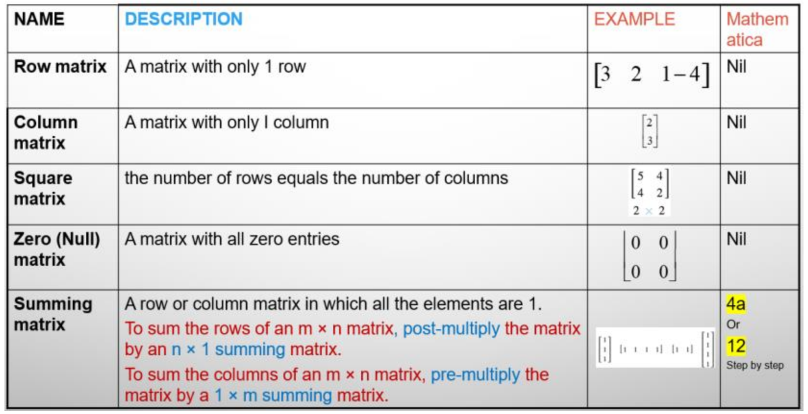
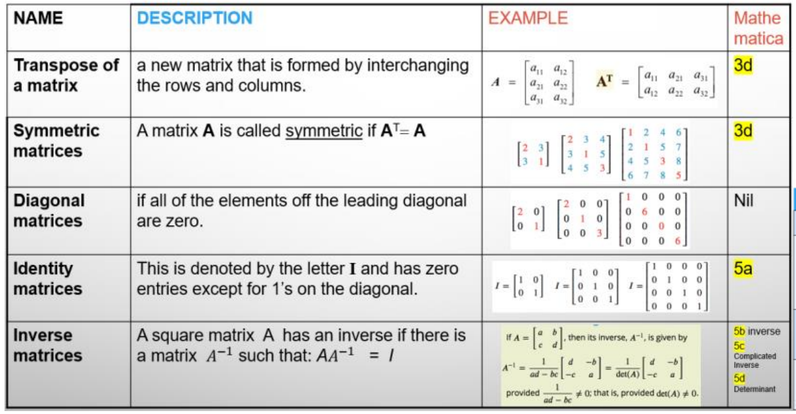
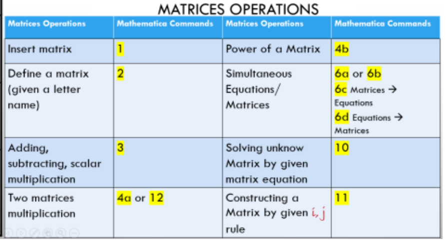


| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Row matrix** | A matrix with only 1 row | $[3 \quad 2 \quad 1-4]$ | Nil |
| **Column matrix** | A matrix with only I column | $\begin{bmatrix}2\\3\end{bmatrix}$ | Nil |
| **Square matrix** | the number of rows equals the number of columns | $\begin{bmatrix}5&4\\4&2\end{bmatrix}$<br>$2 \times 2$ | Nil |
| **Zero (Null) matrix** | A matrix with all zero entries | $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ | Nil |
| **Summing matrix** | A row or column matrix in which all the elements are 1.<br>To sum the rows of an m × n matrix, post-multiply the matrix by an n × 1 summing matrix.<br>To sum the columns of an m × n matrix, pre-multiply the matrix by a 1 × m summing matrix. | [illegible] | 4a<br>Or<br>12<br>Step by step |

