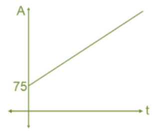
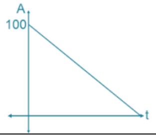
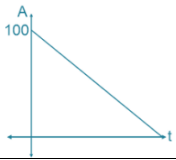
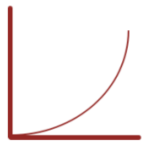
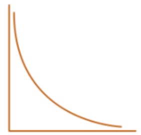
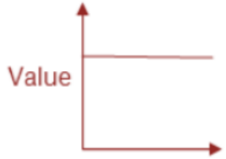
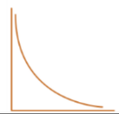
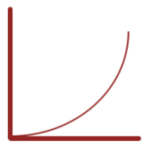
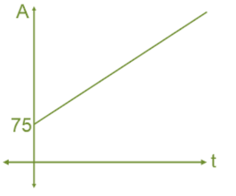
Financial Maths

	Recurrence Relation	Application	Explicit Rules	Graphs	Mathematica command#
Arithmetic Sequences	$v_0 = principal, v_{n+1} = v_n + D$ D = $\frac{r}{100} \times v_0$	Simple Interest	$v_n = v_0 + n * D$		7a Interest in \$ 7b Table graph 7c Future Value
D	$v_0 = Initial\ Value, v_{n+1} = v_n - D$ D = $\frac{r}{100} \times v_0$	Flat Rate Depreciation	$v_n = v_0 - n * D$		9a depreciate \$ 9b Table graph 9c Future Value
Common difference	$v_0 = Initial\ Value, v_{n+1} = v_n - D$ D = $Unit\ cost\ in\ dollars$	Unit Cost Depreciation	$v_n = v_0 - n * D$		10a Table graph 10b Future Value
Geometric Sequences	$v_0 = principal, v_{n+1} = R * v_n$ $R = 1 + \frac{r}{100}$	Compound Interest	$v_n = R^n * v_0$		8a Common Ratio 8b Table graph 8c Future Value
R Common Ratio Growth Factor Decay factor	$v_0 = Initial\ Value, v_{n+1} = R * v_n$ $R = 1 - \frac{r}{100}$	Reduced Balance Depreciation	$v_n = R^n * v_0$		11a Common Ratio 11b Table graph 11c Future Value
Combined Arithmetic & Geometric Sequences	$v_0 = Initial\ Value, v_{n+1} = R * v_n - D$ $R = 1 + \frac{r}{100*p}$ D = $\frac{r}{100} \times v_0$	FV → Interest Only Loan /Perpetuities			6 Common Ratio 3a Common Difference / Interest in \$ / Payment
R Growth Factor	$v_0 = Initial\ Value, v_{n+1} = R * v_n - D$ $R = 1 + \frac{r}{100*p}$	FV↓ Reduced Balance Loans /Annuities	Compound Interest loan extra payment		6 Common Ratio 3b Future Value 3d/12a Final Pymnt 12b Total payment 12c Total Interest 13 Partial Interest
D Repayment	$v_0 = Initial\ Value, v_{n+1} = R * v_n + D$ $R = 1 + \frac{r}{100*p}$	FV↑ Compound Interest Investment Annuity Investment			6 Common Ratio 3c Future Value

Note: Yellow parts need real number, blue parts are formula to calculate, Letter P indicates Compounding monthly etc. Needing rate per month, Monthly Ratio etc.

Core: Recursion and financial mathematics

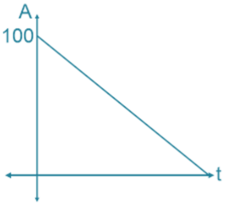
- **Simple interest – (linear growth)**



→ <u>Recurrence model:</u> V_0= **principal**, $V_{n+1} = V_n + D$, where **D = r/100 x V_0**

→ <u>Recurrence rule:</u> $V_n = V_0 + nD$, **D= r/100 x V_0**

- **Flat-rate depreciation – (linear decay)**



→ <u>Recurrence model:</u> V_0= **initial value of asset**, $V_{n+1} = V_n – D$, where **D = r/100 x V_0**

