapter 14

# Flow, matching and scheduling problems



## Chapter objectives

- How do we define a directed graph?
- How do we define flow?
- How do we calculate the maximum flow through a network?
- How do we draw and use a bipartite graph to solve allocation problems?
- How do we find the optimal allocation of multiple groups of objects?
- How do we identify predecessors of an activity?
- How do we draw an activity network and use it to plan for a project?
- How do we account for float times in our project?
- How do we find the earliest starting time and latest finishing time for an activity in a project?
- How do we identify the critical path of an activity network?

In the previous chapter, undirected graphs were used to define and represent situations. In this chapter, directed graphs will be used to model networks and solve problems involving travel, connection, flow, matching, allocation and scheduling.

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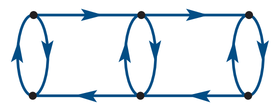
# 14A Flow problems

**Learning intentions**

- To be able to define and describe flow.
- To be able to calculate the maximum flow through a flow network by observation.
- To be able to identify cuts and calculate cut capacities.
- To be able to determine the maximum flow through a flow network by finding the capacity of the minimum cut.
- To be able to solve flow problems by finding minimum cut capacities.

## Directed graphs

In the previous chapter, graphs were used to represent connections between people, places or objects. The vertices of a graph represented objects, such as towns, and edges represented the connections between them, such as roads. **Weighted graphs** included extra numerical information about the connections, such as distance, time or cost. When a graph has this numerical information we call it a **network**.



A **directed graph**, or **digraph**, records directional information on networks using arrows on the edges. The network on the right shows roads around a city. The vertices are the intersections of the roads and the edges are the possible road connections between the intersections. The arrows show that some of the roads only allow traffic in one direction, while others allow traffic in both directions.

## Understanding minimum flow

One of the applications of directed graphs to real-life situations is flow problems. Flow problems involve the transfer or **flow** of material from one point, called the **source**, to another point called the **sink**. Examples of this include water flowing through pipes, or traffic flowing along roads.

source → flow through network → sink

Water flows through pipes in only one direction. In a digraph representing water flow, the vertices are the origin and destination of the water and the edges represent the pipes connecting them. The weights on the edges would be the amount of water that can flow through the pipe in a given time. The weights of flow problem directed graphs are called **capacities**.

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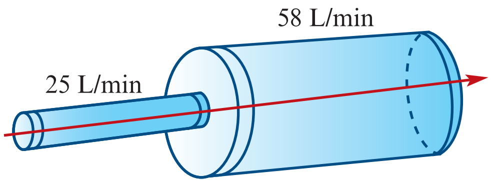
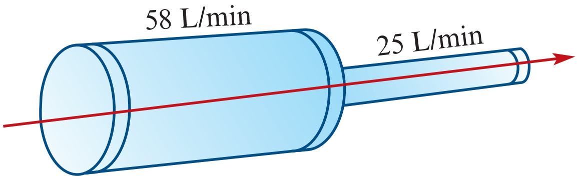
The diagram on the right shows two pipes that are joined together, connecting the source of water to the sink. There is a small pipe with capacity 25 litres per minute joined to a large pipe with capacity 58 litres per minute.




Even though the large pipe has a capacity greater than 25 litres per minute, the small pipe will only allow 25 litres of water through each minute. The flow through the large pipe will never be more than 25 litres per minute. The large pipe will experience flow below its capacity.

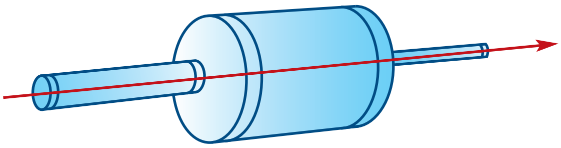
If we reverse the connection and direct water through the large capacity pipe into the smaller capacity pipe, there will be a 'bottleneck' of flow at the junction.




The large capacity pipe is capable of delivering 58 litres of water every minute to the small pipe, but the small pipe will only allow 25 litres per minute to pass.

In both of these situations, the flow through the entire pipe system (both pipes from source to sink) is restricted to a maximum of 25 litres per minute. This is the capacity of the smallest pipe in the connection.

If we connect more pipes together, one after the other, we can calculate the overall capacity or **maximum flow** of the pipe system by looking for the *smallest capacity pipe* in that system.



### Maximum flow

If pipes of different capacities are connected one after the other, the *maximum flow* through the pipes is equal to the *minimum* capacity of the individual pipes.

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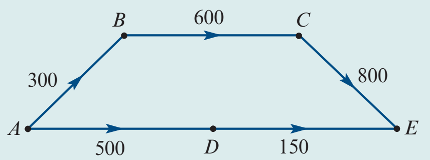
## Example 1 Calculating the maximum flow

In the digraph shown on the right, the vertices $A$, $B$, $C$, $D$ and $E$ represent towns. The edges of the graph represent roads and the weights of those edges are the maximum number of cars that can travel on the road each hour. The roads allow only one-way travel.




**a** Find the maximum traffic flow from $A$ to $E$ through town $C$.

**b** Find the maximum traffic flow from $A$ to $E$ overall.

**c** A new road is being built to allow traffic from town $D$ to town $C$. This road can carry 500 cars per hour.

  **i** Add this road to the digraph.

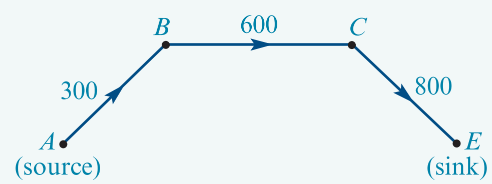
  **ii** Find the maximum traffic flow from $A$ to $E$ overall after this road is built.

### Explanation

**a** Look at the subgraph that includes town $C$. The smallest capacity of the individual roads is 300 cars per hour. This will be the maximum flow through town $C$.

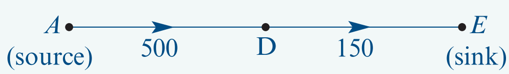
The maximum flow from $A$ to $E$ through town $C$ is equal to the smallest capacity road along that route.

### Solution



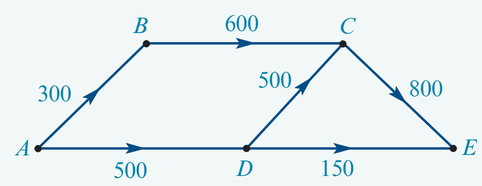

The maximum flow is 300 cars per hour.

**b** Look at the two subgraphs from $A$ to $E$. The maximum flow through $D$ will be 150 cars per hour (minimum capacity). Add the maximum flow through $C$ to the maximum flow through $D$.




The maximum flow from $A$ to $E$ overall is:
$300 + 150 = 450$ cars per hour

**c** **i** Add the edge to the diagram.




**ii** The maximum flow through A–B–C–E is 300. But C–E has capacity 800. If another 500 cars per hour come through A–D–C, they will also be able to travel from C to E.

The new maximum flow is now 800 cars per hour.

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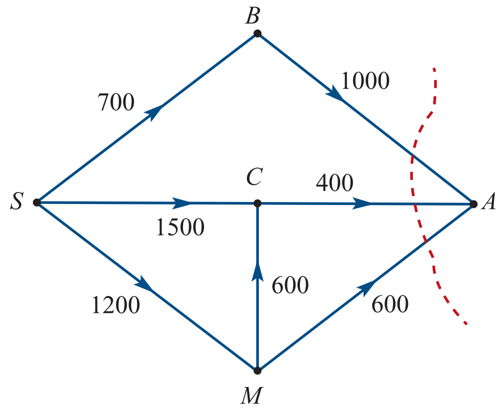
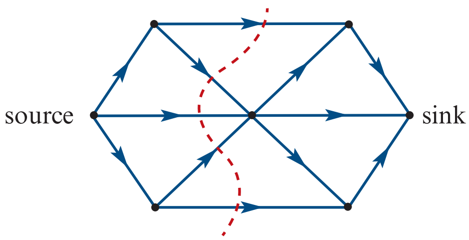
## Cuts

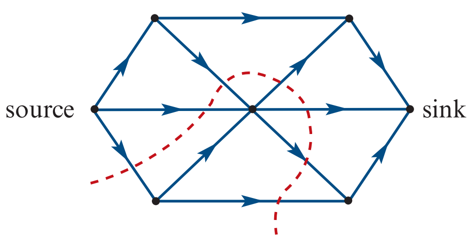
It is difficult to determine the maximum flow by inspection for directed networks that involve many vertices and edges. We can simplify the search for maximum flow by searching for **cuts** within the digraph.

A cut divides the network into two parts, completely separating the source from the sink. It is helpful to think of cuts as imaginary breaks within the network that completely block the flow through that network. For the network or water pipes shown in this diagram, the dotted line is a cut. This cut completely blocks the flow of water from the source ($S$) to the sink ($A$).




The dotted line on the graph above is a valid cut because it separates the source and the sink completely. No material can flow from the source to the sink.

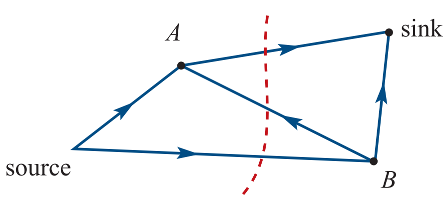



The dotted line on the graph above is *not* a valid cut because material can still flow from the source to the sink. Not all of the pathways from source to sink have been blocked by the cut.

### Capacity of a cut

The **cut capacity** is the sum of all the capacities of the edges that the cut passes through, taking into account the direction of flow. The capacity of an edge is only counted if it flows from the source side to the sink side of the cut.

In the simple network shown, the cut passes through three edges. The edge $B$ to $A$ is not counted in the capacity of the cut because the flow for that edge is from the sink side to the source side of the cut.

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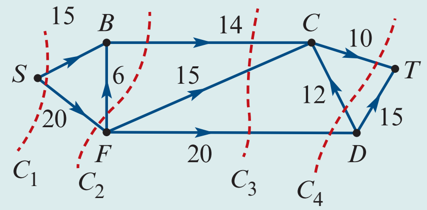
## Example 2 Calculating cut capacity

Calculate the capacity of the four cuts shown in the network on the right. The source is vertex $S$ and the sink is vertex $T$.




**Explanation**

All edges in $C_1$ are counted.

Note that the edge from $F$ to $B$ is not counted in $C_2$.

All edges in $C_3$ are counted.

Note that the edge from $D$ to $C$ is not counted in $C_4$.

**Solution**

The capacity of $C_1 = 15 + 20 = 35$

The capacity of $C_2 = 14 + 20 = 34$

The capacity of $C_3 = 14 + 15 + 20 = 49$

The capacity of $C_4 = 20 + 10 = 30$

The capacity of a cut is important to help determine the maximum flow through any digraph. Look for the smallest, or minimum, cut capacity that exists in the graph. This will be the same as the maximum flow that is possible through that graph. This is known as the *maximum-flow minimum-cut theorem*.

### Cut, cut capacity and minimum cut capacity

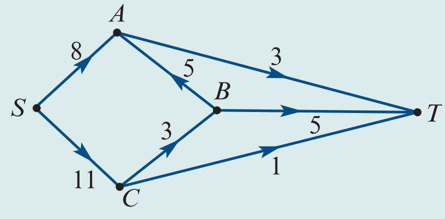
A *cut* is an imaginary line across a directed graph that completely separates the *source* (start of the flow) from the *sink* (destination of the flow).

The *cut capacity* is the sum of the capacities of the edges that are cut. Only edges that flow from the source side of the cut to the sink side of the cut are included in a cut capacity calculation.

The ***minimum cut*** *capacity* possible for a graph equals the *maximum flow* through the graph.

## Example 3 Calculating maximum flow using cuts

Determine the maximum flow from $S$ to $T$ for the digraph shown on the right.

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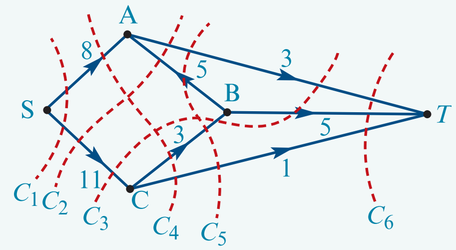
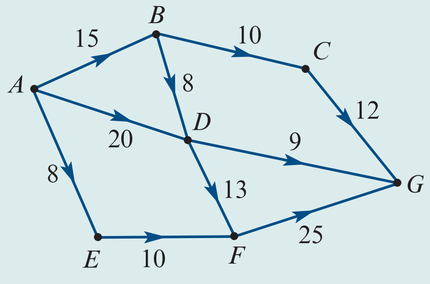
| Explanation | Solution |
|---|---|
| **1** Mark in all possible cuts on the network. |  |
| **2** Calculate the capacity of all the cuts. | The capacity of $C_1 = 8 + 11 = 19$<br>The capacity of $C_2 = 3 + 11 = 14$<br>The capacity of $C_3 = 3 + 5 + 11 = 19$<br>The capacity of $C_4 = 8 + 3 + 1 = 12$<br>The capacity of $C_5 = 3 + 3 + 1 = 7$<br>The capacity of $C_6 = 3 + 5 + 1 = 9$ |
| **3** Identify the minimum cut capacity and write your answer. | The minimum cut capacity is 7 so the maximum flow from S to T is 7. |

## Calculating maximum flow by tracking flow through a network (Optional)

There is another method we can use to calculate the maximum flow through a network. This method relies on tracking the flow through every edge.

### Example 4 Calculating maximum flow

The koala sanctuary in Cowes allows visitors to walk through their park. The park is represented by a network below, where each edge represents one-way tracks for visitors through the park. The direction of travel on each track is shown by an arrow. The numbers on the edges indicate the maximum number of people who are permitted to walk along each track each hour.




**a** Starting at $A$, how many people are permitted to walk to $G$ each hour?

**b** Given that one group of nine people would like to walk from $A$ to $G$ together as a group, list all the different routes they could take so that the entire group of nine will stay together for the duration of their walk.

**c** What is the largest group of people that could walk through the koala sanctuary if they must stay together in a group for the entire duration of the walk?

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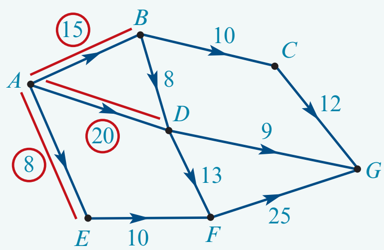
**Explanation**

**a** Firstly, consider the edges coming from the source. When calculating the maximum flow through a network, always assume the initial edges from the source are flowing at their maximum capacity.

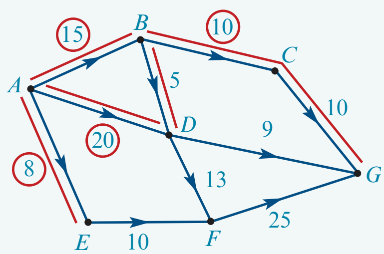
**Solution**

The vertex $A$ is the source in this network. The three edges connected to $A$, flowing towards $B$, $D$ and $E$ will be flowing at maximum capacity because they are coming from the source. Draw lines along these edges and circle their capacities as they are flowing at maximum capacity.

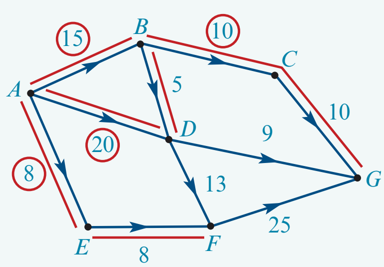



Now, from the vertices accepting flow from the source, start observing the flow through the network and if the maximum capacity of each edge can be achieved via direct flow from another edge or by splitting the flow of one edge into two.

At vertex $B$, although there is a flow of 15 coming in from the source, the edges taking flow towards the sink at $G$ are of different capacities. From $B$ to $C$ only a capacity of 10 can flow, therefore the same 10 can also only flow from $C$ to $G$ as no other edges are connected in this route. From $B$, as 10 flows to $C$, the leftover 5 can flow to $D$. Even though the edge from $B$ to $D$ has a maximum capacity of 8, there is only 5 available to flow through this edge; cross off 8 and write a 5 next to it.




At vertex $E$, a flow of 8 is coming from the source. Although the edge from $E$ to $F$ has a capacity of 10, only a maximum of 8 can flow through it. Just as before, cross out the 10 and write an 8 next to it to signify the flow passing through the edge.

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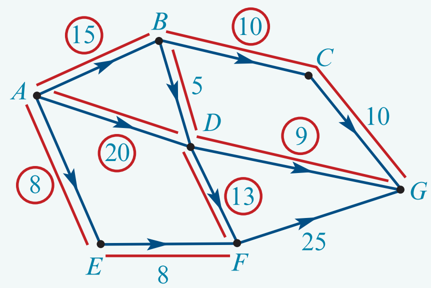
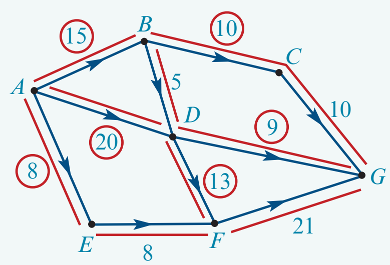
| Explanation | Solution |
|---|---|
| | At vertex $D$ there is a total of 25 flowing into it, coming from vertices $B$ and $A$. This flow of 25 can be redistributed to the two edges coming from $D$ towards the sink at $G$. Of the 25, 9 can flow directly to $G$ and 13 can flow from $D$ to $F$. The maximum capacities of these edges can be achieved, so circle these numbers along the edges.   Finally, the edge from $F$ to $G$ has a maximum capacity of 25, however only 21 $(8 + 13 = 21)$ is coming through. Cross out 25, replace with 21 to indicate the correct flow through the edge.   The maximum flow through the network is the total amount incoming to the *sink* vertex $G$, which is $10 + 9 + 21 = 40$. Therefore a maximum of 40 people can walk through the koala sanctuary each hour. |
| **b** A group of 9 must begin at vertex $A$, only pass through edges with a capacity greater than or equal to 9 and end at vertex $G$. Take note of the direction of the arrow heads. | Starting from vertex $A$ there are only two edges the group of 9 people can walk along; the edges going to vertices $B$ and $D$. Walking through vertex $B$ there is one walk possible: $A - B - C - G$. Walking through vertex $D$ there are two walks possible: $A - D - G$ and $A - D - F - G$. |

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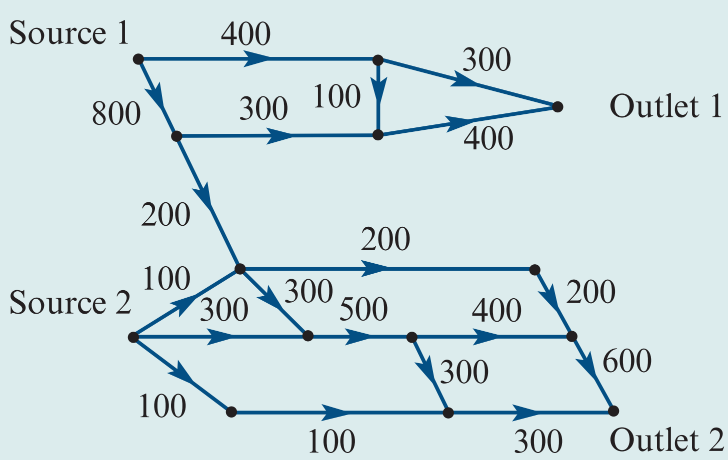
| Explanation | Solution |
| --- | --- |
| **c** From the starting vertex $A$, consider the largest possible group that could start a walk through the sanctuary and then analyse how many of that group could then walk to vertex $G$ given the capacities of each of the edges throughout the network. | Starting at vertex $A$ the largest possible group of people that can enter the sanctuary is 20, however after reaching vertex $D$ the group of 20 cannot stay together as they move towards vertex $G$ because the capacity of the edges from $D$ can only take a maximum of 13 people. From $D$, moving towards $G$ a group of 13 could pass through with no other restrictions. This is the largest group of people that can enter the sanctuary at vertex $A$ and pass through to vertex $G$, together as one group, given the restrictions of the edge capacities. |

## Calculating maximum flow from more than one source

### Example 5 Calculating maximum flow from more than one source

Water enters a network of pipes at either Source 1 or Source 2 and flows out at either Outlet 1 or Outlet 2.