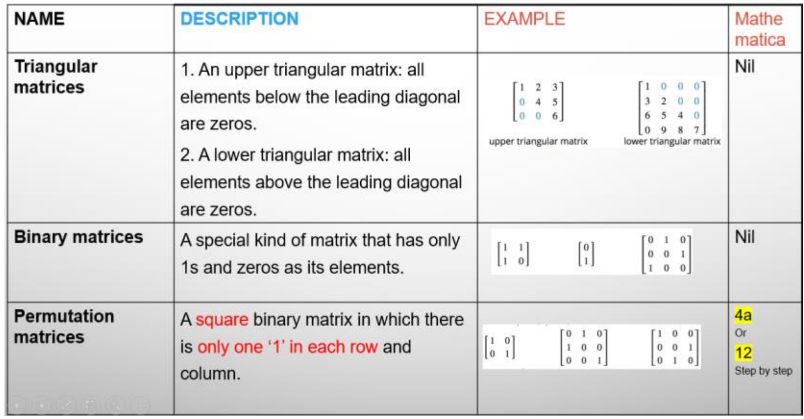


| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Transpose of a matrix** | a new matrix that is formed by interchanging the rows and columns. | $A = \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix} \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\end{bmatrix}$ | 3d |
| **Symmetric matrices** | A matrix **A** is called symmetric if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ | $\begin{bmatrix}2&3\\3&1\end{bmatrix} \begin{bmatrix}2&3&4\\3&1&5\\4&5&3\end{bmatrix} \begin{bmatrix}1&2&4&6\\2&1&5&7\\4&5&3&8\\6&7&8&5\end{bmatrix}$ | 3d |
| **Diagonal matrices** | if all of the elements off the leading diagonal are zero. | $\begin{bmatrix}2&0\\0&1\end{bmatrix} \begin{bmatrix}2&0&0\\0&1&0\\0&0&3\end{bmatrix} \begin{bmatrix}1&0&0&0\\0&6&0&0\\0&0&0&0\\0&0&0&6\end{bmatrix}$ | Nil |
| **Identity matrices** | This is denoted by the letter **I** and has zero entries except for 1's on the diagonal. | $I = \begin{bmatrix}1&0\\0&1\end{bmatrix} \quad I = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad I = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ | 5a |
| **Inverse matrices** | A square matrix A has an inverse if there is a matrix $A^{-1}$ such that: $AA^{-1} = I$ | If $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, then its inverse, $A^{-1}$, is given by<br>$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$<br>provided $\frac{1}{ad-bc} \neq 0$; that is, provided $\det(A) \neq 0$. | 5b inverse<br>5c Complicated Inverse<br>5d Determinant |

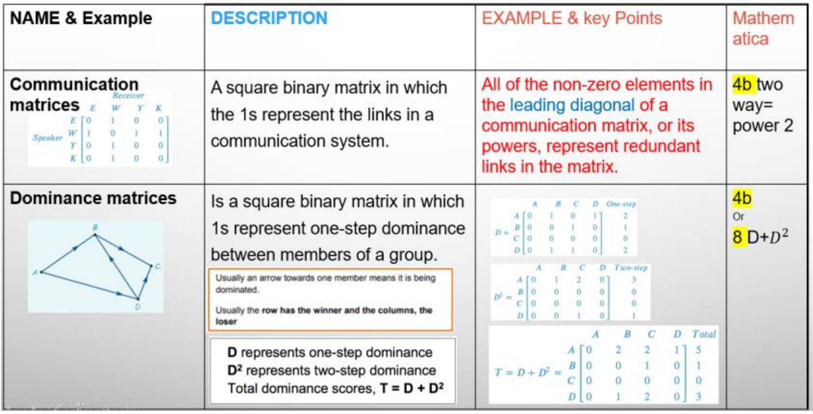


MATRICES OPERATIONS

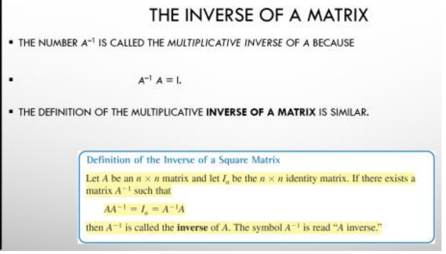
| Matrices Operations | Mathematica Commands | Matrices Operations | Mathematica Commands |
|---|---|---|---|
| Insert matrix | 1 | Power of a Matrix | 4b |
| Define a matrix (given a letter name) | 2 | Simultaneous Equations/ Matrices | 6a or 6b<br>6c Matrices → Equations<br>6d Equations → Matrices |
| Adding, subtracting, scalar multiplication | 3 | Solving unknow Matrix by given matrix equation | 10 |
| Two matrices multiplication | 4a or 12 | Constructing a Matrix by given $i, j$ rule | 11 |



| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Triangular matrices** | 1. An upper triangular matrix: all elements below the leading diagonal are zeros.<br>2. A lower triangular matrix: all elements above the leading diagonal are zeros. | $\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}$ upper triangular matrix<br>$\begin{bmatrix}1&0&0&0\\3&2&0&0\\6&5&4&0\\0&9&8&7\end{bmatrix}$ lower triangular matrix | Nil |
| **Binary matrices** | A special kind of matrix that has only 1s and zeros as its elements. | $\begin{bmatrix}1&1\\1&0\end{bmatrix} \quad \begin{bmatrix}0\\1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ | Nil |
| **Permutation matrices** | A square binary matrix in which there is only one '1' in each row and column. | $\begin{bmatrix}1&0\\0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ | 4a<br>Or<br>12<br>Step by step |