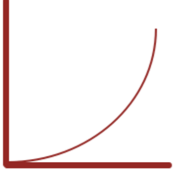
→ <u>Recurrence rule:</u> $V_n = V_0 – nD$, where **D= r/100 x V_0**

- **Unit-cost depreciation – (linear decay)**

→ <u>Recurrence model:</u> V_0= **initial value of asset**, $V_{n+1} = V_n – D$, where **D= cost per unit of use**

→ <u>Recurrence rule:</u> $V_n = V_0 – nD$, where **n = no. of times used**, where **D= cost per unit of use**

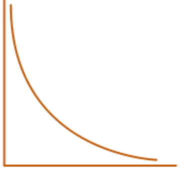
- **Compound interest investments and loans – (geometric growth)**



→ <u>Recurrence model:</u> V_0= **principal**, $V_{n+1} = RV_n$, where **R = 1 + r/100**

→ <u>Recurrence rule:</u> $V_n = (1+ r/100)^n \times V_0$

- **Reducing balance depreciation – (geometric decay)**



→ Another type of depreciation

→ <u>Recurrence model:</u> V_0= **initial value**, $V_{n+1} = RV_n$, where **R = 1 - r/100**

→ <u>Recurrence rule:</u> $V_n = (1- r/100)^n \times V_0$

- **Nominal interest rate**

→ **The rate per annum / number of compounding periods**

→ <u>Example:</u> 3.6 % p.a converted into a monthly rate:
= 3.6/ 12 = 0.3% per month

> <u>Note:</u> Increasing the number of compounding periods per year will increase the total interest earned / paid.

2. Calculating effective interest rates

r := nominal rate (% p.a.)

n := number of time periods

$$effectiverate = \left(\left(1 + \frac{\frac{r}{n}}{100}\right)^{n} - 1.\right) * 100$$

Clear[r, n]

Effective rate of interest

Effective interest rate

The effective interest rate of a loan or investment is the interest earned after one year expressed as a percentage of the amount borrowed or invested.

Let:

- r be the nominal interest rate per annum
- $r_{\text{effective}}$ be the effective annual interest rate
- n be the number of times the interest compounds each year.

The effective annual interest rate is given by: $r_{\text{effective}} = \left(\left(1 + \frac{r/n}{100}\right)^{n} - 1\right) \times 100\%$

Effective Interest Rate in Mathematica

– due to regular payment to make FV ↓

- # Reducing balance loans

→ Recurrence model: **V_0= principal**, **$V_{n+1} = RV_n$ - D**, where **R = 1 + r /100**, where **D= payment made**, where **r = interest rate per compounding period**

→ Amortisation: reducing the balance of the loan until it reaches a value of **zero**

→ Worked example: *Interest on a $1000 loan was charged at the rate of 1.25% per month and the loan was to be repaid with four monthly payments of $257.85.*

Payment number	Payment amount	Interest paid	Principal reduction	Balance of loan
0	0	0	0	1000.00
1	257.85	12.50	245.35	754.65
2	257.85	9.43	248.42	506.23
3	257.85	6.33	251.52	254.71
4	257.89*	3.18	254.71	0.00
	Total	31.44	1000.00	

- Future value↓
- Rate = r = $\frac{\text{Interest}}{\text{Previous balance}} \times 100$
- Regular Payment = Interest + Principal Reduction
- Previous Balance – Principal Reduction = New Balance of Loan
- Total Interest = Total Payment – Principal Reduction

Properties of a reducing balance loan:

At each step of the loan:

→ **Interest paid = interest rate per payment period × unpaid balance.**
For example, when payment 1 is made, interest paid = 1.25% of 1000 = $12.50

→ **principal reduction = payment made − interest paid**
For example, when payment 1 is made, principal reduction = 257.85 − 12.50 = $245.35

→ **balance of loan = balance owing − reduction in balance**
For example, when payment 1 is made, reduction in balance = 1000 − 245.35 = $754.65.