The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which water can flow through each pipe.




Determine the maximum rate, in kilolitres per minute, at which water can flow from these pipes into the ocean at Outlet 1 and Outlet 2.
Note that although this method gives us the maximum flow for each outlet, we cannot always add these values up to find the total maximum flow through the system, because we might not be able to achieve maximum flow for every outlet at the same time.

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**Explanation**

The outlets need to be considered separately.

### Outlet 1

Look for the minimum cut that prevents water reaching Outlet 1.

Note: The pipe with capacity 200 leading towards Outlet 2 does not need to be considered in any cut because this pipe *always* prevents water from reaching Outlet 1.

**Solution**

The capacity of $C_1$ is: 400 + 800 = 1200

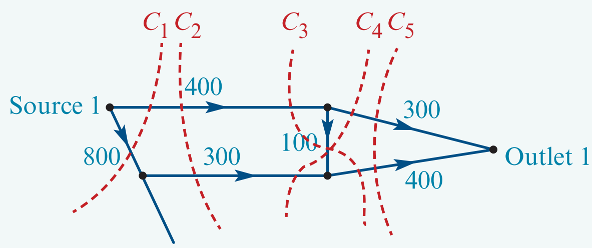
The capacity of $C_2$ is: 400 + 300 = 700

The capacity of $C_3$ is: 400 + 400 = 800

The capacity of $C_4$ is: 300 + 100 + 300 = 700

The capacity of $C_5$ is: 300 + 400 = 700

The minimum cut/maximum flow is 700 kilolitres per minute.




### Outlet 2

Look for the minimum cut that prevents water reaching Outlet 2.

Note: The pipe with capacity 200 leading towards Outlet 2 will need to be considered in any cut because this pipe delivers water towards Outlet 2 and must be 'cut' like all the others. Other cuts are possible, but have not been included in the diagram.

The capacity of $C_1$ is: 200 + 100 + 300 + 100 = 700

The capacity of $C_2$ is: 200 + 300 + 300 + 100 = 900

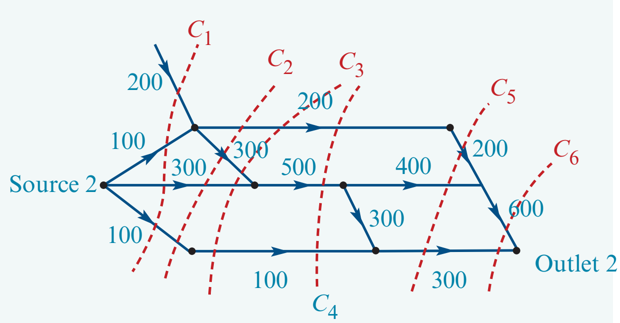
The capacity of $C_3$ is: 200 + 300 + 300 + 100 = 900

The capacity of $C_4$ is: 200 + 500 + 100 = 800

The capacity of $C_5$ is: 200 + 400 + 300 = 900

The capacity of $C_6$ is: 600 + 300 = 900

The minimum cut/maximum flow is 700 kilolitres per minute.




Note: The maximum flow through a network with two sources can also be determined by tracking the flow as outlined in Example 4.

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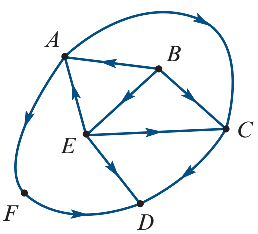
Skill-sheet

# Exercise 14A

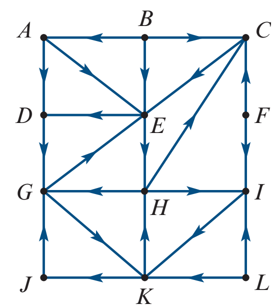
## Directed graphs

**1** Find the number of vertices that can be reached from vertex $A$ in each of the directed graphs below.

**a**



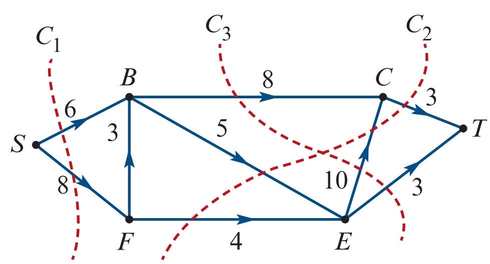

**b**



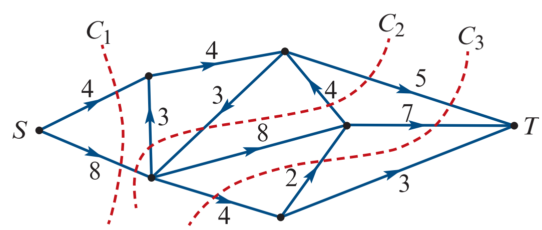

## Cuts

Example 2

**2** Determine the capacity of each of the cuts in the digraph opposite. The source is vertex $S$ and the sink is vertex $T$.




**3** Determine the capacity of each of the cuts in the digraph opposite. The source is vertex $S$ and the sink is vertex $T$.

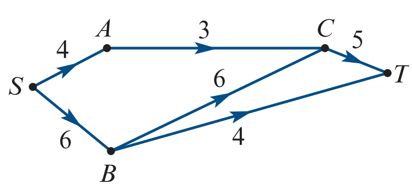



## Maximum-flow

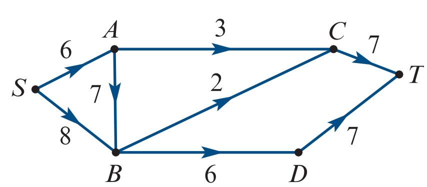
Example 3
Example 4

**4** Find the maximum flow for each of the following graphs. The source is vertex $S$ and the sink is vertex $T$.

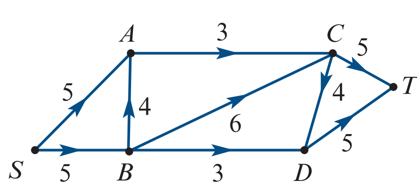
**a**



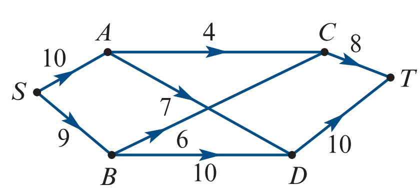

**b**




**c**




**d**

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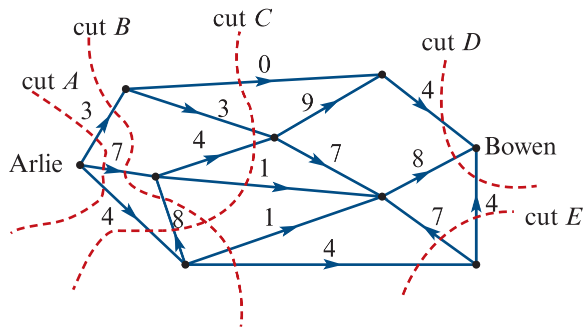
### Minimum-cut maximum-flow

**5** A train journey consists of a connected sequence of stages formed by edges on the directed network opposite from Arlie to Bowen. The number of available seats for each stage is indicated beside the corresponding edge, as shown on the diagram on the right.




The five cuts, $A$, $B$, $C$, $D$ and $E$, shown on the network are attempts to find the maximum number of available seats that can be booked for a journey from Arlie to Bowen.

**a** Write down the capacity of cut $A$, cut $B$, cut $C$ and cut $D$.

**b** Explain why cut $E$ is not a valid cut when trying to find the minimum cut between Arlie and Bowen.

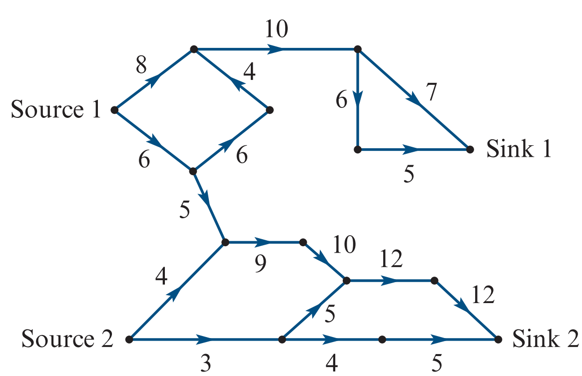
**c** Find the maximum number of available seats for a train journey from Arlie to Bowen.

### Networks with more than one source

Example 5

**6** In each of the following, water pipes of different capacities are connected to two water sources and two sinks. Networks of water pipes are shown in the diagrams below. The numbers on the edges represent the capacities, in kilolitres per minute, of the pipes. For each of the following, find the maximum flow, in kilolitres per minute, to each of the sinks in this network.

**a**

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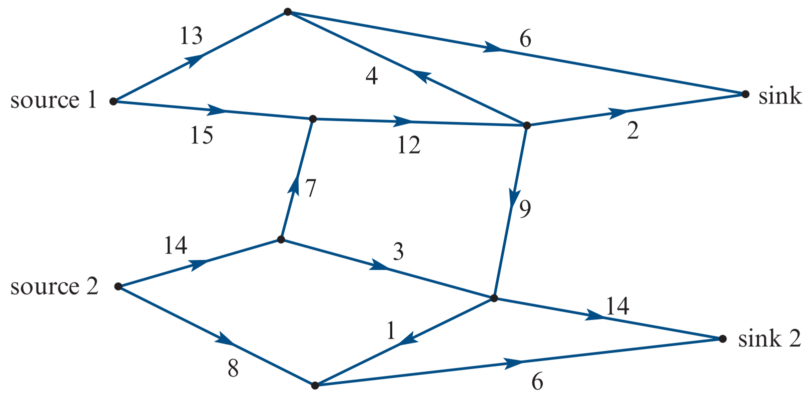
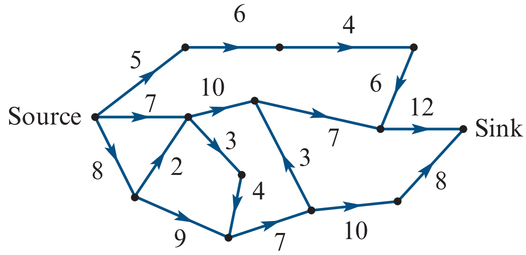
**b**




### Analysis of maximum-flow problems

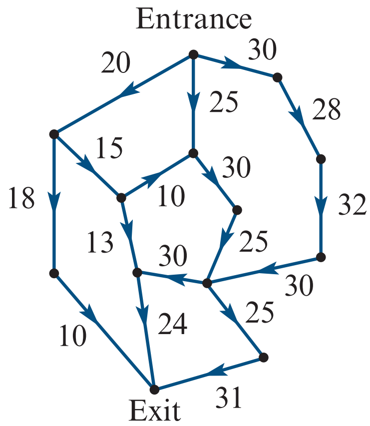
**7** The flow of water through a series of pipes, in litres per minute, is shown in the directed network below.




**a** How many different routes from the *source* to the *sink* are possible?

**b** Determine the maximum flow from the *source* to the *sink*.

**8** The corridors people can walk through to visit different exhibits in a museum are given as a directed network opposite. To avoid congestion around every exhibit, the museum imposes a maximum capacity policy throughout each corridor between exhibits. The numbers on the edges represent the maximum number of people that can walk through each corridor of the museum every 30 minutes.




**a** On the network opposite, identify a cut that has a capacity of 80.

**b** Determine the maximum flow of people from the entrance to the exit of the museum.

**c** One group of primary school students would like to walk through the museum. The teacher explains that this can happen unsupervised if all students in the group remain together, not separating to explore different routes. Given that a group of students must stay together from the entrance to the exit, what is the largest group of students possible that can pass through the museum every 30 minutes?

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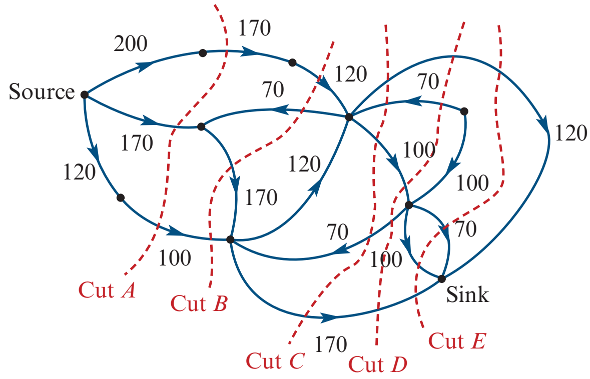
## Exam 1 style questions

*Questions 9 and 10 refer to the diagram opposite*

The flow of liquid through a series of pipes, in litres per minute, is shown in the directed network opposite. Five cuts labelled $A$ to $E$ are shown on the network.



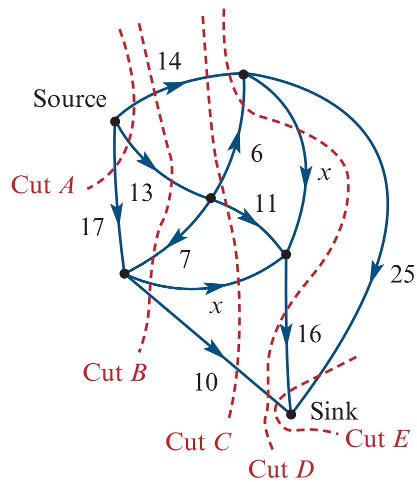

**9** The capacity of Cut $E$ is

**A** 70 **B** 170 **C** 290 **D** 390 **E** 460

**10** The number of these cuts with a capacity equal to the maximum flow of liquid from the source to the sink, in litres per minute, is

**A** 0 **B** 1 **C** 2 **D** 3 **E** 4

**11** The flow of water through a series of pipes, in litres per minute, is shown in the network opposite. The weightings of two edges are labelled $x$. Five cuts labelled $A$ to $E$ are shown on the network. The maximum flow of water from the source to the sink, in litres per minute, is given by the capacity of




**A** Cut $A$ if $x = 4$ **B** Cut $B$ if $x = 6$ **C** Cut $C$ if $x = 8$

**D** Cut $D$ if $x = 6$ **E** Cut $E$ if $x = 8$

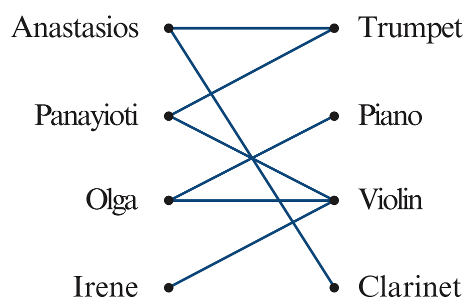
# 14B Matching and allocation problems

**Learning intentions**

- ▶ To be able to define and describe bipartite graphs.
- ▶ To be able to solve assignment problems using the Hungarian Algorithm.

## Bipartite graphs

In some situations, the vertices of a graph belong in two separate sets. Consider a music school that has four teachers: Anastasios, Panayioti, Olga and Irene. These teachers, between them can teach four different instruments: trumpet, piano, violin and clarinet. The teachers and instruments are represented by vertices, arranged vertically as shown in the diagram opposite.




The edges of the diagram connect the teachers to the instruments they can teach.

This type of graph is called a **bipartite graph.** Each edge in a bipartite graph joins a vertex from one group to a vertex in the other group.

In the situation described above, the school would need to match each teacher to one instrumental class; this is an example of an **allocation problem**. The bipartite graph above graphically shows the instrument(s) that each teacher can teach and can help the school assign each teacher to an instrument. Anastasios is the only teacher who can teach clarinet, Irene can only teach violin, therefore Olga must teach piano and Panayioti must teach trumpet.

### Example 6

Nick, Maria, David and Subitha are presenters on a TV travel show. Each presenter will be assigned a story to film about one country that they have visited before.

- Nick has visited Greece, Malaysia and Brazil
- Maria has visited Greece and France
- Subitha has visited Malaysia and Brazil
- David has visited Malaysia

Construct a bipartite graph of the information above and use it to decide on the assignment of each presenter to one country.

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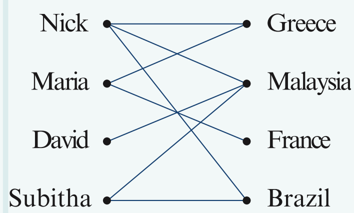
**Explanation**

The two groups of items are: Presenters and Countries. Draw a vertex for each presenter in one column and each country in another. Join the vertex of each presenter to the vertex of each country they have visited.

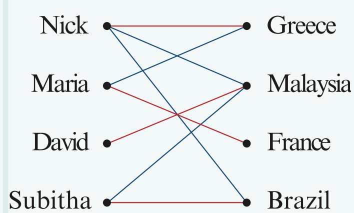
**Solution**




To allocate a presenter to each country, begin by identifying the vertices with only one edge connected to them.

- David is the only presenter who has visited Malaysia, so he must visit that country.
- France has only been visited by one presenter: Maria. She must be allocated to France, therefore she cannot be allocated to Greece, the other country she has visited before.

Highlight definite allocations on the bipartite graph




Given that each presenter can only visit one country, the final allocations can be deduced by eliminating impossible allocations.