| NAME & Example | DESCRIPTION | EXAMPLE & key Points | Mathematica |
|---|---|---|---|
| **Communication matrices**<br>$\begin{array}{cc} & \text{Receiver} \\ \text{Speaker} & \begin{array}{c\|cccc} & E & W & Y & K \\ \hline E & 0&1&0&0 \\ W & 1&0&1&1 \\ Y & 0&1&0&0 \\ K & 0&1&0&0 \end{array}\end{array}$ | A square binary matrix in which the 1s represent the links in a communication system. | All of the non-zero elements in the leading diagonal of a communication matrix, or its powers, represent redundant links in the matrix. | 4b two way= power 2 |
| **Dominance matrices** | Is a square binary matrix in which 1s represent one-step dominance between members of a group.<br>Usually an arrow towards one member means it is being dominated.<br>Usually the **row has the winner and the columns, the loser**<br>**D** represents one-step dominance<br>**D²** represents two-step dominance<br>Total dominance scores, **T = D + D²** | $D = \begin{array}{c\|cccc\|c} & A & B & C & D & \text{One-step} \\ \hline A & 0&1&0&1&2 \\ B & 0&0&1&0&1 \\ C & 0&0&0&0&0 \\ D & 0&1&1&0&2 \end{array}$<br>$D^2 = \begin{array}{c\|cccc\|c} & A & B & C & D & \text{Two-step} \\ \hline A & 0&1&2&0&3 \\ B & 0&0&0&0&0 \\ C & 0&0&0&0&0 \\ D & 0&0&1&0&1 \end{array}$<br>$T = D + D^2 = \begin{array}{c\|cccc\|c} & A & B & C & D & \text{Total} \\ \hline A & 0&2&2&1&5 \\ B & 0&0&1&0&1 \\ C & 0&0&0&0&0 \\ D & 0&1&2&0&3 \end{array}$ | 4b<br>Or<br>8 $\mathrm{D}+D^2$ |



THE INVERSE OF A MATRIX

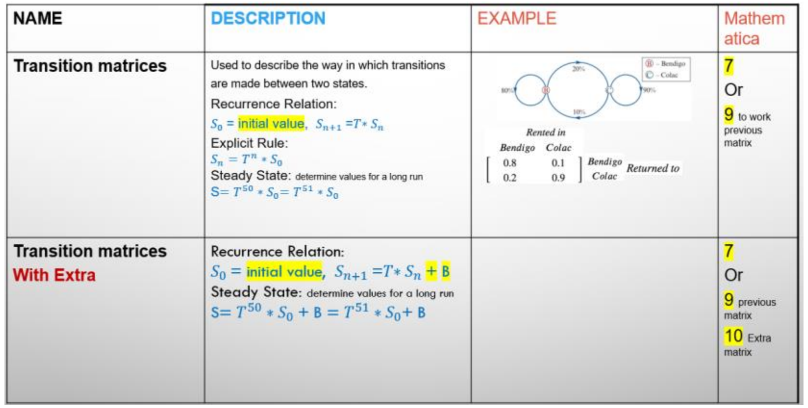
- THE NUMBER $A^{-1}$ IS CALLED THE *MULTIPLICATIVE INVERSE* OF *A* BECAUSE
- $A^{-1}A = I.$
- THE DEFINITION OF THE MULTIPLICATIVE **INVERSE OF A MATRIX** IS SIMILAR.

Definition of the Inverse of a Square Matrix

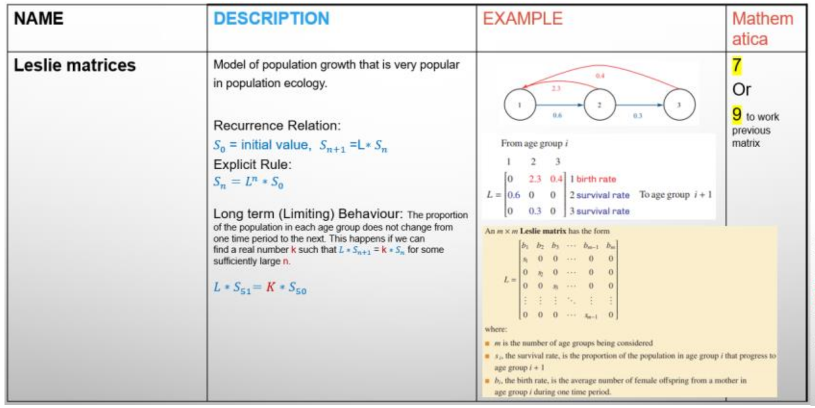
Let $A$ be an $n \times n$ matrix and let $I_n$ be the $n \times n$ identity matrix. If there exists a matrix $A^{-1}$ such that