→ **cost of repaying the loan = the sum of the payments**
For this loan, the total cost of repaying the loan = 3 × 257.85 + 257.89 = $1031.44.

→ **total interest paid = total cost of repaying the loan − principal**
For this loan, the total interest paid = 1031.44 − 1000 = $31.44.

Reducing Balance Loan in Mathematica

To find Payment for Reduced Balance Loan or Annuity

$$futurevalue[prin\_, r\_, n\_, t\_, payt\_] := prin\left(1 + \frac{\left(\frac{r}{n}\right)}{100}\right)^{(n*t)} - payt * \left(\frac{\left(\left(1 + \frac{\left(\frac{r}{n}\right)}{100}\right)^{(n*t)} - 1\right)}{\left(\frac{\left(\frac{r}{n}\right)}{100}\right)}\right)$$

```
NSolve[futurevalue[prin, r, n, t, payt] == Future Value, Reals]
```

– due to regular payment to make FV ↓

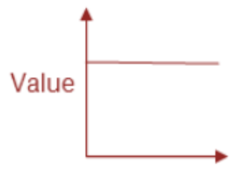
- *Interest only loans:* borrower repays only the interest charged, the value of the loan remains the same for the duration of the loan

→ Recurrence relation: **V_0 = initial amount, $V_{n+1} = RV_n - D$**
where **$R = 1 + r/100$, $D = r/100 \times V_0$**

→ Payment (D) = interest charged:
$D = r/100 \times V_0$

→ Example: $D = RV_n$
V_1 : $50 = 1.05 \times 1000$

(if the payments on the loan are \$50, then the amount owed on the loan will remain the same as \$1000)



$D = 50$
$V_0 = 1000,$
$V_{n+1} = 1.05V_n - 50$
$V_0 = 1000$
$V_1 = 1000$
$V_2 = 1000$
$V_3 = 1000$
$V_4 = 1000$
The amount owed stays constant.

Interest Only Loan in Mathematica

Repayment for Interest Only Loan or Perpetuity

```
repayment[prin_, r_, n_] := ((r/n)/100) prin
NSolve[repayment[prin, r, n] == payt, Reals]
```

- *Annuities*

→ Recurrence relation: **V_0 = principal, $V_{n+1} = RV_n - D$**, where **$R = 1 + r/100$,**
where **D = payment received**

→ Amortisation Worked example: *\$12 000 was invested in annuity that earns interest at the rate of 6% p.a, providing a monthly income of \$2035 per month for 6 months.*

→ Note: *after the 6th payment, the balance actually ends up being 0.88 (88 cents). But since we need our balance to be zero, we just add the 88 cents to the payment amount.*

2035.00+0.88= \$ 2035.88 payment for payment 6.

Payment number	Payment received	Interest earned	Principal reduction	Balance of annuity
0	0	0.00	0.00	12000.00
1	2035.00	60.00	1975.00	10025.00
2	2035.00	50.13	1984.88	8040.13
3	2035.00	40.20	1994.80	6045.33
4	2035.00	30.23	2004.77	4040.55
5	2035.00	20.20	2014.80	2025.76
6	2035.88	10.13	2024.87	0.00

Annuity in Mathematica

To find Payment for Reduced Balance Loan or Annuity

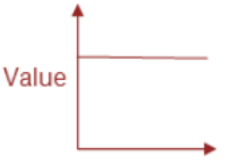
```
futurevalue[prin_, r_, n_, t_, payt_] := prin (1 + (r/n)/100)^(n*t) - payt * (((1 + (r/n)/100)^(n*t) - 1)/((r/n)/100))
NSolve[futurevalue[prin, r, n, t, payt] == Future Value, Reals]
```

– due to regular payment to make FV ↓

- **Perpetuities:** an annuity where the regular payments are the same as the interest earned

→ Recurrence relation: **V_0 = principal**, **$V_{n+1} = RV_n - D$**, where **$R = 1 + r/100$**, where **D= payment received**

→ Payment (D) = interest charged:
$D = r/100 \times V_0$



Perpetuity in Mathematica

Repayment for Interest Only Loan or Perpetuity

```
repayment[prin_, r_, n_] := ((r/n)/100) prin
NSolve[repayment[ prin , r , n ] == payt , Reals]
```