- Maria is allocated to France, therefore she cannot visit Greece. Nick is the only other presenter who has visited Greece, therefore he must be allocated that country.
- As David is allocated to Malaysia, Subitha has only one other country available to visit.
- Subitha must be allocated to Brazil and this is also supported by the fact Nick must be allocated to Greece.

Write the assignments.

- Nick will be allocated Greece
- Maria will be allocated France
- David will be allocated Malaysia
- Subitha will be allocated Brazil

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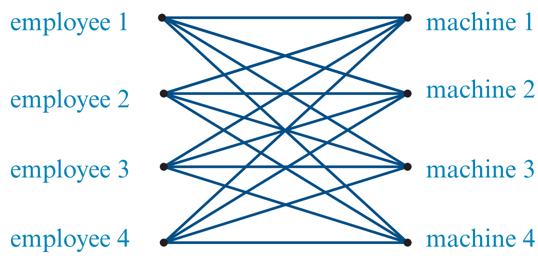
## The Hungarian algorithm

The graph on the right shows four employees in a factory. There are four different machines that are used in the production of an item. Every employee can use every machine and so this is a **complete bipartite graph**. Employees and machines can be matched in many different ways.




Rather than just assign an employee to a machine randomly, the factory could use information about how well each employee uses each machine, perhaps in terms of how quickly each performs the task. The times taken would be the weights on the edges of the bipartite graph. Rather than writing all of the weights on a complete bipartite graph (which would be a very complicated diagram), we can summarise the time information in a table and then use an algorithm, called the **Hungarian algorithm**, to allocate employees to machines in order to minimise the time taken to finish the tasks.

The table on the right shows the four employees: Wendy, Xenefon, Yolanda and Zelda. The machines in a factory are represented by the letters $A$, $B$, $C$ and $D$.

The numbers in the table are the times, in minutes, it takes each employee to finish the task on each machine.

| *Employee* | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| Wendy | 30 | 40 | 50 | 60 |
| Xenefon | 70 | 30 | 40 | 70 |
| Yolanda | 60 | 50 | 60 | 30 |
| Zelda | 20 | 80 | 50 | 70 |

The table is called a **cost matrix**. Even though the numbers do not represent money value, this table contains information about the cost, in terms of time, of employees using each machine. The cost matrix can be used to determine the best way to allocate an employee to a machine so that the overall cost, in terms of the time taken to finish the work, is minimised. The Hungarian algorithm is used to do this.

### Performing the Hungarian algorithm

#### Step 1: Subtract the lowest value in each row, from every value in that row.

- 30 has been subtracted from every value in the row for Wendy.
- 30 has been subtracted from every value in the row for Xenefon.
- 30 has been subtracted from every value in the row for Yolanda.
- 20 has been subtracted from every value in the row for Zelda.

| *Employee* | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| Wendy | 0 | 10 | 20 | 30 |
| Xenefon | 40 | 0 | 10 | 40 |
| Yolanda | 30 | 20 | 30 | 0 |
| Zelda | 0 | 60 | 30 | 50 |

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**Step 2: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 3.**

- The zeros can be covered with three lines. This is less than the number of allocations to be made (4).
- Continue to step 3.

| *Employee* | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| Wendy | 0 | 10 | 20 | 30 |
| Xenefon | 40 | 0 | 10 | 40 |
| Yolanda | 30 | 20 | 30 | 0 |
| Zelda | 0 | 60 | 30 | 50 |

**Step 3: If a column does not contain a zero, subtract the lowest value in that column from every value in that column.**

- Column *C* does not have a zero.
- 10 has been subtracted from every value in column *C*.

| *Employee* | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| Wendy | 0 | 10 | 10 | 30 |
| Xenefon | 40 | 0 | 0 | 40 |
| Yolanda | 30 | 20 | 20 | 0 |
| Zelda | 0 | 60 | 20 | 50 |

**Step 4: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 5a.**

- The zeros can be covered with three lines. This is less than the number of allocations to be made (4).
- Continue to step 5a.

| *Employee* | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| Wendy | 0 | 10 | 10 | 30 |
| Xenefon | 40 | 0 | 0 | 40 |
| Yolanda | 30 | 20 | 20 | 0 |
| Zelda | 0 | 60 | 20 | 50 |

**Step 5a: Add the smallest uncovered value to any value that is covered by two lines. Subtract the smallest uncovered value from all the uncovered values.**

- The smallest uncovered element is 10.
- 10 has been *added* to Xenefon–*A* and Xenefon–*D* because these values are covered by two lines.
- 10 has been *subtracted* from all the uncovered values.

| *Employee* | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| Wendy | 0 | 0 | 0 | 30 |
| Xenefon | 50 | 0 | 0 | 50 |
| Yolanda | 30 | 10 | 10 | 0 |
| Zelda | 0 | 50 | 10 | 50 |

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### Step 5b: Repeat from step 4.

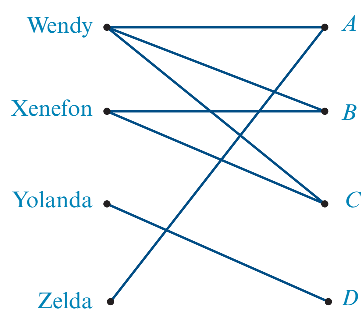
- The zeros can be covered with a minimum of four lines. This is the same as the number or allocations to make.
- Continue to step 6.

| *Employee* | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| Wendy | 0 | 0 | 0 | 30 |
| Xenefon | 50 | 0 | 0 | 50 |
| Yolanda | 30 | 10 | 10 | 0 |
| Zelda | 0 | 50 | 10 | 50 |

### Step 6: Draw a bipartite graph with an edge for every zero value in the table.

In the bipartite graph:

- Wendy will be connected to *A*, *B* and *C*
- Xenefon will be connected to *B* and *C*
- Yolanda will be connected to *D*
- Zelda will be connected to *A*.




### Step 7: Make the allocation and calculate minimum cost

- Zelda must operate machine *A* (20 minutes).
- Yolanda must operate machine *D* (30 minutes).
- Wendy can operate either machine *B* (40 minutes) or *C* (50 minutes).
- Xenefon can operate either machine *B* (30 minutes) or *C* (40 minutes).

Note: Because Wendy and Xenefon can operate either B or C, there are two possible allocations. Both allocations will have the same minimum cost.

The minimum time taken to finish the work = 20 + 30 + 50 + 30 = 130 minutes.

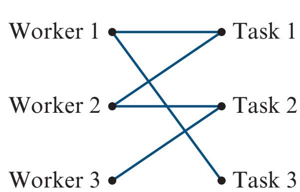
Skill-sheet

## Exercise 14B

### Bipartite graphs

Example 6

**1 a** On Monday, three workers are each to be allocated one task at work. The bipartite graph below shows which task(s) each person is able to complete.




If each person completes a different task, write down the task each worker must be allocated to on Monday.

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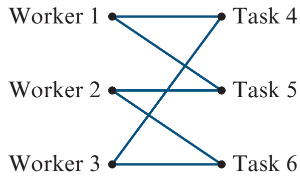
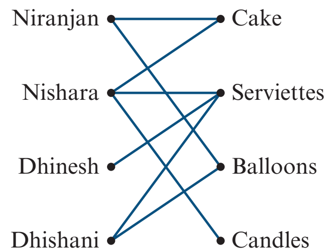
**b** On Tuesday, the same three workers will be allocated to a new set of tasks. The bipartite graph below shows which task(s) each person is able to complete.



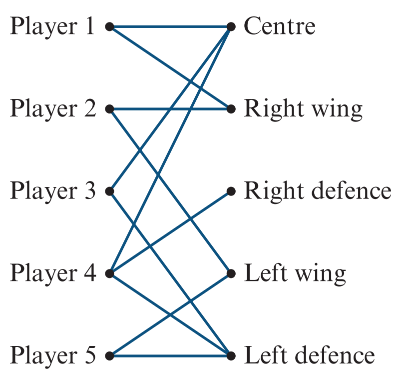

Given that Worker 2 must complete Task 6, write down the new task each worker must be allocated to on Tuesday.

**2** It is Miko's birthday and his sister Aria has asked some of his friends to assist with the celebrations by purchasing some items for a party. The bipartite graph below shows which item(s) each person is able to purchase on their way to the party.




Each friend must purchase an item. Write down which item each friend must purchase.

**3** The sport of ice hockey has six player positions: goalie, left defence, right defence, right wing, left wing and centre. A group of six have decided to play. Only one person is happy to play goalie. The other five people must be allocated to the other five positions.
The bipartite graph below shows which positions each of the five players can play.




Each player plays a different position. Write down two possible allocations, describing which position each player must play.

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**4** Gloria, Minh, Carlos and Trevor are buying ice-cream. They have a choice of five flavours: chocolate, vanilla, peppermint, butterscotch and strawberry. Gloria likes vanilla and butterscotch, but not the others. Minh only likes strawberry. Carlos likes chocolate, peppermint and butterscotch. Trevor likes all flavours.

**a** Explain why a bipartite graph can be used to display this information.

**b** Draw a bipartite graph with the people on the left and flavours on the right.

**c** What is the degree of the vertex representing Trevor?

## The Hungarian algorithm

**5** **a** A cost matrix is shown. Find the allocation(s) by the Hungarian algorithm that will give the minimum cost.

| | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| *W* | 110 | 95 | 140 | 80 |
| *X* | 105 | 82 | 145 | 80 |
| *Y* | 125 | 78 | 140 | 75 |
| *Z* | 115 | 90 | 135 | 85 |

**b** Find the minimum cost for the given cost matrix and give a possible allocation.

| | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| *W* | 2 | 4 | 3 | 5 |
| *X* | 3 | 5 | 3 | 4 |
| *Y* | 2 | 3 | 4 | 2 |
| *Z* | 2 | 4 | 2 | 3 |

**6** A school is to enter four students in four track events: 100 m, 400 m, 800 m and 1500 m. The four students' times (in seconds) are given in the table. The rules permit each student to enter only one event. The aim is to obtain the minimum total time.

| *Student* | *100 m* | *400 m* | *800 m* | *1500 m* |
|---|---|---|---|---|
| Dimitri | 11 | 62 | 144 | 379 |
| John | 13 | 60 | 146 | 359 |
| Carol | 12 | 61 | 149 | 369 |
| Elizabeth | 13 | 63 | 142 | 349 |

Use the Hungarian algorithm to select the 'best' student for each event.

**7** Three volunteer workers, Joe, Meg and Ali, are available to help with three jobs. The time (in minutes) in which each worker is able to complete each task is given in the table opposite. Which allocation of workers to jobs will enable the jobs to be completed in the minimum time?

| | *Job* | | |
|---|---|---|---|
| *Student* | *A* | *B* | *C* |
| Joe | 20 | 20 | 36 |
| Meg | 16 | 20 | 44 |
| Ali | 26 | 26 | 44 |

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**8** A company has four machine operators and four different machines that they can operate. The table shows the hourly cost in dollars of running each machine for each operator. How should the machinists be allocated to the machines to minimise the hourly cost from each of the machines with the staff available?

| *Operator* | *Machine* | | | |
|---|---|---|---|---|
| | *W* | *X* | *Y* | *Z* |
| *A* | 38 | 35 | 26 | 54 |
| *B* | 32 | 29 | 32 | 26 |
| *C* | 44 | 26 | 23 | 35 |
| *D* | 20 | 26 | 32 | 29 |

**9** A football association is scheduling football games to be played by three teams (the Champs, the Stars and the Wests) on a public holiday. On this day, one team must play at their Home ground, one will play Away and one will play at a Neutral ground.

The costs (in \$'000s) for each team to play at each of the grounds are given in the table below.

Determine a schedule that will minimise the total cost of playing the three games and determine this cost.

Note: There are two different ways of scheduling the games to achieve the same minimum cost. Identify both of these.

| *Team* | *Home* | *Away* | *Neutral* |
|---|---|---|---|
| Champs | 10 | 9 | 8 |
| Stars | 7 | 4 | 5 |
| Wests | 8 | 7 | 6 |

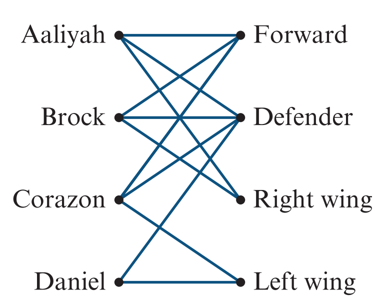
**10** A roadside vehicle assistance organisation has four service vehicles located in four different places. The table below shows the distance (in kilometres) of each of these service vehicles from four motorists in need of roadside assistance.

| *Service vehicle* | *Motorist* | | | |
|---|---|---|---|---|
| | *Jess* | *Mark* | *Raj* | *Karla* |
| *A* | 18 | 15 | 15 | 16 |
| *B* | 7 | 17 | 11 | 13 |
| *C* | 25 | 19 | 18 | 21 |
| *D* | 9 | 22 | 19 | 23 |

Determine a service vehicle assignment that will ensure that the total distance travelled by the service vehicles is minimised. Determine this distance.

## Exam 1 style questions

**11** The sport of futsal has five player positions: goalkeeper, forward, defender, right wing and left wing. In a group of five friends, Ezekiel will always play goalkeeper, but the other four friends Aaliyah, Brock, Corazon and Daniel will rotate their responsibilities and are able to play a number of positions each.
The bipartite graph below shows which positions each of the four friends can play.

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Based on the bipartite graph, which one of the following allocations is **not** possible?

**A**

| Friend | Position |
|---|---|
| Aaliyah | Right wing |
| Brock | Forward |
| Corazon | Defender |
| Daniel | Left wing |

**B**

| Friend | Position |
|---|---|
| Aaliyah | Right wing |
| Brock | Defender |
| Corazon | Forward |
| Daniel | Left wing |

**C**

| Friend | Position |
|---|---|
| Aaliyah | Forward |
| Brock | Right wing |
| Corazon | Left wing |
| Daniel | Defender |

**D**

| Friend | Position |
|---|---|
| Aaliyah | Forward |
| Brock | Defender |
| Corazon | Left wing |
| Daniel | Right wing |

**E**

| Friend | Position |
|---|---|
| Aaliyah | Forward |
| Brock | Right wing |
| Corazon | Defender |
| Daniel | Left wing |

*Use the following information to answer questions 12 and 13*

Five people work at a bank. Each person will perform one task. The time taken for each person to complete tasks 1, 2, 3, 4 and 5, in hours, is shown in the table below.

| | Anita | Brad | Carmen | Dexter | Electra |
|---|---|---|---|---|---|
| **Task 1** | 1 | 2 | 2 | 5 | 4 |
| **Task 2** | 4 | 9 | 7 | 11 | 6 |
| **Task 3** | 5 | 3 | 3 | 9 | 4 |
| **Task 4** | 8 | 5 | 6 | 6 | 7 |
| **Task 5** | 5 | 8 | 4 | 6 | 9 |

**12** The manager of the bank wants to allocate the tasks so as to minimise the total time taken to complete the five tasks. If each person starts their allocated task at the same time, then the first person to finish could be either

**A** Anita or Brad **B** Anita or Elektra **C** Brad or Carmen

**D** Brad or Dexter **E** Brad or Elektra

**13** Before the tasks are performed, it is found that Elektra will only require 4 hours to complete Task 5 rather than 9 hours. If the tasks are allocated based on this new information, the minimum total time for all tasks will

**A** increase by 4 days. **B** decrease by 4 days. **C** decrease by 3 days.

**D** decrease by 2 days. **E** decrease by 1 day.

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**14** Four people, Xena, Wilson, Yasmine, Zachary, are each assigned a different job by their manager. The table below shows the time, in hours, that each person would take to complete each of the four jobs.

| | Job 1 | Job 2 | Job 3 | Job 4 |
|---|---|---|---|---|
| **Xena** | 5 | 3 | 7 | $p$ |
| **Wilson** | 1 | 2 | 5 | 6 |
| **Yasmine** | 1 | 7 | 1 | 5 |
| **Zachary** | 4 | 7 | 6 | $p$ |

Wilson takes 6 minutes to complete Job 4, while Yasmine only takes 5 minutes to complete Job 4. Both Xena and Zachary take $p$ minutes to complete Job 4.
The manager will allocate the jobs as follows:

- Job 1 to Wilson
- Job 2 to Xena
- Job 3 to Yasmine
- Job 4 to Zachary

This allocation will achieve the minimum total completion time if the value of $p$ is not greater than

**A** 6 **B** 7 **C** 8 **D** 9 **E** 10

# 14C Precedence tables and activity networks

Learning intentions

- To be able to identify activities in a project.
- To be able to understand the precedence that some activities have over others in a project.
- To be able to identify immediate predecessors of activities from an activity network.
- To be able to draw an activity network from a precedence table.
- To be able to understand and explain the need for dummy activities in projects.
- To be able to include dummy activities in activity networks as required.

## Drawing activity networks from precedence tables

Building a house, manufacturing a product, organising a wedding and other similar projects all require many individual **activities** to be completed before the project is finished. The individual activities often rely upon each other and some can't be performed until other activities are complete.

In the organisation of a wedding, invitations would be sent out to guests, but a plan for seating people at the tables during the reception can't be completed until the invitations are accepted. When building a house, the plastering of the walls can't begin until the house is sealed from the weather.

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For any project, if activity $A$ must be completed before activity $B$ can begin then activity $A$ is said to be an **immediate predecessor** of activity $B$. The activities within a project can have multiple immediate predecessors and these are usually recorded in a table called a **precedence table**.

This precedence table shows some of the activities involved in a project and their immediate predecessors.

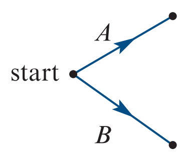
The information in the precedence table can be used to draw a network diagram called an **activity network**.

Activity networks do not have labelled vertices, other than the *start* and *finish* of the project. The activities in the project are represented by the edges of the diagram and *so it is the edges that must be labelled*, not the vertices.

| *Activity* | *Immediate predecessors* |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | $A$ |
| $D$ | $B$ |
| $E$ | $B$ |
| $F$ | $C, D$ |
| $G$ | $E, F$ |

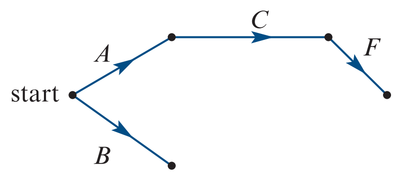
Activities $A$ and $B$ have no immediate predecessors.

These activities can start immediately and can be completed at the same time.



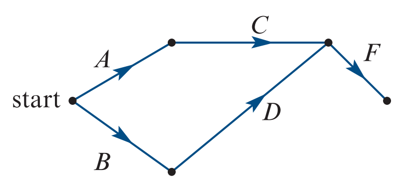

Activity $A$ is an immediate predecessor of activity $C$, so activity $C$ must follow immediately after activity $A$.