$$AA^{-1} = I_n = A^{-1}A$$

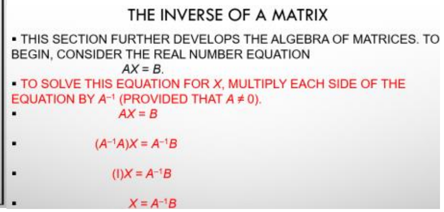
then $A^{-1}$ is called the **inverse** of $A$. The symbol $A^{-1}$ is read "$A$ inverse."



| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Transition matrices** | Used to describe the way in which transitions are made between two states.<br>Recurrence Relation:<br>$S_0$ = initial value, $S_{n+1} = T * S_n$<br>Explicit Rule:<br>$S_n = T^n * S_0$<br>Steady State: determine values for a long run<br>$S = T^{50} * S_0 = T^{51} * S_0$ | $\begin{array}{ccc} \text{Rented in} & & \\ \text{Bendigo} \quad \text{Colac} & & \\ \begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix} & \begin{array}{c}\text{Bendigo}\\\text{Colac}\end{array} & \text{Returned to}\end{array}$ | 7<br>Or<br>9 to work previous matrix |
| **Transition matrices With Extra** | Recurrence Relation:<br>$S_0$ = initial value, $S_{n+1} = T * S_n + \mathrm{B}$<br>Steady State: determine values for a long run<br>$S = T^{50} * S_0 + \mathrm{B} = T^{51} * S_0 + \mathrm{B}$ | | 7<br>Or<br>9 previous matrix<br>10 Extra matrix |



| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Leslie matrices** | Model of population growth that is very popular in population ecology.<br>Recurrence Relation:<br>$S_0$ = initial value, $S_{n+1} = L * S_n$<br>Explicit Rule:<br>$S_n = L^n * S_0$<br>Long term (Limiting) Behaviour: The proportion of the population in each age group does not change from one time period to the next. This happens if we can find a real number k such that $L * S_{n+1} = k * S_n$ for some sufficiently large n.<br>$L * S_{51} = K * S_{50}$ | From age group $i$<br>$L = \begin{array}{ccc} 1 & 2 & 3 \\ \hline 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{array} \begin{array}{l} \\ 1 \text{ birth rate} \\ 2 \text{ survival rate} \\ 3 \text{ survival rate} \end{array}$ To age group $i+1$<br>An $m \times m$ **Leslie matrix** has the form<br>$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$<br>where:<br>$m$ is the number of age groups being considered<br>$s_i$, the survival rate, is the proportion of the population in age group $i$ that progress to age group $i+1$<br>$b_i$, the birth rate, is the average number of female offspring from a mother in age group $i$ during one time period. | 7<br>Or<br>9 to work previous matrix |



THE INVERSE OF A MATRIX

- THIS SECTION FURTHER DEVELOPS THE ALGEBRA OF MATRICES. TO BEGIN, CONSIDER THE REAL NUMBER EQUATION
  $AX = B.$
- TO SOLVE THIS EQUATION FOR $X$, MULTIPLY EACH SIDE OF THE EQUATION BY $A^{-1}$ (PROVIDED THAT $A \neq 0$).
- $AX = B$
- $(A^{-1}A)X = A^{-1}B$
- $(I)X = A^{-1}B$
- $X = A^{-1}B$

# Module 1 – Matrices

- ## Types of matrices:

→ Square matrix: a matrix that has the same number of rows and columns

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

→ Transpose matrix: swapping the rows and columns of the matrix an using the new order to make a new one

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

→ Binary matrix: a matrix made up of only 2 values

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

→ Zero matrix: all elements are zero

→ Permutation matrix: a matrix with 1 in every row that only occurs once

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Identity matrix: a matrix where the 1s are in the diagonal and the remaining elements are zeros

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Symmetric matrix: where the elements reflect each other across the diagonal

$$\begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 1 & 5 & 7 \\ 4 & 5 & 3 & 8 \\ 6 & 7 & 8 & 5 \end{bmatrix}$$

→ Triangular matrix: where the elements above the leading diagonal are zeros (upper triangular) or below the diagonal (lower triangular)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

→ Communication matrix: a square binary matrix where the 1s represent the direct (one-step) communication links.