- **Compound interest investment**

+ due to regular payment to make FV ↑

→ Recurrence relation: **V_0= the principal**, **$V_{n+1} = RV_n + D$**, where **$R = 1 + r/100$**, **D= payment made**

→ Amortisation Worked example: $V_0 = 1200$, $V_{n+1} = 1.0025V_n + 50$

Payment number	Payment made	Interest earned	Principal increase	Balance of investment
0	0.00	0.00	0.00	1200.00
1	50.00	3.00	53.00	1253.00
2	50.00	3.13	53.13	1306.13
3	50.00	3.27	53.27	1359.40
4	50.00	3.40	53.40	1412.80
5	50.00	3.53	53.53	1466.33
6	50.00	3.67	53.67	1519.99
7	50.00	3.80	53.80	1573.79
8	50.00	3.93	53.93	1627.73
9	50.00	4.07	54.07	1681.80
10	50.00	4.20	54.20	1736.00
11	50.00	4.34	54.34	1790.34
12	50.00	4.48	54.48	1844.82

- Future value↑
- Rate = r = $\frac{\text{Interest}}{\text{Previous balance}} \times 100$
- Regular Payment + Interest = Principal Increase
- Previous Balance+ Principal Increase = New Balance
- Total Interest = Final Balance – Principal – additional paym

Properties of an investment:

At each step of the investment:

→ **interest earned = interest rate per compounding period × previous balance**
For example, when payment 1 is made, interest paid = 0.25% of 1200 = $3.00

→ **principal increase = payment made + interest earned**
For example, when payment 1 is made principal increase = 3.00 + 50.00 = $53.00

→ **Balance of investment = previous balance + principal increase**
For example, when payment 1 is made, the new balance is 1200.00 + 3.00 + 50.00 = $1253.00.

→ **total interest earned = balance of loan − (principal + additional payments)**
After 12 months, the total interest earned = 1844.82 − (1200 + 12 × 50) = $44.82
Note: This amount can also be obtained by summing the interest column.

Notes: If calculating interest during **midway the loan (i.e. loan has not reached a value of 0 yet),** then the calculation is as follows: **Total number of repayments – reduction in principle (initial loan amount – current loan balance)**

Compound Interest Investment or Annuity Investment in Mathematica

To find Payment for Compound interest investments with regular additions to the principal (annuity investment)

```
futurevalue[prin_, r_, n_, t_, payt_] := prin (1 + (r/n)/100)^(n*t) + payt * (((1 + (r/n)/100)^(n*t) - 1) / ((r/n)/100))
NSolve[futurevalue[ prin , r , n , t , payt ] == Future Value , Reals]
```

+ due to regular payment to make FV ↑

3. Financial Solver for Future Value and Regular Payment

FV ↓ Final Payment, Time = $\frac{\text{Number of REGULAR payment}}{\text{Number of payment per year}}$ = Number of Regular payment/ Number of payment per year

The final payment for Reduced Balance Loan or Annuity

$$finalpayment[prin\_, r\_, n\_, t\_, payt\_] := \left(\left(prin\left(1+\frac{\left(\frac{r}{n}\right)}{100}\right)^{(n*t)} - payt*\left(\frac{\left(\left(1+\frac{\left(\frac{r}{n}\right)}{100}\right)^{(n*t)}-1\right)}{\left(\frac{\left(\frac{r}{n}\right)}{100}\right)}\right)\right) + \left(prin\left(1+\frac{\left(\frac{r}{n}\right)}{100}\right)^{(n*t)} - payt*\left(\frac{\left(\left(1+\frac{\left(\frac{r}{n}\right)}{100}\right)^{(n*t)}-1\right)}{\left(\frac{\left(\frac{r}{n}\right)}{100}\right)}\right)\right)*\left(\frac{\left(\frac{r}{n}\right)}{100}\right)\right)$$

```
NSolve[finalpayment[ Principal , Rate per annum , Number of payment per year , Time in YEARS , Payment ] == The Final Payment , Reals] // FullForm
```