Activity $C$ is an immediate predecessor of activity $F$, so activity $F$ must follow immediately after activity $C$.



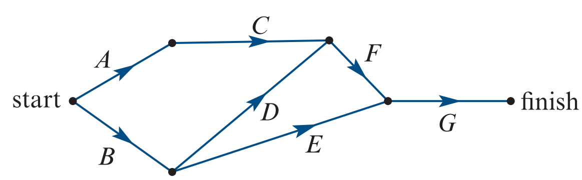

Activity $D$ has immediate predecessor activity $B$ so it follows immediately after activity $B$.

Activity $D$ is also an immediate predecessor of activity $F$ so activity $F$ must follow immediately after activity $D$.




Activity $E$ has immediate predecessor activity $B$ so it will follow immediately after activity $B$.

Activity $G$ has immediate predecessor activity $F$ and activity $E$ and so it must follow immediately after both of these activities.




Activity $G$ is not an immediate predecessor for any activity and so the project is finished after this activity is complete.

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## Activity networks

When activity $A$ must be completed before activity $B$ can begin, activity $A$ is called an immediate *predecessor* of activity $B$.

A table containing the activities of a project, and their immediate predecessors, is called a *precedence table*.

An *activity network* can be drawn from a precedence table. Activity networks have edges representing activities. The vertices are not labelled, other than the start and finish vertices.

### Example 7 Drawing an activity network from a precedence table.

Draw an activity network from the precedence table shown on the right.

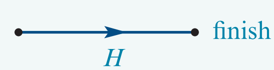
In this solution, the activity network will be drawn from the finish back to the start.

| *Activity* | *Immediate predecessors* |
|---|---|
| $A$ | – |
| $B$ | $A$ |
| $C$ | $A$ |
| $D$ | $A$ |
| $E$ | $B$ |
| $F$ | $C$ |
| $G$ | $D$ |
| $H$ | $E, F, G$ |

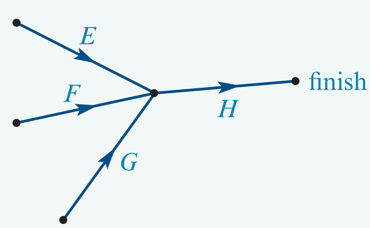
**Explanation**

**Solution**

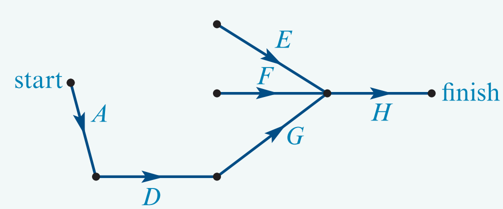
$H$ is not an immediate predecessor for any other activity so it will lead to the finish of the project.



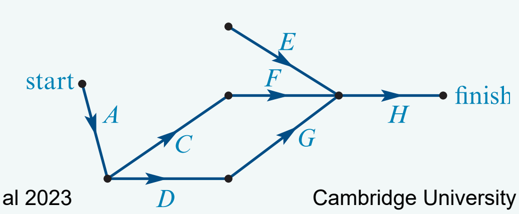

$H$ has immediate predecessors $E$, $F$ and $G$ and so these three activities will lead into activity $H$.




Activity $D$ is an immediate predecessor of activity $G$ *and* has immediate predecessor activity $A$. There will be a path through activity $A$, activity $D$ and then activity $G$.




Activity $C$ is an immediate predecessor of activity $F$ *and* has immediate predecessor activity $A$. There will be a path through activity $A$, activity $C$ and then activity $F$.

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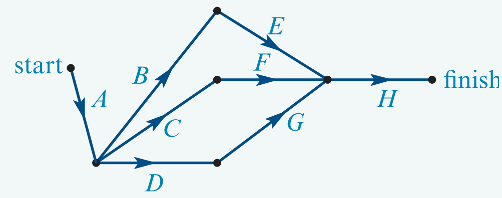
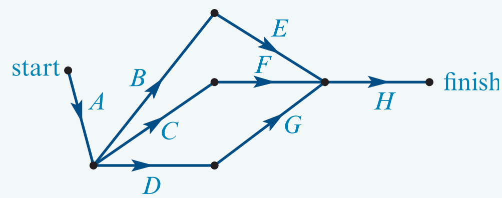
**Explanation**

Activity $B$ is an immediate predecessor of activity $E$ *and* has immediate predecessor activity $A$. There will be a path through activity $A$, activity $B$ and then activity $E$.

**Solution**




Activity $A$ has no immediate predecessors, so it is the start of the project.




**Sketching activity networks**

Activities that have no immediate predecessors follow from the start vertex.

Activities that are not immediate predecessors for other activities lead to the finish vertex.

For every other activity, look for:

- which activities it has as immediate predecessors
- which activities it is an immediate predecessor for.

Construct the activity network from this information.

## Dummy activities

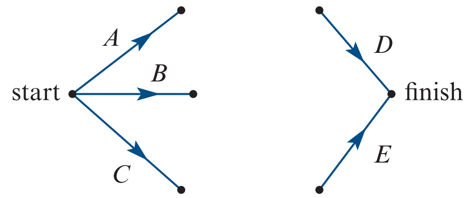
Sometimes two activities will have some of the same immediate predecessors, but not all of them. In this very simple precedence table, activity $D$ and activity $E$ share the immediate predecessor activity $B$, but they both have an immediate predecessor activity that the other does not.

| *Activity* | *Immediate predecessors* |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | – |
| $D$ | $A$, $B$ |
| $E$ | $B$, $C$ |

This overlap of predecessors presents some difficulty when constructing the activity network, but this difficulty is easily overcome.

Activity $D$ and activity $E$ are not immediate predecessors for any other activity, so they will lead directly to the finish vertex of the project.

Activites $A$, $B$ and $C$ have no immediate predecessors, so they will follow directly from the start vertex of the project.

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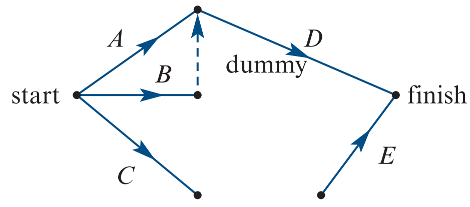
The start and finish of the activity network are shown in the diagram above. We need to use the precedence information for activity $D$ and activity $E$ to join these two parts together.

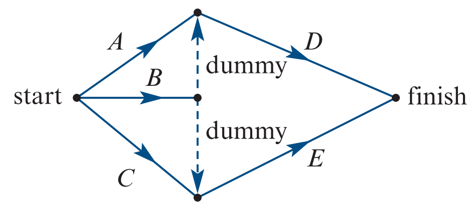
Activity $D$ needs to follow directly from activity $A$ and activity $B$, but we can only draw one edge for activity $D$. Activity $E$ needs to follow directly from both activity $B$ and activity $C$, but again we only have one edge for activity $E$, not two.

The solution is to draw the diagram with activity $D$ starting after one of its immediate predecessors, and using a **dummy activity** for the other. The dummy activities are represented by dotted edges and are, in effect, imaginary. They are not real activities, but they allow all of the predecessors from the table to be correctly represented.




The dummy activity for $D$ allows activity $D$ to directly follow both activity $A$ and $B$.

A dummy activity is also needed for activity $E$ because it, too, has to start after two different activities, activity $B$ and $C$.




## Dummy activities

A dummy activity is required if two activities share some, but not all, of their immediate predecessors.

A dummy activity will be required *from the end* of each shared immediate predecessor *to the start* of the activity that has additional immediate predecessors.

Dummy activities are represented in the activity network using *dotted lines*.

## Example 8 Using a dummy activity in an activity network

Draw an activity network from the precedence table shown on the right.

| *Activity* | *Immediate predecessors* |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | $A$ |
| $D$ | $B$ |
| $E$ | $C$, $D$ |
| $F$ | $C$ |
| $G$ | $E$, $F$ |

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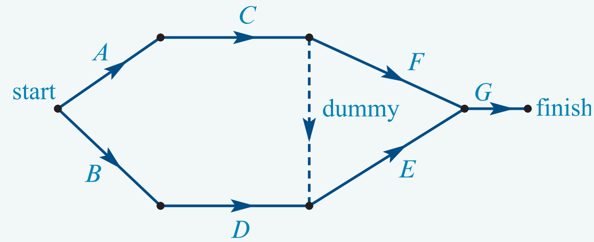
**Explanation**

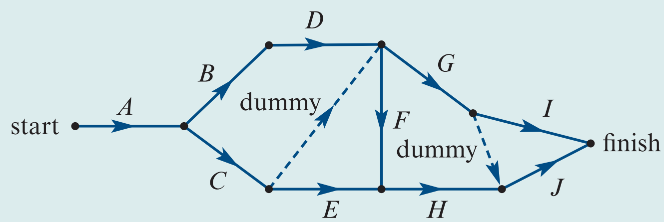
- *A* and *B* will lead from the start vertex.
- *G* will lead to the end vertex.
- A dummy will be required from the end of activity *C* (shared immediate predecessor) to the start of activity *E* (the activity with an additional immediate predecessor.)

**Solution**




## Example 9 Creating a precedence table from an activity network

Write down a precedence table for the activity network shown on the right.




**Explanation**

1. Create a table with a row for each activity.
2. Look at the start of an activity. Write down all of the activities that lead directly to this activity in the immediate predecessor column.
3. Activity *C* is a predecessor of activity *E*, and the dummy activity makes it also a predecessor of *F* and *G*.
4. Activity *G* is a predecessor of activity *I*, and the dummy activity makes it also a predecessor of *J*.

**Solution**

| *Activity* | *Immediate predecessors* |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *B* |
| *E* | *C* |
| *F* | *D*, *C* |
| *G* | *D*, *C* |
| *H* | *E*, *F* |
| *I* | *G* |
| *J* | *G*, *H* |

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Skill-sheet

# Exercise 14C

## Constructing activity networks from precedence tables

Example 7

1 Draw an activity network for each of the precedence tables below.

a

| Activity | Immediate predecessors |
|---|---|
| $A$ | – |
| $B$ | $A$ |
| $C$ | $A$ |
| $D$ | $B$ |
| $E$ | $C$ |

b

| Activity | Immediate predecessors |
|---|---|
| $P$ | – |
| $Q$ | – |
| $R$ | $P$ |
| $S$ | $Q$ |
| $T$ | $R, S$ |

c

| Activity | Immediate predecessors |
|---|---|
| $T$ | – |
| $U$ | – |
| $V$ | $T$ |
| $W$ | $U$ |
| $X$ | $V, W$ |
| $Y$ | $X$ |
| $Z$ | $Y$ |

d

| Activity | Immediate predecessors |
|---|---|
| $F$ | – |
| $G$ | – |
| $H$ | – |
| $I$ | $F$ |
| $J$ | $G, I$ |
| $K$ | $H, J$ |
| $L$ | $K$ |

e

| Activity | Immediate predecessors |
|---|---|
| $K$ | – |
| $L$ | – |
| $M$ | $K$ |
| $N$ | $M$ |
| $O$ | $N, L$ |
| $P$ | $O$ |
| $Q$ | $P$ |
| $R$ | $M$ |
| $S$ | $R, Q$ |

f

| Activity | Immediate predecessors |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | – |
| $D$ | $B$ |
| $E$ | $A, D$ |
| $F$ | $E, C$ |
| $G$ | $F$ |
| $H$ | $G$ |
| $I$ | $E, C$ |
| $J$ | $G$ |
| $K$ | $H, I$ |

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### Constructing activity networks requiring dummy activities from precedence tables

Example 8

**2** Draw an activity network for the following precedence tables. Dummy activities will need to be used.

**a**

| *Activity* | *Immediate predecessors* |
|---|---|
| $F$ | – |
| $G$ | – |
| $H$ | $F$ |
| $I$ | $H$, $G$ |
| $J$ | $G$ |

**b**

| *Activity* | *Immediate predecessors* |
|---|---|
| $A$ | – |
| $B$ | $A$ |
| $C$ | $A$ |
| $D$ | $B$ |
| $E$ | $B$, $C$ |

**c**

| *Activity* | *Immediate predecessors* |
|---|---|
| $P$ | – |
| $Q$ | – |
| $R$ | $P$ |
| $S$ | $Q$ |
| $T$ | $Q$ |
| $U$ | $R$, $S$ |
| $V$ | $R$, $S$, $T$ |

**d**

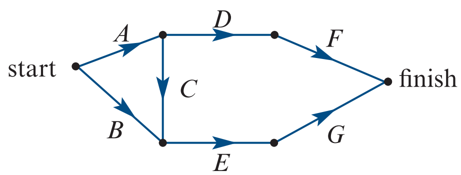
| *Activity* | *Immediate predecessors* |
|---|---|
| $A$ | – |
| $B$ | $A$ |
| $C$ | $A$ |
| $D$ | $B$, $C$ |
| $E$ | $C$ |
| $F$ | $E$ |
| $G$ | $D$ |
| $H$ | $F$, $G$ |

### Constructing precedence tables from activity networks

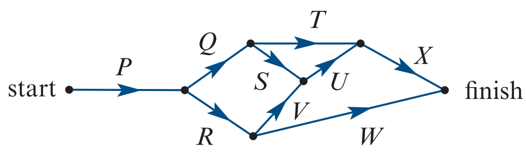
Example 9

**3** Write down a precedence table for the activity networks shown below.

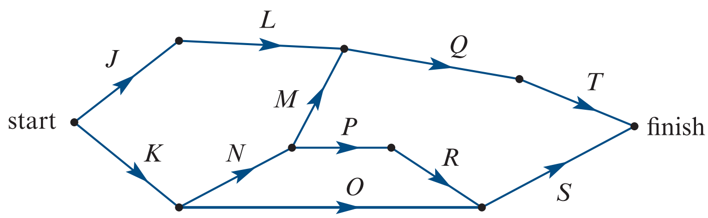
**a**




**b**




**c**

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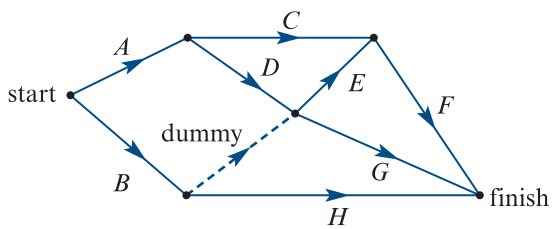
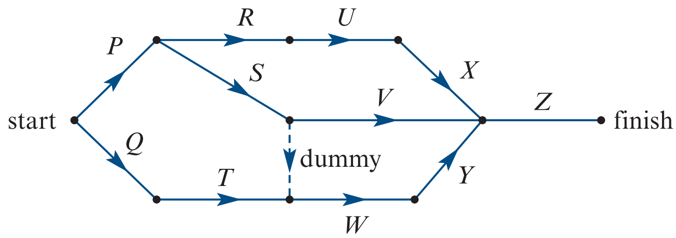
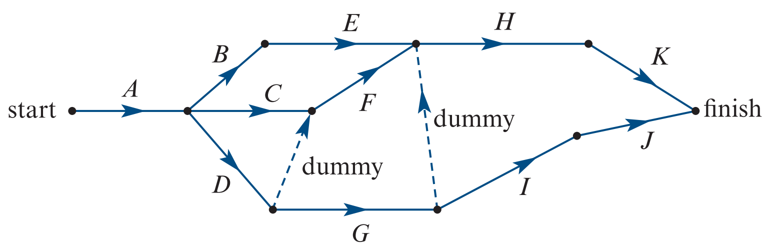
**d**



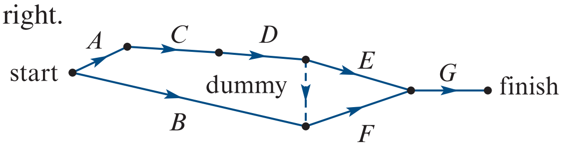

**e**




**f**




**4** The following activity network shows the activities in a project to repair a dent in a car panel. The activities are listed in the table on the right.



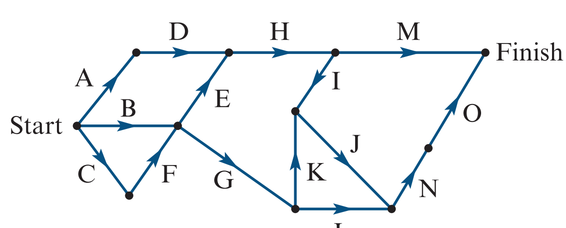

| *Activity* | *Description* |
|---|---|
| *A* | Remove panel |
| *B* | Order component |
| *C* | Remove broken component |
| *D* | Pound out dent |
| *E* | Repaint |
| *F* | Install new component |
| *G* | Replace panel |

**a** Which activity or activities are the immediate predecessors of the event 'remove broken component'?

**b** Which activities are the immediate predecessors of the activity 'install new component'?

## Analysis of activity networks and precedence tables

**5** Consider the following activity network for a project.

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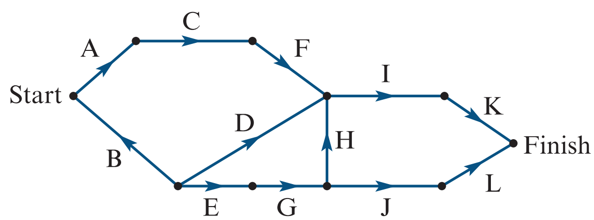
**a** Write down a precedence table for the network above.

**b** Write down the two paths from *start* to *finish* that begin with activity $A$.

**c** Write down the four paths from *start* to *finish* that begin with activity $B$.

**d** Write down the four paths from *start* to *finish* that begin with activity $C$.

**6** Consider the following activity network.




**a** Which activities are immediate predecessors of activity $I$?

**b** Which activities must be completed before activity $I$ can commence?

**7** The information in the table opposite can be used to complete a directed network. This network will require a dummy activity.

**a** Which activity will the dummy activity be drawn from the end of?

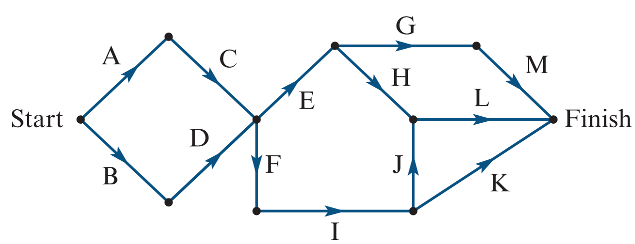
**b** Which activity will the dummy activity be drawn to the start of?