- To identify two-step communication links (how to communicate through somebody to get to someone), square the original matrix.
- Add C and $C^2$ to find the overall one- step and two-step communication matrix, T.

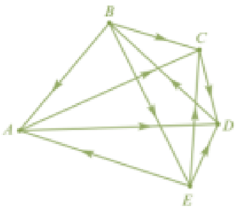
$$\begin{array}{cc} & \text{Receiver} \\ \text{Speaker} & \begin{array}{c|cccc} & E & W & Y & K \\ \hline E & 0 & 1 & 0 & 0 \\ W & 1 & 0 & 1 & 1 \\ Y & 0 & 1 & 0 & 0 \\ K & 0 & 1 & 0 & 0 \end{array} \end{array}$$

- Redundant communication link: when the sender and receiver are the same person.

All of the non-zero elements in the leading diagonal of a communication matrix, or its powers, represent redundant links in the matrix.

→ Dominance matrix: an square binary matrix where the 1s represent the one step dominances between members of the group.

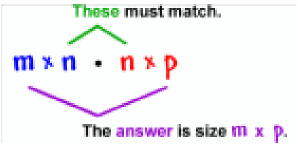
- To identify two- step dominance, just square the one step dominance matrix.



- Add D and $D^2$ to find the overall one-step and two – step dominance matrix, T.

$$D = \begin{array}{c|ccccc|c} & A & B & C & D & E & \text{Dominance} \\ \hline A & 0 & 0 & 1 & 1 & 0 & 2 \\ B & 1 & 0 & 1 & 0 & 1 & 3 \\ C & 0 & 0 & 0 & 1 & 0 & 1 \\ D & 0 & 1 & 0 & 0 & 0 & 1 \\ E & 1 & 0 & 1 & 1 & 0 & 3 \end{array}$$

> Usually an arrow towards one member means it is being dominated.
>
> Usually the **row has the winner and the columns, the loser**



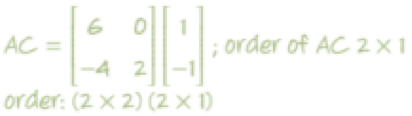
## • Matrix arithmetic

→ Addition and subtraction: matrices need to be the same order if they are added or subtracted.

$$A+B=\begin{bmatrix}2&3&0\\1&4&2\end{bmatrix}+\begin{bmatrix}1&2&3\\2&-2&1\end{bmatrix}$$

→ Multiplication: using scalar multiplication (multiplying the matrix by a number)

- The number of columns of the first matrix must = number of rows of the second matrix
- Matrix 1 order: 2x3 and Matrix 2 order: 3x4
  So order of matrix 1 x matrix 2 = 2x4



$$AC=\begin{bmatrix}6&0\\-4&2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}; \text{order of } AC\ 2\times 1$$

order: $(2\times 2)(2\times 1)$

→ Summing matrix:

- To **sum the rows** of an m x n matrix, **post-multiply** the matrix by an **n x 1** summing matrix.
- To **sum the columns** of an m x n matrix, **pre-multiply** the matrix by a **1 x m** summing matrix.

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}1&1&1&1\end{bmatrix}$$

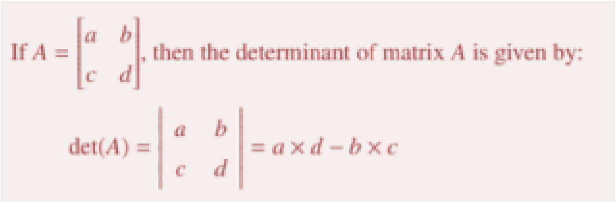
→ Squaring matrices: only **square matrices** can be given powers

## • The inverse matrix

→ When the inverse matrix is multiplied with its inverse, the product is an identity matrix.
→ Only square matrices can be inversed.
→ Inverse of A = $A^{-1}$

## • The determinant

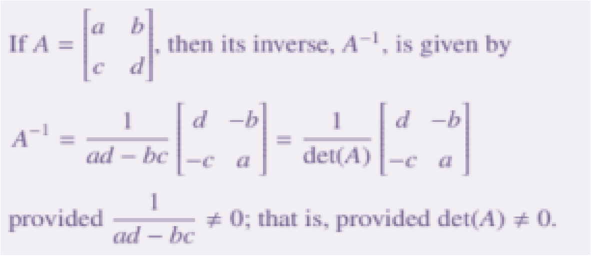
→ When **determinant is 0 = there is no inverse**
→ When determinant is a negative or positive number = has an inverse
→