7. Simple Interest

Four values involved: rate per annum, number of payment per year, initial value, **interest in dollars**

Finding interest in dollars (Common difference) for Simple Interest

$$difference[V0\_, r\_, p\_, D\_] := \frac{r}{100\,p} * V0 \; // \; N$$

```
NSolve[difference[ Principal , Rate per annum , Number of payment per year , Interest in dollars (Common Difference) ] == Interest in dollars (Common Difference) , Reals]
```

Generating Table values and Graphs

Generate a table and graph of Simple Interest--**Recurrence Relation** V_0= Principal , $V_{n+1}=V_n+$ Interest in dollars (Common Difference) .
Explicit Rule: $V_n= V_0+$ Number of Payment X Interest in dollars (Common Difference)

```
f[x_] := x + Interest in dollars (Common Difference)
NestList[f, Principal , ( Number of payment per year * Time in YEARS )]
ListLinePlot[NestList[f, Principal , ( Number of payment per year * Time in YEARS )]]
```

Four values involved: rate per annum, number of payment per year, initial value, **Time in Years**

Explicit Rule Calculation for Simple Interest

$$nthvalue[V0\_, r\_, p\_, n\_] := V0 + n * p * \frac{r}{100\,p} * V0$$

```
NSolve[nthvalue[ Principal , Rate per annum , Number of payment per year , Time in YEARS ] == Future Value , Reals]
```

7. Simple Interest

8. Compound Interest

9. Flat Rate Depreciation

10. Unit Cost Depreciation

11. Reduced Balance Depreciation

Commands needed for comparing tables or graphs

Example: **Reduced Balance Depreciation**
initial Value=30000, Common Ratio 0.92, depreciating monthly, Future values after 3 years in table and graph

Generate a table and graph of Reduced balance Depreciation--**Recurrence Relation:** V_0= Initial Value, V_{n+1}= Common Ratio V_n.

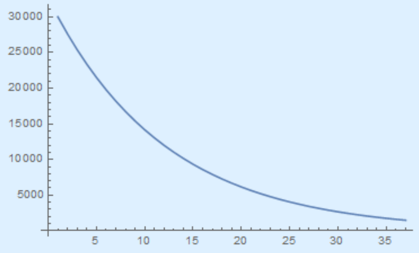
Explicit Rule: $V_n = V_0$ Common Ratio$^{\text{Number of payment per year} * \text{Time in YEARS}}$

```
f[x_] := Common Ratio x
NestList[f, Initial Value, (Number of payment per year * Time in YEARS)]
ListLinePlot[NestList[f, Initial Value, (Number of payment per year * Time in YEARS)]]
```

In[ ]:=
```
f[x_] := 0.92 x
NestList[f, 30000, (12 * 3)]
ListLinePlot[NestList[f, 30000, (12 * 3)]]
```

Out[ ]=
```
{30000, 27600., 25392., 23360.6, 21491.8, 19772.4, 18190.7, 16735.4, 15396.6, 14164.8, 13031.7, 11989.1, 11030., 10147.6, 9335.78, 8588.92, 7901.81, 7269.66,
6688.09, 6153.04, 5660.8, 5207.94, 4791.3, 4408., 4055.36, 3730.93, 3432.45, 3157.86, 2905.23, 2672.81, 2458.99, 2262.27, 2081.29, 1914.78, 1761.6, 1620.67, 1491.02}
```

Out[ ]=



Amortisation table

- Future value↓
- Rate = r = $\frac{\text{Interest}}{\text{Previous balance}} \times 100$
- Regular Payment = Interest + Principal Reduction
- Previous Balance – Principal Reduction = New Balance of Loan
- Total Interest = Total Payment – Principal Reduction
- Future value↑
- Rate = r = $\frac{\text{Interest}}{\text{Previous balance}} \times 100$
- Regular Payment + Interest = Principal Increase
- Previous Balance+ Principal Increase = New Balance
- Total Interest = Final Balance – Principal – additional payment