**c** Why is it necessary to include a dummy activity in this network?

| *Activity* | *Immediate predecessors* |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | $A$ |
| $D$ | $B$ |
| $E$ | $A$ |
| $F$ | $C, D$ |
| $G$ | $B$ |
| $H$ | $F$ |
| $I$ | $G$ |
| $J$ | $I$ |
| $K$ | $G, H, J$ |
| $L$ | $E$ |

## Exam 1 style questions

**8** The activity network below shows the sequence of activities required to complete a project.




Beginning with activity $B$, the number of paths from start to finish is

**A** 1 **B** 2 **C** 3 **D** 4 **E** 5

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*Use the following information to answer Questions 9 and 10*

A project involves eight activities, $A$ to $H$.

The immediate predecessor(s) of each activity is shown in the table opposite. A directed network for this project will require a dummy activity.

| Activity | Immediate predecessors |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | – |
| $D$ | $A$ |
| $E$ | $C$ |
| $F$ | $B, E$ |
| $G$ | $B$ |
| $H$ | $G$ |

**9** The dummy activity will be drawn from the end of

**A** activity $B$ to the start of activity $F$.
**B** activity $B$ to the start of activity $E$.
**C** activity $G$ to the start of activity $F$.
**D** activity $G$ to the start of activity $E$.
**E** activity $E$ to the start of activity $F$.

**10** The number of paths from start to finish is

**A** 4 **B** 5 **C** 6 **D** 7 **E** 8

# 14D Scheduling problems

**Learning intentions**

- To be able to understand activity networks that include weights (durations) of each activity.
- To be able to determine the EST for activities using forward scanning.
- To be able to determine the LST for activities using backward scanning.
- To be able to calculate and understand the existence of float times for some activities.
- To be able to identify the critical path and completion time of a project.

## Scheduling

Projects that involve multiple activities are usually completed against a time schedule. Knowing how long individual activities within a project are likely to take allows managers of such projects to hire staff, book equipment and also to estimate overall costs of the project. Allocating time to the completion of activities in a project is called *scheduling*. Scheduling problems involve analysis to determine the minimum overall time it would take to complete a project.

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## Weighted precedence tables

The estimated time to complete activities within a project can be recorded in a precedence table, alongside the immediate predecessor information.

A precedence table that contains the estimated duration, in days, of each activity is shown on the right.

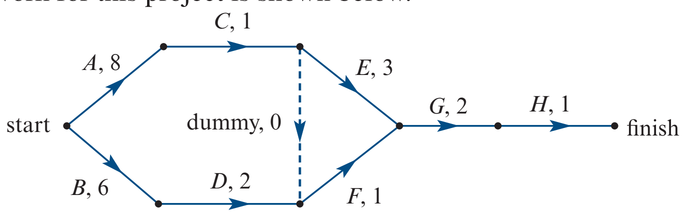
These estimated times are the weights for the edges of the activity network and need to be recorded alongside the name of the activity on the graph.

A dummy activity is required from the end of activity *C* (*C* is repeated in immediate predecessors) to the start of activity *F* (*F* is the activity that has an extra immediate predecessor).

| *Activity* | *Estimated completion time (days)* | *Immediate predecessors* |
|---|---|---|
| *A* | 8 | – |
| *B* | 6 | – |
| *C* | 1 | *A* |
| *D* | 2 | *B* |
| *E* | 3 | *C* |
| *F* | 1 | *C*, *D* |
| *G* | 2 | *E*, *F* |
| *H* | 1 | *G* |

The weight (duration) of dummy activities is always zero.

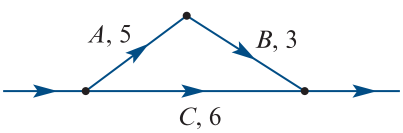
The activity network for this project is shown below.




## Float times

The diagram below shows a small section of a different activity network. There are three activities shown, with their individual durations, in hours.

Activity *B* and activity *C* are both immediate predecessors to the next activity, so the project cannot continue until both of these tasks are finished. Activity *B* cannot begin until activity *A* is finished.




Activity *C* can be completed at the same time as activity *A* and activity *B*.

Activity *A* and *B* will take a total of $5 + 3 = 8$ hours, while activity *C* only requires 6 hours. There is some flexibility around when activity *C* needs to start. There are $8 - 6 = 2$ hours spare for the completion of activity *C*. This value is called the **float time** for activity *C*.

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The flexibility around the starting time for activity $C$ can be demonstrated with the following diagram.

| | A | A | A | A | A | B | B | B |
|---|---|---|---|---|---|---|---|---|
| *Start at same time* | C | C | C | C | C | C | Slack | Slack |
| *Delay C by 1 hour* | Slack | C | C | C | C | C | C | Slack |
| *Delay C by 2 hour* | Slack | Slack | C | C | C | C | C | C |

The five red squares represent the 5 hours it takes to complete activity $A$. The three green squares represent the 3 hours it takes to complete activity $B$.

The six yellow squares represent the 6 hours it takes to complete activity $C$. Activity $C$ does not have to start at the same time as activity $A$ because it has some slack time available (2 hours).

Activity $C$ should not be delayed by more than 2 hours because this would cause delays to the project. The next activity requires $B$ and $C$ to be complete before it can begin.

## Calculating and recording earliest starting times (EST)

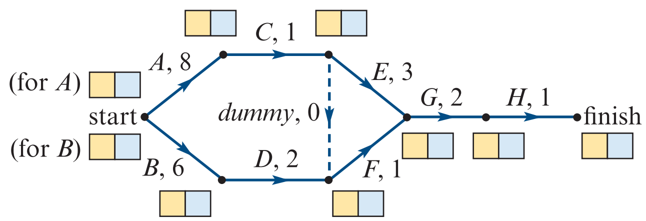
In order for a project to be completed in the shortest time possible, it is important that certain key activities start at the earliest possible time. The **earliest starting time**, or **EST**, for each activity is the earliest time after the start of the entire project that the individual activity can start. An EST of 8 means an activity can start 8 hours (or whatever time period is given) after the start of the project.

The EST for each activity is found by a process called **forward scanning**.

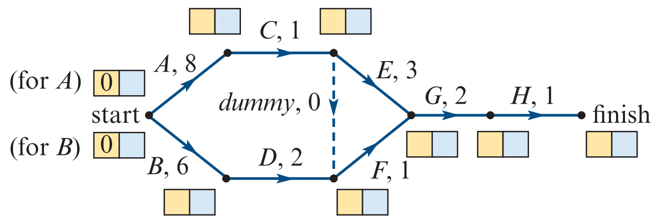
### Forward scanning

Forward scanning will be demonstrated using the activity network below. The durations of each are in days.

**1** Draw a box, split into two cells, next to each vertex of the activity network, as shown in the diagram opposite. If more than one activity begins at a vertex, draw a box for each of these activities.




**2** Activities that begin at the start of the project have an EST of zero (0). Write this in the left box, shown shaded yellow in the diagram.

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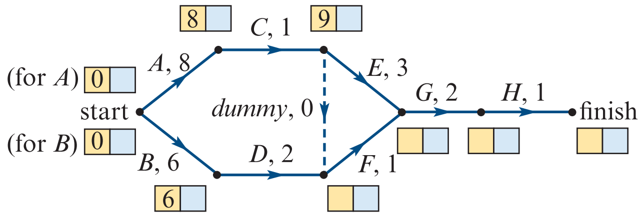
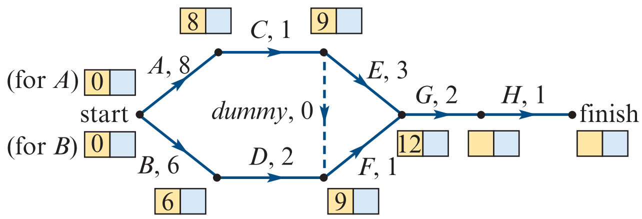
**3** Calculate the EST of each activity of the project by adding the EST of the immediate predecessor to the duration of the immediate predecessor.



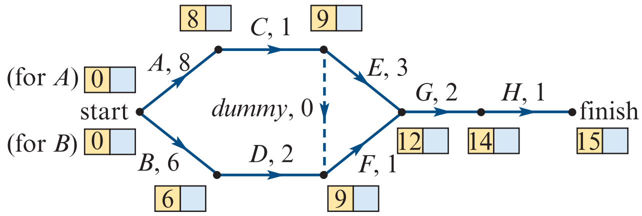

Notes: EST of $C$ = EST of $A$ + duration of $A$ (EST of $C = 0 + 8 = 8$)
EST of $D$ = EST of $B$ + duration of $B$ (EST of $D = 0 + 6 = 6$)
EST of $E$ = EST of $C$ + duration of $C$ (EST of $C = 8 + 1 = 9$)

**4** If an activity has more than one predecessor, calculate the EST using each of the predecessors and choose the *largest* value.




Notes: EST of $F = 6 + 2 = 8$ or EST of $F = 9 + 0 = 9$. Use 9.
EST of $G = 9 + 1 = 10$ or EST of $G = 9 + 3 = 12$. Use 12.

**5** The EST value at the finish of the project is the minimum time it takes to complete the project.




Notes: The minimum time to complete this project is 15 days.

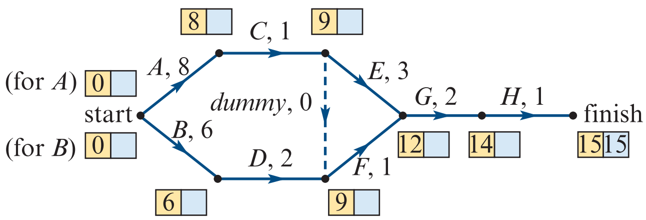
## Calculating and recording latest starting times (LST)

Some activities, as we saw earlier, have some flexibility around the time that they can start. The **latest start time**, or LST, for each activity is the latest time after the start of the entire project that the individual activity can start. LSTs for each activity are calculated using the reverse of the process used to calculate the ESTs. This process is called **backward scanning**.

### Backward scanning

Backward scanning will be demonstrated using the activity network with completed forward scanning from above.

**1** Copy the minimum time to complete the project into the right cell shown shaded blue in the diagram.

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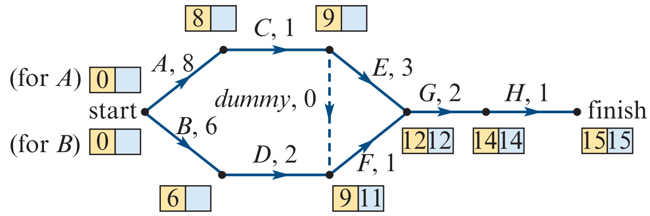
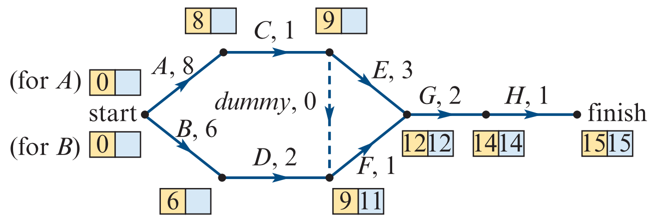
**2** Calculate the LST for each activity by subtracting the duration of the activity from the LST of the following activity.



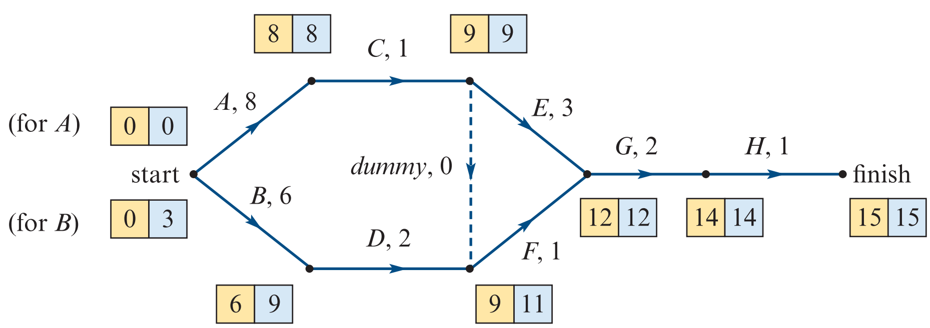

Notes: **1** LST of $H$ = LST of finish – duration of $H$ (LST of $H = 15 - 1 = 14$).
**2** LST of $G$ = LST of $H$ – duration of $G$ (LST of $G = 14 - 2 = 12$).
**3** LST of $F$ = LST of $G$ – duration of $F$ (LST of $F = 12 - 1 = 11$).
**4** LST of $E$ = LST of $G$ – duration of $E$ (LST of $E = 12 - 3 = 9$).

**3** If more than one activity have the same predecessor, calculate the LST using each of the activities that follow and choose the *smallest* value.



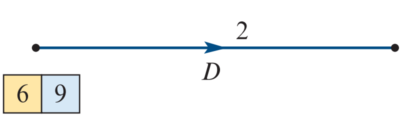

Note: LST of $E = 12 - 3 = 9$ (from duration of $E$) or LST of $E = 11 - 0 = 11$ (from duration of *dummy*). Use 9.

The completed activity network with all EST and LST is shown below.



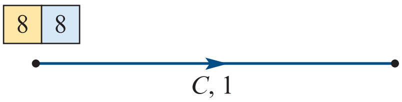

## Identifying float times and the critical path

The boxes at the vertices in the activity network above give the EST and LST for the activity that begins at that vertex.




The EST for activity $D$ is 6 and the LST for activity $D$ is 9. This means activity $D$ has a float time of $9 - 6 = 3$ hours.

Activity $D$ can be delayed by 3 hours without delaying the rest of the project.




The EST for activity $C$ is 8 and the LST for activity $C$ is 8. This means activity $C$ has a float time of $8 - 8 = 0$ hours.

Activity $C$ has no flexibility around its starting time at all. Any delay to the start of this activity will delay the whole project and extend the minimum time for completion.

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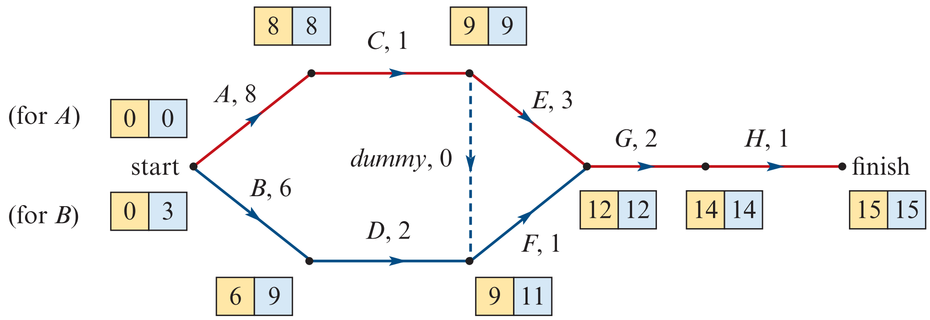
Activities that have no float time are critical ones for completion of the project. Tracking through the activity network along the edges of critical activities gives the **critical path** for the project. The critical path for this project is highlighted in red on the diagram below.




## Critical path

- A critical path is the longest or equal longest path in an activity network.
- There can be more than one critical path in an activity network.
- The critical path is the sequence of activities that cannot be delayed without affecting the overall completion time of the project.

The process for determining the critical path is called **critical path analysis**.

## Critical path analysis

- Draw a box with two cells next to each vertex of the activity network.
- Calculate the EST for each activity by forward scanning:

  EST = EST of predecessor + duration of predecessor
- If an activity has more than one predecessor, the EST is the *largest* of the alternatives.
- The minimum overall completion time of the project is the EST value at the end vertex.
- Calculate the LST for each activity by backward scanning:

  LST = LST of following activity − duration of activity
- If an activity has more than one following activity, the LST is the *smallest* of the alternatives.
- Float = LST − EST
- If float time = 0, the activity is on the critical path.

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## Example 10 Finding the critical path from a precedence table

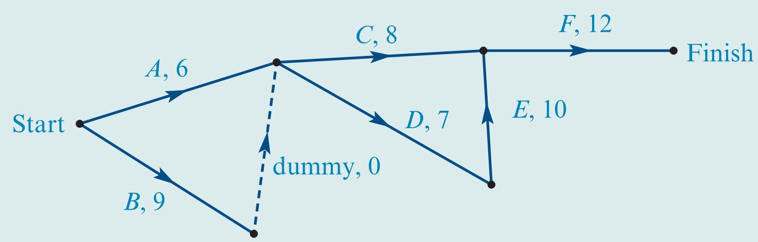
A project has six activities as shown in the precedence table opposite and the associated activity network is shown below.

**a** Complete the critical path analysis to calculate the EST and LST for each activity.

**b** Write down the critical path of this project.

**c** What is the minimum time required to complete the project?

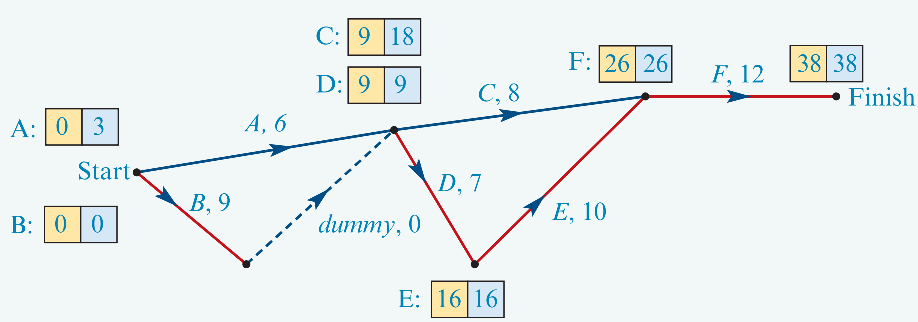
| Activity | Duration (days) | Immediate predecessors |
|---|---|---|
| $A$ | 6 | – |
| $B$ | 9 | – |
| $C$ | 8 | $A, B$ |
| $D$ | 7 | $A, B$ |
| $E$ | 10 | $D$ |
| $F$ | 12 | $C, E$ |




### Explanation

### Solution

**a**




**b** The critical path is highlighted in red.
Note: The dummy is not included in the critical path.

The critical path of this project is $B \rightarrow D \rightarrow E \rightarrow F$.

**c** The minimum completion time is in EST of the end box.

The minimum completion time of this project is 38 days.

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## Example 11 Finding the critical path

A project has eight activities as shown in the precedence table opposite.

| Activity | Duration (weeks) | Immediate predecessors |
|---|---|---|
| *A* | 3 | – |
| *B* | 5 | – |
| *C* | 12 | – |
| *D* | 4 | *A* |
| *E* | 7 | *A* |
| *F* | 5 | *B*, *D* |
| *G* | 6 | *E*, *F* |
| *H* | 2 | *C* |

**a** Draw an activity network for this project.

**b** Complete the critical path analysis to calculate the EST and LST for each activity.

**c** What is the earliest starting time for activity H?

**d** What is the latest starting time for activity H?

**e** What is the float time of activity H?

**f** Write down the critical path of this project.

**g** What is the minimum time required to complete the project?

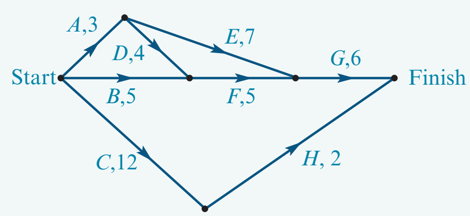
**h** The person responsible for completing activity E falls sick three weeks into the project. If he will be away from work for two weeks, will this cause the entire project to be delayed?

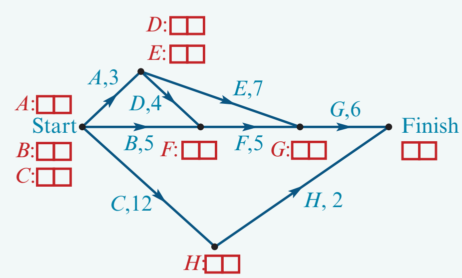
### Explanation

**a** *A*, *B* and *C* have no predecessors and so can begin at the same time. Continue drawing the network as outlined by the table above, including arrowheads, activity labels, duration labels and correct immediate predecessors.

**b** Begin by drawing boxes, split into two cells, at the beginning of each activity. Label them with the name of each activity. You must also include a box, split into two cells, at the final vertex where the project finishes.

### Solution

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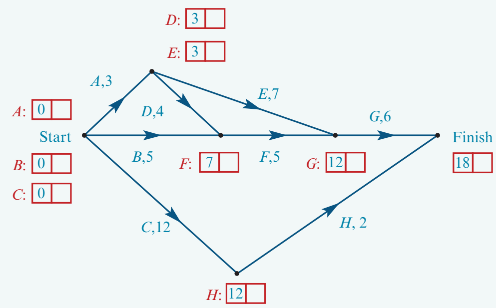
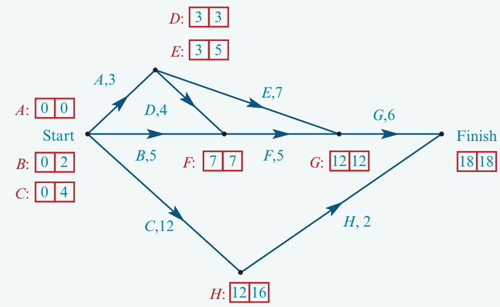
Use forward scanning to identify the EST for each activity. Activities with no immediate predecessors always have an EST of zero. Add the left cell value at the start of the activity to the duration and write the result in the left cell at the end of the activity. Use the largest of the possibilities if there is more than one activity ending at the same vertex. Identify the minimum project completion time as the left cell value at the finish vertex




Use backward scanning to identify the LST for each activity. Subtract the duration from the right cell value at the end of the activity and write the result in the right cell at the start of the activity. Use the smallest of the possibilities if there is more than one activity beginning at the same vertex.




**c** EST values are in the left cell at the start of each activity.

The EST for activity $H$ is 12 weeks

**d** LST values are in the right cell at the end of each activity.

The EST for activity $H$ is 16 weeks

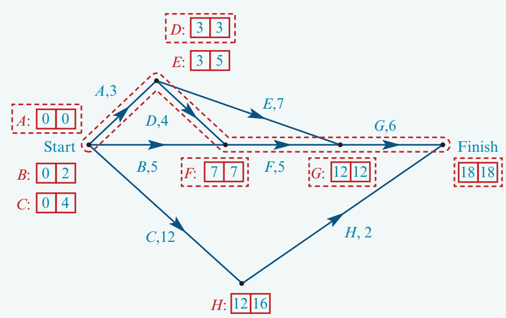
**e** Float = $LST - EST$.

Float $H = LST - EST$

$= 16 - 12$

$= 4$ weeks

**f** The critical path joins all of the activities that have the same EST and LST, and therefore which have zero float time.




The critical path for this project is $A - D - F - G$

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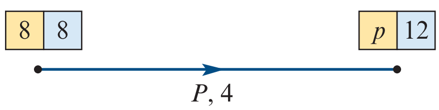
| | |
|---|---|
| **g** The minimum time required to complete the project is the EST (also, always equal to the LST) at the finish vertex. | 18 weeks |
| **h** If the float time is more or equal to the delay in the start of activity E, the project will not be affected. | The person will be away for two weeks, starting 3 weeks into the project. This is equal to the float time for activity E, and so delaying the start of activity E until the person comes back to work will not affect the overall completion time of the project. |

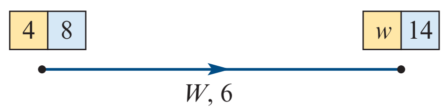
Skill-sheet

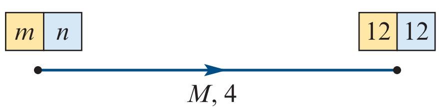
## Exercise 14D

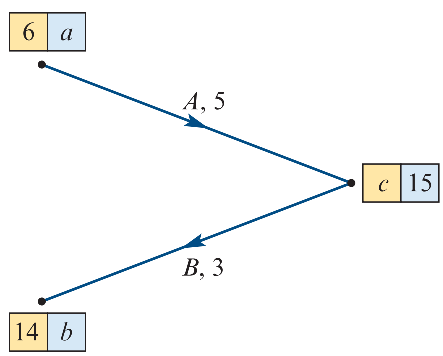
### Calculations from elements of an activity network

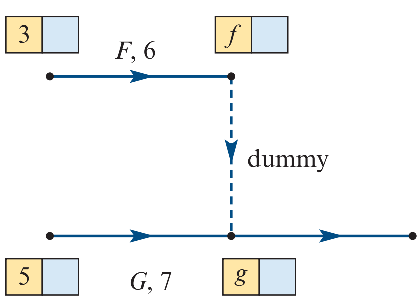
**1** Write down the value of each pronumeral in the sections of activity networks below.

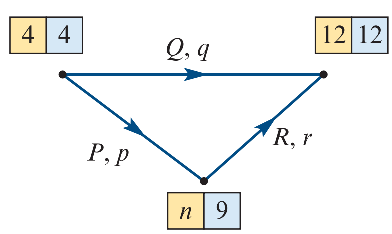
**a**



**b**



**c**



**d**



**e**



**f**
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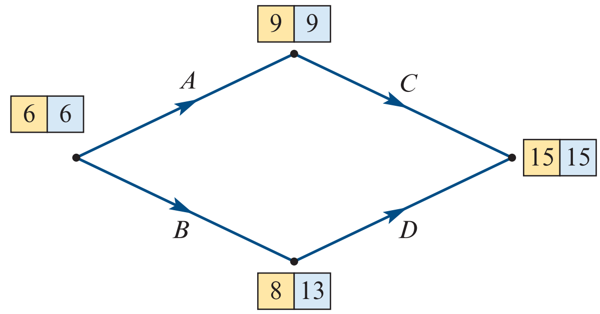
**2** Consider the section of an activity network shown in the diagram opposite.




**a** What is the duration of activity *A*?

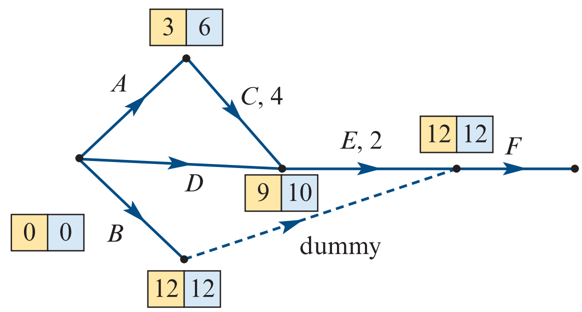
**b** What is the critical path through this section of the activity network?

**c** What is the float time of activity *B*?

**d** What is the latest time that activity *D* can start?

**e** What is the duration of activity *D*?

**3** Consider the section of an activity network shown in the diagram below.




**a** What is the duration of activity *B*?

**b** What is the latest start time for activity *E*?

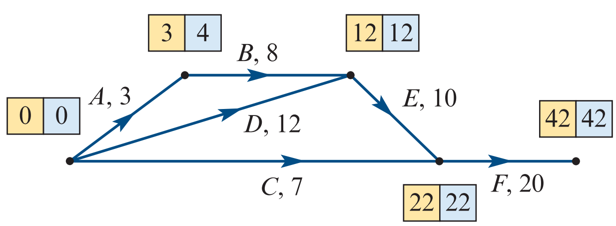
**c** What is the earliest time that activity *E* can start?

**d** What is the float time for activity *E*?

**e** What is the duration of activity *A*?

**f** What is the duration of activity *D*?

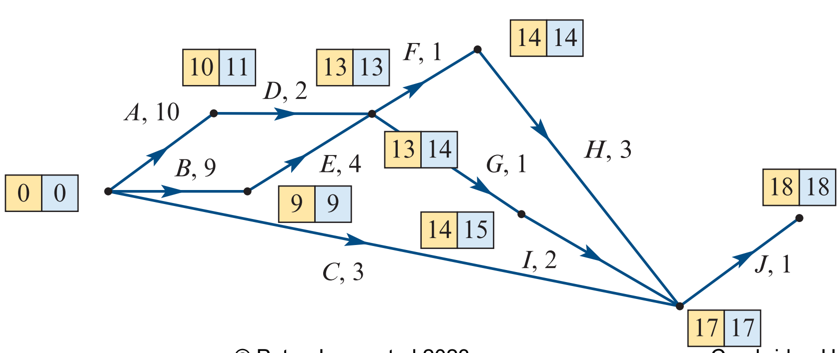
**4** An activity network is shown in the diagram opposite.




**a** Write down the critical path for this project.

**b** Calculate the float times for non-critical activities.

**5** An activity network is shown in the diagram below.

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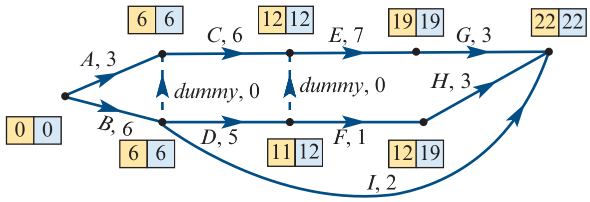
**a** Write down the critical path.

**b** Write down the float times for all non-critical activities.

**6** A precedence table and activity network for a project are shown below. The precedence table is incomplete.

| *Activity* | *Duration (weeks)* | *Immediate predecessors* |
|---|---|---|
| *A* | 3 | – |
| *B* | 6 | – |
| *C* | 6 | |
| *D* | | *B* |
| *E* | 7 | |
| *F* | 1 | *D* |
| *G* | | *E* |
| *H* | 3 | |
| *I* | 2 | *B* |



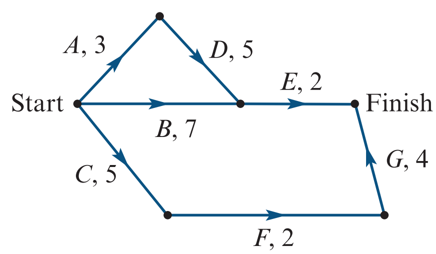

**a** Complete the table above.

**b** Write down the critical path for this project.

### Critical path analysis from a given activity network

Example 10

**7** Consider the following activity network for a project. The duration of each activity is given in the network, in days.



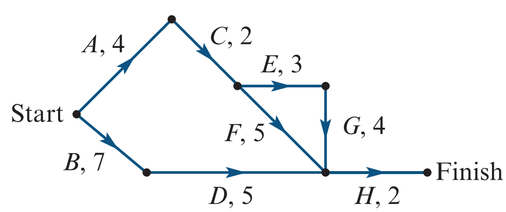

**a** Determine the earliest start time for activity E.

**b** Find the minimum completion time for this project.

**c** Write down the critical path for this project.

**d** Which activity has a float time of two days?

**8** Consider the following activity network for a project.

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**a** Write down the three activities that are immediate predecessors of activity $H$.

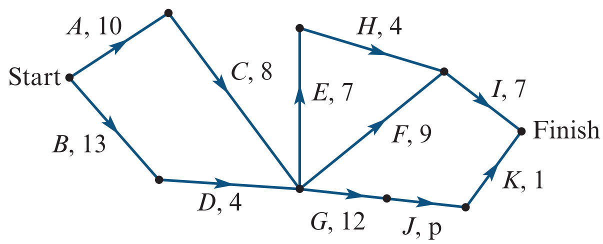
**b** Determine the earliest start time of activity $H$.

**c** For activity $H$ the earliest start time and the latest start time are the same. What does this tell us about activity $H$?

**d** Determine the minimum completion time, in hours, for this project.

**e** Which activity could be delayed for the longest time without affecting the minimum completion time of the project?

**9** Consider the following activity network for a project.



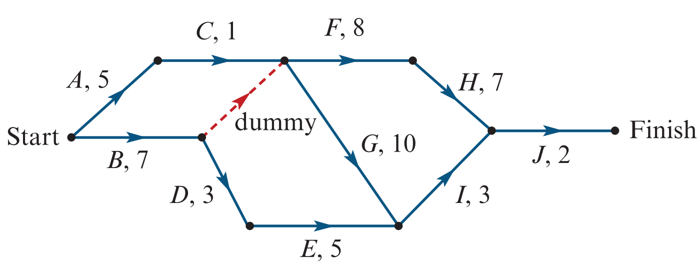

**a** Determine the earliest start time for the following activities:

**i** $H$ **ii** $I$ **iii** $J$

**b** Determine the value of $p$, in weeks, that would create more than one critical path.

**c** If the value of $p$ is 3 weeks, what will be the float time, in weeks, of activity $G$.

**10** Consider the following activity network for a project.




**a** Write down the two immediate predecessors of activity $G$.

**b** Which of the ten activities must be completed before activity $I$ can begin?

**c** Write down the critical path of for this project.

**d** Determine the float time of activity $E$.

**e** Which three activities could have their completion times increased by two days without altering the minimum completion time?

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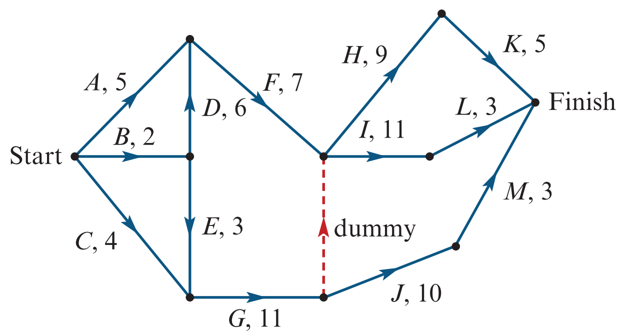
**11** Consider the following activity network for a project.




**a** Complete a precedence table for this network, using two columns, one column for the activities and a second column for the Immediate predecessors.

**b** How many activities have an earliest start time of 16 hours?

**c** Find the latest start time of activity $F$.

**d** There are two critical paths. Write down both critical paths.

**e** How many activities can be delayed by 1 hour without increasing the minimum completion time of the project?

## Critical path analysis from precedence table only

Example 11

**12** The precedence table for a project is shown opposite.

**a** Draw an activity network for this project.

**b** Complete the critical path analysis to calculate the EST and LST for each activity.

**c** Write down the critical path of this project.

**d** What is the minimum time required to complete the project?

| *Activity* | *Duration (weeks)* | *Immediate predecessors* |
|---|---|---|
| $P$ | 4 | – |
| $Q$ | 5 | – |
| $R$ | 12 | – |
| $S$ | 3 | $P$ |
| $T$ | 6 | $Q$ |
| $U$ | 3 | $S$ |
| $V$ | 4 | $R$ |
| $W$ | 8 | $R$, $T$, $U$ |
| $X$ | 13 | $V$ |
| $Y$ | 6 | $W$, $X$ |

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**13** The precedence table for a project is shown opposite.

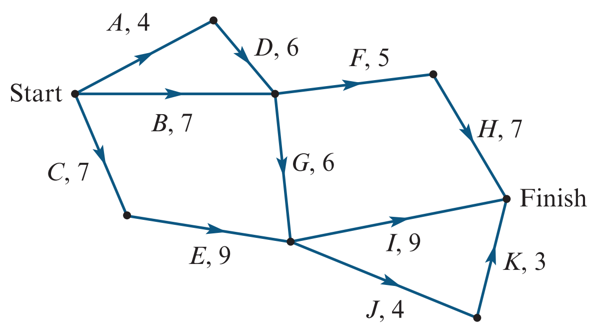
- **a** Draw an activity network for this project.
- **b** Complete the critical path analysis to calculate the EST and LST for each activity.
- **c** Write down the critical path of this project
- **d** What is the minimum time required to complete the project?

| *Activity* | *Duration (weeks)* | *Immediate predecessors* |
|---|---|---|
| $I$ | 2 | – |
| $J$ | 3 | – |
| $K$ | 5 | – |
| $L$ | 4 | $I$ |
| $M$ | 8 | $J$, $N$ |
| $N$ | 1 | $K$ |
| $O$ | 6 | $L$, $M$ |
| $P$ | 6 | $J$, $N$ |
| $Q$ | 7 | $J$, $N$ |
| $R$ | 5 | $K$ |
| $S$ | 1 | $O$ |
| $T$ | 9 | $Q$, $R$ |

## Exam 1 style questions

*Use the following information to answer questions 14, 15 and 16*

The directed network opposite shows the sequence of eleven activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.




**14** The earliest starting time, in days, for activity $J$ is

**A** 12 **B** 13 **C** 14 **D** 15 **E** 16

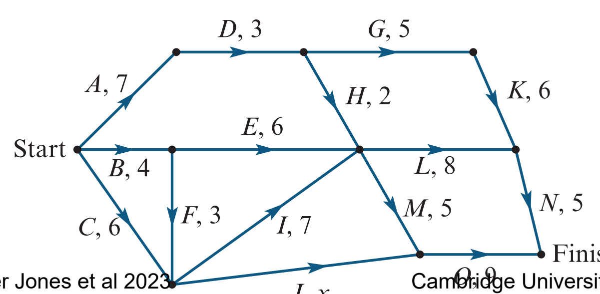
**15** The number of activities that have exactly two immediate predecessors is

**A** 1 **B** 2 **C** 3 **D** 4 **E** 5

**16** How many of these activities could be delayed without affecting the minimum completion time of the project?

**A** 3 **B** 4 **C** 5 **D** 6 **E** 7

**17** The directed graph opposite shows the sequence of activities required to complete a project. The time taken to complete each activity, in weeks, is also shown.

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The minimum completion time for this project is 28 weeks. The time taken to complete activity $J$ is labelled $x$. The maximum value of $x$ is

**A** 12 **B** 10 **C** 8 **D** 4 **E** 2

**18** A project consists of ten activities, $A$ to $J$. The table below shows the immediate predecessor(s) and earliest start time, in days, of each activity.

| *Activity* | *Immediate predecessors* | *Earliest starting time* |
|---|---|---|
| $A$ | – | 0 |
| $B$ | – | 0 |
| $C$ | – | 0 |
| $D$ | $A$ | 6 |
| $E$ | $B$ | 5 |
| $F$ | $B$ | 5 |
| $G$ | $C$ | 4 |
| $H$ | $D, E$ | 13 |
| $I$ | $F, G$ | 14 |
| $J$ | $H, I$ | 25 |

It is known that activity $H$ has a completion time of ten days. The project can still be completed in minimum time if activity $D$ is delayed. The maximum length of the delay for activity $D$ is

**A** one day **B** two days **C** seven days **D** eight days **E** nine days

# 14E Crashing

**Learning intentions**

- To be able to use crashing to reduce the completion time of a project.
- To be able to minimise the cost of crashing activities to achieve the maximum reduction in completion time of a project.

## Altering completion times

The minimum time it takes to complete a project depends upon the time it takes to complete the individual activities of the project, and upon the predecessors each of the activities have. Critical path analysis can be completed to find the overall minimum completion time.

Sometimes, the managers of a project might arrange for one or more activities within the project to be completed in a shorter time than originally planned. Changing the conditions of an activity within a project, and recalculating the minimum completion time for the project, is called **crashing**.

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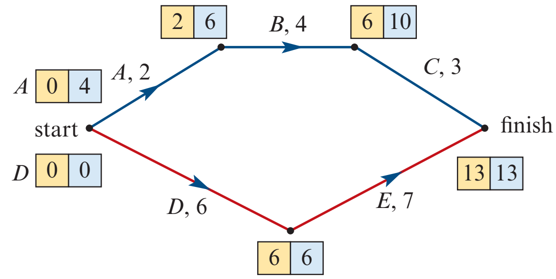
An individual activity could be crashed by employing more staff, sourcing alternate materials or simply because weather or other factors allow the activity to be completed in a shorter time than usual.

## A simple crashing example

A simple activity network is shown in the diagram on the right. The forwards and backwards scanning processes have been completed and the critical path has been determined. The critical path is shown in red on the diagram.




The minimum time for completion is currently 13 hours. In order to reduce this overall time, the manager of the project should try to complete one, or more, of the activities in a shorter time than normal. Reducing the time taken to complete activity $A$, $B$ or $C$ would not achieve this goal however. These activities are not on the critical path and so they already have slack time. Reducing their completion time will not shorten the overall time taken to complete the project.

Activity $D$ and $E$, on the other hand, lie on the critical path. Reducing the duration of these activities will reduce the overall time for the project. If activity $D$ was reduced in time to 4 hours instead, the project will be completed in 11, not 13, hours.

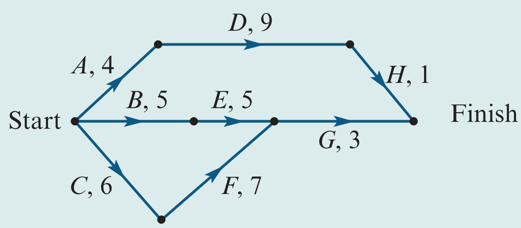
## Crashing with cost

Shortening the completion time for any individual activity could result in an extra cost for the project. In the simple example above, the cost of reducing the completion time of activity $D$ by 1 hour is \$150, while the cost of reducing the completion time of activity $E$ by 1 hour is \$18.

Clearly it is best to reduce the completion time, or crash, the activity that will cost the least.

### Example 12 Crashing one activity with cost

The directed network below shows the sequence of 8 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.




**a** Write down the critical path for this project.

**b** What is the minimum completion time for the project?

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Activity F can be reduced by a maximum of 3 days at a cost of $100 per day.

**c** What is the new minimum completion time for the project?

**d** What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

**Explanation**

**a** In crashing problems, we first need to identify the critical path, or paths. We will do this by remembering that a critical path is the longest or equal longest path in the activity network. Using this method, set up a table, list all possible paths from *Start* to *Finish* for the directed network and calculate the length of each path. The critical path is the path with the longest time from *Start* to *Finish*.

**Solution**

| *Path* | *Duration(days)* |
|---|---|
| $A - D - H$ | 14 |
| $B - E - G$ | 13 |
| $C - F - G$ | 16 |

The critical path is $C - F - G$

**Explanation**

**b** Write the duration of the critical path identified in the previous part.

**c** Crash all possible activities by the maximum reduction. Add a new column to the summary table to get an overview of the new duration of each path. This may result in a new critical path. Consider the cost of crashing and whether it is worth applying the maximum reduction.

**Solution**

16 days

| *Path* | *Duration* (days) | *New duration with maximum reduction* ($F$ by 3) |
|---|---|---|
| $A - D - H$ | 14 | 14 |
| $B - E - G$ | 13 | 13 |
| $C - F - G$ | 16 | 13 |

The new minimum completion time for the project is 14 days.

Activity $F$ originally took 7 days to complete. It can be crashed, which means activity $F$ may be reduced by a maximum of 3 days, to result in a completion time of 4 days. It is possible to choose to reduce acitvity F by 0, 1, 2 or 3 days. Reducing activity $F$ by the maximum 3 days would result in the original critical path to be reduced from a total of 16 days, down to 13 days. Considering there is a cost of $100 per day, this is **not** a desirable outcome; crashing activity $F$ by 3 days results in a new critical path $A - D - H$ with a total completion time of 14 days. If we crash activity $F$ by 2 days only, we create 2 equal critical paths requiring 14 days to complete the project.

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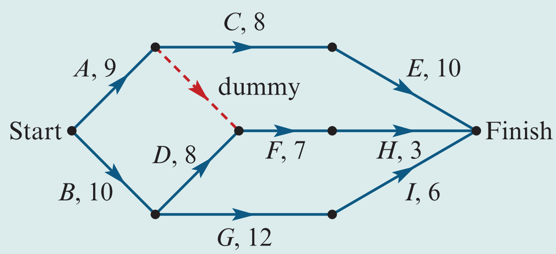
| | |
|---|---|
| **d** Reducing activity $F$ by 2 days allows us to reduce the overall completion time of the project at minimum cost. | \$200. |

## Example 13 Crashing multiple activities with cost

The directed network below shows the sequence of 9 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.




The minimum completion time for the project is 28 days. It is possible to reduce the completion time for activities $B, E, G, H$ and $I$. The completion time for each of these five activities can be reduced by a maximum of two days.

**a** What is the new minimum completion time, in days, that the project could take?

The reduction in completion time for each of these five activities will incur an additional cost. The table opposite shows the five activities that can have their completion times reduced and the associated daily cost, in dollars.

| Activity | Daily cost(\$) |
|---|---|
| $B$ | 1500 |
| $E$ | 2000 |
| $G$ | 700 |
| $H$ | 900 |
| $I$ | 800 |

**b** What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

### Explanation

**a** List all possible paths from *Start* to *Finish*, including the completion time of each. Crash all activities by their maximum reduction. Identify the new critical path (path with the longest completion time) after the maximum reductions are applied.

### Solution

| Path | Duration (days) | New duration after maximum reduction (B,E,G,H,I by 2) |
|---|---|---|
| $A - C - E$ | 27 | 25 |
| $A - F - H$ | 19 | 17 |
| $B - D - F - H$ | 28 | 24 |
| $B - G - I$ | 28 | 22 |

$A - C - E$ is the new critical path with a duration of 25 days.

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**Explanation**

**b** **1** Begin with the new critical path $A - C - E$. The reduction of activity $E$ must occur to achieve the new minimal completion time, therefore reducing activity $E$ by 2 days is essential.

**2** Ignore the path $A - F - H$ because its completion time of 19 days is already lower than the critical path.

**3** Consider the path $B - D - F - H$. It has a completion time of 28 days and must be reduced to 25 days to equal the critical path of $A - C - E$. There are two options; reduce $B$ by 2 and $H$ by 1 or reduce $B$ by 1 and $H$ by 2. From the table, it is more expensive per day to reduce $B$ than $H$, however by choosing to reduce $B$ this will also reduce the completion time of the final path $B - G - I$, which is more cost effective; reducing activity $B$ reduces the completion time of *two* different paths. So reduce activity $B$ by 2 and $H$ by 1 day to reduce the overall completion time of $B - D - F - H$ down to 25 days.

**4** The final path $B - G - I$ has already been reduced by 2 days due to the reduction of activity $B$ previously chosen. One more activity must be reduced for this path to equal the critical path. Activity $G$ has a lower cost of reduction than activity $I$ per day, so include this in your calculation.

**5** Calculate your total cost of crashing.

**Solution**

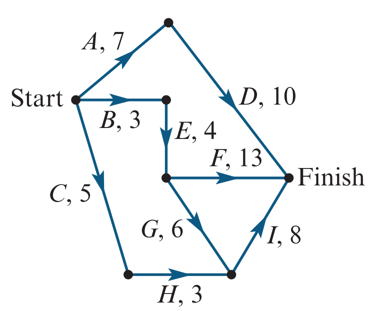
**b** Cost of crashing $= E$ by 2 days $+ B$ by 2 days $+ H$ by 1 day $+ G$ by 1 day

$$= 2000 \times 2 + 1500 \times 2 + 900 + 700$$

$$= \$8600$$

## Exercise 14E

**1** The activity network for a project is shown in the diagram below. The duration for each activity is in hours.

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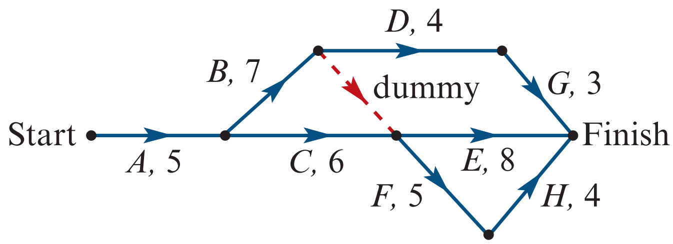
**a** List all four paths from the Start to the Finish of the project, with their respective completion times.

**b** Identify the critical path and the minimum completion time for the project.

**c** If Activity E is reduced by 3 hours, identify the new minimum completion time for this project.

Example 11

**2** The directed network below shows the sequence of 8 activities that are needed to complete a project. The time, in days, that is takes to complete each activity is also shown.




**a** Write down the critical path for this project.

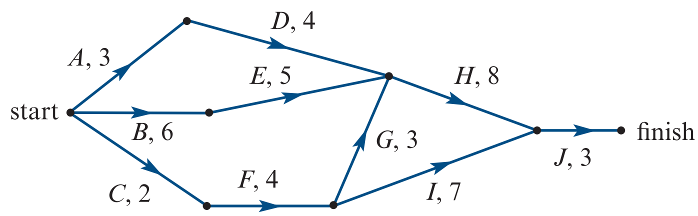
**b** What is the minimum completion time for the project?

Activity B can be reduced by a maximum of 3 days at a cost of $100 per day.

**c** What is the new minimum completion time for the project?

**d** What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

**3** The activity network for a project is shown in the diagram on the right. The duration for each activity is in hours.




**a** Identify the critical path for this project.

**b** What is the maximum number of hours that the completion time for activity $E$ can be reduced by without changing the minimum completion time of the project?

**c** What is the maximum number of hours that the completion time for activity $H$ can be reduced without changing the minimum completion time of the project?

**d** Every activity can be reduced in duration by a maximum of 2 hours. If every activity was reduced by the maximum amount possible, what is the new minimum completion time for the project?

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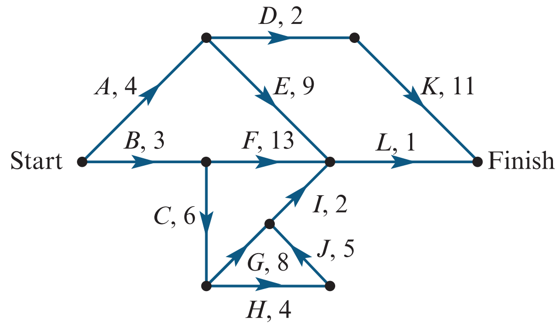
**4** The activity network for a project is shown in the diagram below. The duration for each activity is in hours.

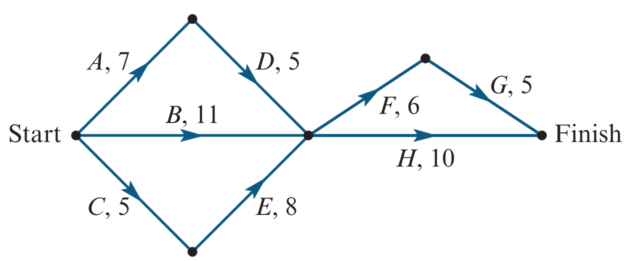



**a** How many activities could be delayed by 4 hours without altering the minimum completion time for the project?

**b** If the project is to be crashed by reducing the completion time of one activity only, what is the minimum time, in hours, that the project can be completed in?

**c** Activity G can be reduced in time at a cost of $200 per hour. Activity J can be reduced in time at a cost of $150 per hour. What is the cost of reducing the completion time of this project as much as possible?

Example 13 **5** The directed network below shows the sequence of 8 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.




The minimum completion time for the project is 24 days. It is possible to reduce the completion time for activities *D*, *E* and *H*. The completion time for each of these three activities can be reduced by a maximum of two days.

**a** What is the new minimum completion time, in days, of the project?

The reduction in completion time for each of these three activities will incur an additional cost. The table opposite shows the three activities that can have their completion times reduced and the associated daily cost, in dollars.

| Activity | Daily cost($) |
|---|---|
| *D* | 170 |
| *E* | 350 |
| *H* | 200 |

**b** What is the minimum cost that will achieve the greatest reduction in time taken to complete the project?

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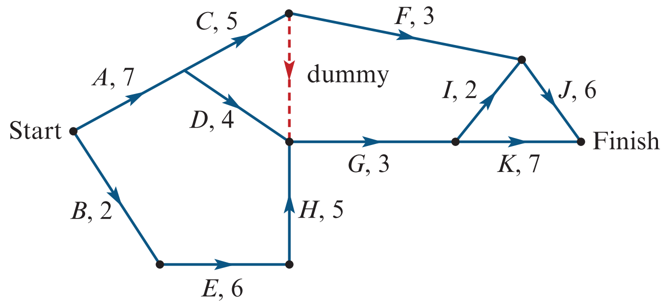
**6** The directed network below shows the sequence of 11 activities that are needed to complete a project. The time, in days, that it takes to complete each activity is also shown.




**a** Which activities are immediate predecessors to activity G?

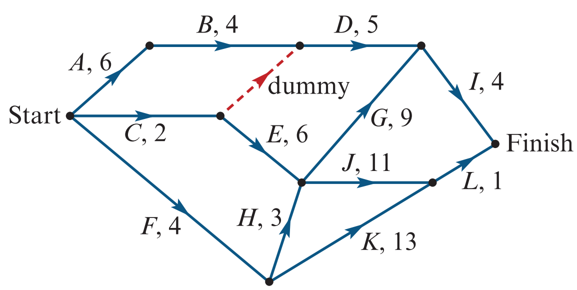
**b** Which activities, if crashed, would create more than one critical path?

**c** The project could finish earlier if some activities were crashed. Five activities, $B, E, G, H$ and $I$, can all be reduced by one hour. The cost of this crashing is \$150 per hour.

**i** What is the minimum number of hours in which the project could now be completed?

**ii** What is the minimum cost of completing the project in this time?

**7** The directed network below shows the sequence of 12 activities that are needed to complete a project. The time, in weeks, that it takes to complete each activity is also shown.




**a** Determine the shortest time in which this project can be completed.

**b** Determine the earliest start time for activity $D$.

**c** Determine the latest start time for activity $H$.

**d** Which activity has a float time of more than two weeks?

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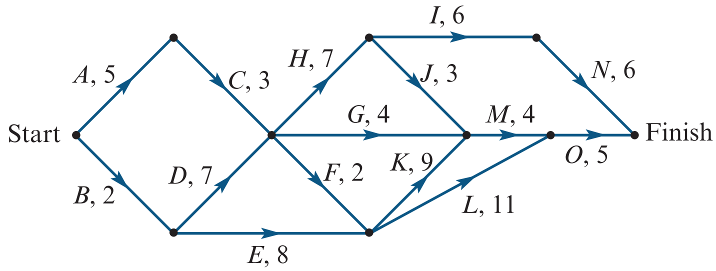
**e** The completion times for activities $D, E, G, H$ and $J$ can each be reduced by a maximum of two weeks. The table opposite shows the five activities than can have their completion time reduced and the associated weekly cost, in dollars.
What is the minimum cost to complete the project in the shortest time possible?

| Activity | Weekly cost($) |
|---|---|
| $D$ | 2000 |
| $E$ | 1000 |
| $G$ | 500 |
| $H$ | 1500 |
| $J$ | 3000 |

**8** The directed network below shows the sequence of 15 activities that are needed to complete a maintenance project at the MCG. The time, in days, that it takes to complete each activity is also shown.




**a** What is the minimum completion time?

**b** How many activities are on the critical path?

**c** How many paths have a completion time of 28 days?

**d** The completion times for activities $H, J, K, L$ and $M$ can each be reduced by a maximum of two days. The cost of reducing the time of each activity is \$500 per day. The MCG requires the overall completion time for the maintenance project to be reduced by three days at minimum cost. Complete the table below, showing the reductions in individual activity completion times that would achieve this.

| Activity | *Reduction in completion time* (0, 1 or 2 days) |
|---|---|
| $H$ | |
| $J$ | |
| $K$ | |
| $L$ | |
| $M$ | |

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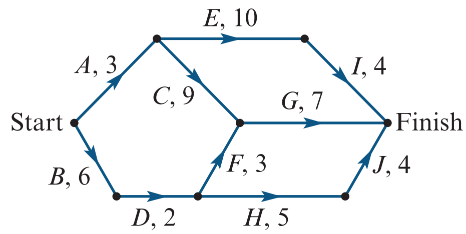
## Exam 1 style questions

*Questions 9 and 10 refer to the diagram opposite.*

The directed graph opposite shows the sequence of activities required to complete a project. All times are in hours.



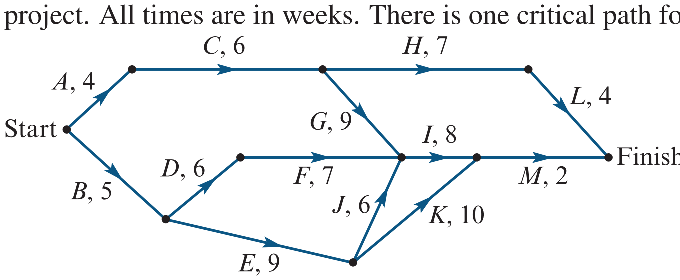

**9** There is one critical path for this project.
The critical path is

**A** $A-E-I$
**B** $A-C-G$
**C** $B-D-F-G$
**D** $B-D-H-J$
**E** $A-C-F-H-J$

**10** Four critical paths would exist if the duration of activities

**A** *A* and *B* were reduced by one hour.
**B** *C* and *G* were reduced by one hour.
**C** *A* and *C* were reduced by two hours.
**D** *B* and *G* were reduced by two hours.
**E** *D* and *H* were reduced by three hours.

**11** The directed graph below shows the sequence of activities required to complete a project. All times are in weeks. There is one critical path for this project.




The total completion time of the project can be reduced by four weeks by reducing

**A** activity *B* by four weeks
**B** activity *F* by four weeks.
**C** activity *J* by four weeks.
**D** activity *I* by three weeks and activity *J* by one week.
**E** activity *D* by three weeks and activity *E* by one week.

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## Key ideas and chapter summary

Assignment

**Weighted graph** — A **weighted graph** is a graph in which a number representing the size of same quantity is associated with each edge. These numbers are called weights.

**Network** — A **network** is a weighted graph in which the weights are physical quantities, for example distance, time or cost.

**Directed graph (digraph)** — A **directed graph** is a graph where direction is indicated for every edge. This is often abbreviated to **digraph**.

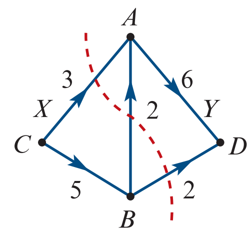
**Flow** — The transfer of material through a directed network. **Flow** can refer to the movement of water or traffic.

**Capacity** — The maximum flow of substance that an edge of a directed graph can allow during a particular time interval. The **capacity** of water pipes is the amount of water (usually in litres) that the pipe will allow through per time period (minutes, hours, etc.). Other examples of capacity are number of cars per minute or number of people per hour.

**Source** — The **source** is the origin of the material flowing through a network.

**Sink** — The **sink** is the final destination of the material flowing through a network.

**Cut** — A **cut** is a line dividing a directed graph into two parts (shown as a broken line dividing the graph below into two sections, labelled $X$ and $Y$).




**Cut capacity** — The sum of the capacities (weights) of the edges directed from $X$ to $Y$ that the cut passes through. For the weighted digraph shown, the capacity of the cut is 7.

**Minimum cut** — The **minimum cut** is the cut with the minimum capacity. The cut must separate the source from the sink.

**Maximum flow** — The **maximum flow** through a directed graph is equal to the capacity of the minimum cut.

**Bipartite graph** — A **bipartite graph** has two distinct groups or categories for the vertices. Connections exist between a vertex or vertices from one group with a vertex or vertices from the other group. There are no connections between the vertices within a group.

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**Allocation** An **allocation** is made when each of the vertices in one group from a bipartite graph are matched with one of the vertices in the other group from that graph. An allocation is possible when both groups have exactly the same number of vertices. The vertices in each group are matched to only one vertex from the other group.

**Cost matrix** A table that contains the costs of allocating objects from one group (such as people) to another (such as tasks). The 'cost' can be money, or other factors such as the time taken.

**Hungarian algorithm** The **Hungarian algorithm** is an algorithm that is used to determine the best allocation to minimise the overall cost.

**Activity network** An **activity network** is a directed graph that shows the required order of completing individual activities that make up a project.

**Immediate predecessor** If activity *A* is an **immediate predecessor** to activity *B*, activity *A* must be completed before activity *B* can begin.

**Precedence table** A **precedence table** is a table that records the activities of a project and their immediate predecessors. Precedence tables can also contain the duration of each activity.

**Dummy activity** A **dummy activity** has zero cost. It is required if two activities share some, but not all, of the same immediate predecessors. It allows the network to show all precedence relationships in a project correctly.

**Earliest starting time (EST)** **EST** is the earliest time an activity in a project can begin.

**Latest starting time (LST)** **LST** is the latest time an activity in a project can begin, without affecting the overall completion time for the project.

**Float (slack) time** **Float (slack) time** is the difference between the latest starting time and the earliest starting time.

Float = LST – EST

The float time is sometimes called the slack time. It is the largest amount of time that an activity can be delayed without affecting the overall completion time for the project.

**Forward scanning** **Forward scanning** is a process of determining the EST for each activity in an activity network. The EST of an activity is added to the duration of that activity to determine the EST of the next activity. The EST of any activity is equal to the largest forward scanning value determined from all immediate predecessors.

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Review

**Backward scanning** — **Backward scanning** is a process of determining the LST for each activity in an activity network. The LST of an activity is equal to the LST of the activity that follows, minus the duration of the activity.

**Critical path** — The **critical path** is the series of activities that cannot be delayed without affecting the overall completion time of the project. Activities on the critical path have no slack time. Their EST and LST are equal.

**Critical path analysis** — **Critical path analysis** is a project planning method in which activity durations are known with certainty.

**Crashing** — **Crashing** is the process of shortening the length of time in which a project can be completed by reducing the time required to complete individual activities. Reducing the individual activity completion times often costs money; this increases the overall cost of a project.

## Skills checklist

Check-list

*Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.* ☑

14A **1** I can define and describe a directed graph. ☐

See Example 1, and Exercise 14A Question 1

14A **2** I can determine the maximum flow for any section of sequential edges of a directed graph. ☐

See Example 4, and Exercise 14A Question 4

14A **3** I can determine cut capacities. ☐

See Example 2, and Exercise 14A Question 2

14A **4** I can determine the maximum flow as equal to the minimum cut capacity. ☐

See Example 3, and Exercise 14A Question 5

14B **5** I can draw directed and weighted bipartite graphs. ☐

See Exercise 14B Question 4

14B **6** I can use the Hungarian algorithm to determine an optimum allocation in order to minimise cost. ☐

See Exercise 14B Question 2

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14C **7** I can create an activity network from a precedence table. ☐

See Example 5, and Exercise 14C Question 5

14C **8** I can write down a precedence table from an activity network. ☐

See Exercise 14C Question 1

14C **9** I can decide when to use dummy activities in an activity network. ☐

See Example 6, and Exercise 14C Question 3

14D **10** I can use forward scanning to determine the earliest starting time of activities in an activity network. ☐

See Example 8, and Exercise 14D Question 12

14D **11** I can use backward scanning to determine the latest starting time of activities in an activity network. ☐

See Example 8, and Exercise 14D Question 12

14D **12** I can determine the float time for activities in an activity network. ☐

See Example 8, and Exercise 14D Question 7

14D **13** I can determine the overall minimum completion time for a project using critical path analysis. ☐

See Example 8, and Exercise 14D Question 8

14D **14** I can determine the critical path for an activity network. ☐

See Example 8, and Exercise 14D Question 9

14E **15** I can use crashing to reduce the completion time of a project. ☐

See Example 9, and Exercise 14E Question 1

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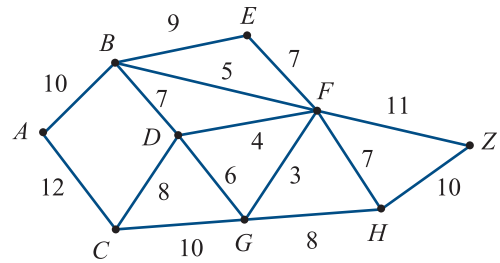
## Multiple-choice questions

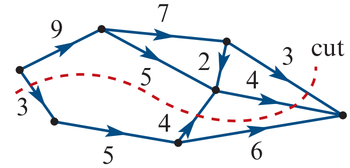
**1** The shortest path from $A$ to $Z$ in the network on the right has length:

**A** 10 **B** 15
**C** 22 **D** 26
**E** 28



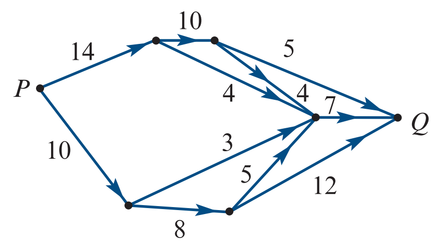

**2** For the network shown on the right, the capacity of the cut is:

**A** 3 **B** 6 **C** 9
**D** 10 **E** 14



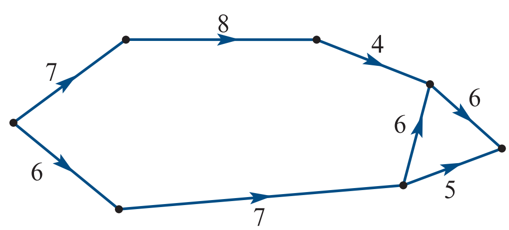

**3** In the communications network shown, the numbers represent transmission capacities for information (data) in scaled units. What is the maximum flow of information from station $P$ to station $Q$?

**A** 20 **B** 22
**C** 23 **D** 24
**E** 30

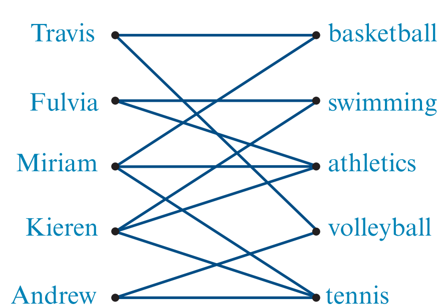



**4** The maximum flow in the network opposite, from source to sink, is:

**A** 10 **B** 11
**C** 12 **D** 13
**E** 14




**5** A group of five students represent their school in five different sports. The information is displayed in a bipartite graph. From this graph we can conclude that:




**A** Travis and Miriam played all the sports between them.
**B** In total, Miriam and Fulvia played fewer sports than Andrew and Travis.
**C** Kieren and Miriam each played the same number of sports.
**D** In total, Kieren and Travis played fewer different sports than Miriam and Fulvia.
**E** Andrew played fewer sports than any of the others.

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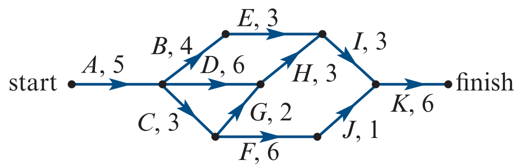
**6** This activity network is for a project where the component times in days are shown. The critical path for the network of this project is given by:




**A** $A–B–E–I–K$

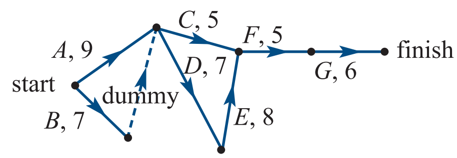
**B** $A–D–H–I–K$

**C** $A–C–G–H–I–K$

**D** $A–C–F–J–K$

**E** $A–D–G–F–J–K$

**7** The activity network shown represents a project development with activities and their durations (in days) listed on the edges of the graph. Note that the dummy activity takes zero time.




The earliest time (in days) that activity $F$ can begin is:

**A** 0 **B** 12 **C** 14 **D** 22 **E** 24

**8** The table opposite lists the seven activities in a project and the earliest start time, in hours, and the predecessor(s) of each task. The time taken for activity $F$ is five hours. Without affecting the time taken for the entire project, the time taken for activity $D$ could be increased by:

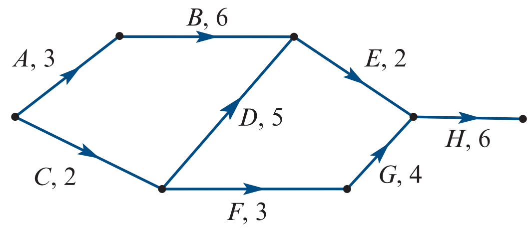
**A** 0 hours **B** 2 hours

**C** 3 hours **D** 4 hours

**E** 12 hours

| *Task* | *Immediate predecessor* | *EST* |
|---|---|---|
| *A* | – | 0 |
| *B* | – | 0 |
| *C* | *A* | 24 |
| *D* | *B* | 29 |
| *E* | *C* | 39 |
| *F* | *D* | 41 |
| *G* | *E*, *F* | 50 |

**9** The edges in this activity network correspond to the tasks involved in the preparation of an examination. The numbers indicate the time, in weeks, needed for each task. The total number of weeks needed for the preparation of the examination is:




**A** 14 **B** 15 **C** 16 **D** 17 **E** 27

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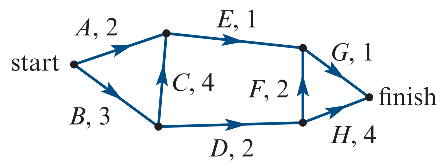
Review

**10** The activity network represents a manufacturing process with activities and their duration (in hours) listed on the edges of the graph. The earliest time (in hours) after the start that activity *G* can begin is:




**A** 3 **B** 5 **C** 6 **D** 7 **E** 8

## Written-response questions

**1** An English class recently performed poorly in their essay writing assessment. To help them improve, the teacher separated the class into groups of five and assigned one of the following tasks to each person: Introduction, Body paragraph 1, Body paragraph 2, Body paragraph 3 and Conclusion. Each task will be completed by one person. The table below shows the time, in minutes, that each person would take to complete each of the five tasks.

| | Intro | Par 1 | Par 2 | Par 3 | Con |
|---|---|---|---|---|---|
| **Alvin** | 16 | 14 | 19 | 9 | 9 |
| **Billy** | 17 | 18 | 10 | 9 | 9 |
| **Chloe** | 9 | 8 | 6 | 15 | 8 |
| **Danielle** | 11 | 12 | 11 | 16 | 6 |
| **Elena** | 10 | 10 | 8 | 15 | 8 |

The tasks will be allocated so that the total time of completing the five tasks is a minimum

**a** Complete the sentences below by clearly stating which task each student should write in order for the essay to be completed in the minimum time possible.

- Alvin should write the...
- Billy should write the...
- Chloe should write the...
- Danielle should write the...
- Elena should write the...

**b** What is the minimum total time the group will dedicate to completing the essay?

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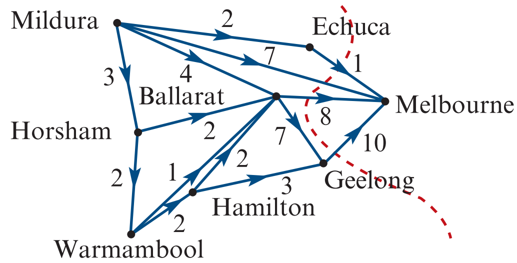
**2** WestAir Company flies routes in western Victoria. The network shows the layout of connecting flight paths for WestAir, which originate in Mildura and terminate in either Melbourne or on the way to Melbourne. On this network, the available spaces for passengers flying out of various locations on one morning are shown.




The network has one cut shown.

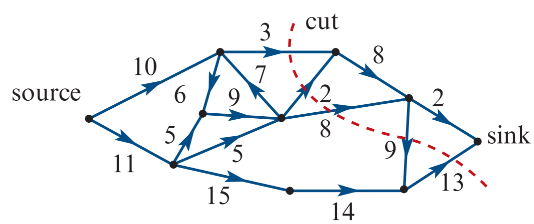
**a** What is the capacity of this cut?

**b** What is the maximum number of passengers who could travel from Mildura to Melbourne for the morning?

**3** A school swimming team wants to select a $4 \times 200$ metre relay team. The fastest times of its four best swimmers in each of the strokes are shown in the table below. Which swimmer should swim which stroke to give the team the best chance of winning, and what would be their time to swim the relay?

| *Swimmer* | *Backstroke* | *Breaststroke* | *Butterfly* | *Freestyle* |
|---|---|---|---|---|
| Rob | 76 | 78 | 70 | 62 |
| Joel | 74 | 80 | 66 | 62 |
| Henk | 72 | 76 | 68 | 58 |
| Sav | 78 | 80 | 66 | 60 |

**4** In the network opposite, the values on the edges give the maximum flow possible between each pair of vertices. The arrows show the direction of flow in the network. Also shown is a cut that separates the source from the sink.




**a** Determine the capacity of the cut shown.

**b** Determine the maximum flow through this network.

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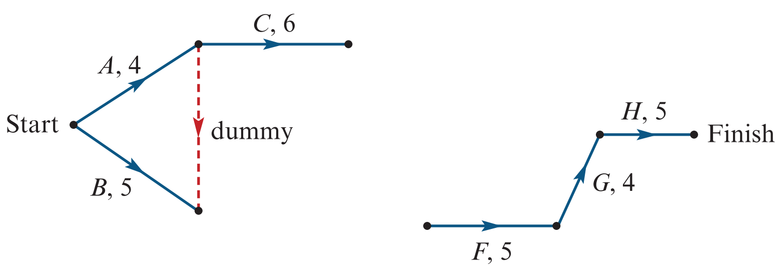
**5** A project requires eight activities (A – H) to be completed. The duration, in hours, and the immediate predecessor(s) of each activity are shown in the table below.

| *Activity* | *Duration (hours)* | *Immediate predecessor(s)* |
|---|---|---|
| $A$ | 4 | – |
| $B$ | 5 | – |
| $C$ | 6 | $A$ |
| $D$ | 7 | $A$, $B$ |
| $E$ | 10 | $C$, $D$ |
| $F$ | 5 | $D$ |
| $G$ | 4 | $F$ |
| $H$ | 5 | $E$, $G$ |

**a** The directed network that shows these activities is shown below. Add the three missing features to the network.




**b** Determine the earliest start time for activity $E$.

**c** What is the float time of activity $G$?

**d** How many of these activities have a non-zero float time?

**e** Write down the critical path for this project.

**f** What is the minimum completion time for this project?

**g** The project could finish earlier if some activities were crashed.

  **i** Activity $E$ can be crashed by two hours. If this occurs, what will be the new critical path for the project?

  **ii** In addition to activity $E$ crashing by two hours, activities, $A$, $B$ and $D$ can also be crashed by one hour each. What is the minimum number of days in which the project can now be completed?

**h** After careful deliberation, it is decided that crashing the original directed network is not possible. Alternatively, an upgrade will be made to this project, where one extra activity will be added. This activity has a duration of five hours, an earliest starting time of twelve hours and a latest starting time of seventeen hours. Complete the following sentence by filling in the boxes provided.
The extra activity could be represented on the network above by a directed edge from the end of activity ☐ to the start of activity ☐

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