If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then the determinant of matrix $A$ is given by:

$$\det(A)=\begin{vmatrix}a&b\\c&d\end{vmatrix}=a\times d-b\times c$$

→

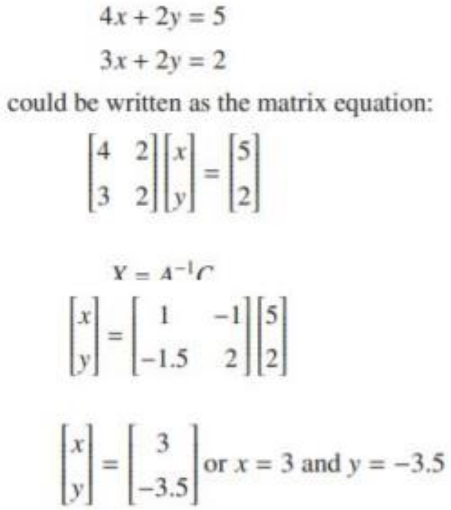


If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then its inverse, $A^{-1}$, is given by

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\frac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

provided $\frac{1}{ad-bc}\neq 0$; that is, provided $\det(A)\neq 0$.

## • Solving simultaneous equations with matrices



→ $4x + 2y = 5$

$3x + 2y = 2$

could be written as the matrix equation:

$$\begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$X = A^{-1}C$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1.5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

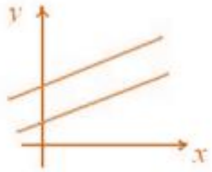
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3.5 \end{bmatrix} \text{ or } x = 3 \text{ and } y = -3.5$$

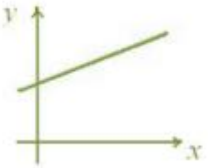
**AX = B**

To solve for X **pre-multiply** both sides by inverse of A so, $\mathbf{X = A^{-1} B}$

**XA = B**

To solve for X **post-multiply** both sides by inverse of A so, $\mathbf{X = BA^{-1}}$



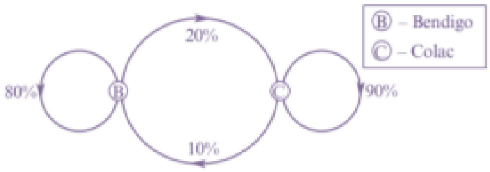


→ <u>When equations are **inconsistent:**</u> their graphs do not cross (**parallel**)
- Determinant = 0

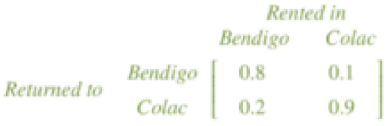
→ <u>When equations are **dependent**</u>: the graphs coincide infinitely (**no unique solution**)
- Determinant = 0

## • Transition matrices

→ <u>Example:</u>



Transition matrix for the diagram:



$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{\textit{Rented in}} \\ & & \textit{Bendigo} & \textit{Colac} \\ \textit{Returned to} & \textit{Bendigo} & 0.8 & 0.1 \\ & \textit{Colac} & 0.2 & 0.9 \end{array}$$

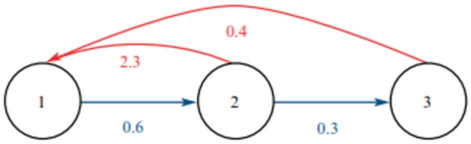
→ <u>Recurrence relation:</u> **$S_0$ = initial matrix, $S_{n+1} = T \times S_n$**

→ <u>Recurrence rule:</u> **$S_n = T^n \times S_0$**

→ <u>Steady state matrix:</u> determining values for the long-run

- Given by $S = T^{50} / T^{51} \times S_0$ (note that n may be higher values for predicting larger numbers)

# LIFE CYCLE TRANSITION DIAGRAM



From age group $i$

$$\begin{array}{cll} & \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ L = & \left[\begin{array}{ccc} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{array}\right] \begin{array}{l} 1 \text{ birth rate} \\ 2 \text{ survival rate} \\ 3 \text{ survival rate} \end{array} & \text{To age group } i+1 \end{array}$$

### Leslie matrices

An $m \times m$ **Leslie matrix** has the form

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

where:

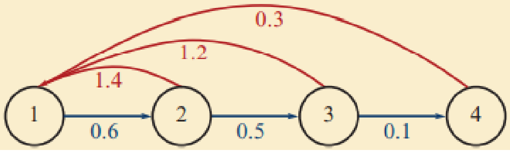
- $m$ is the number of age groups being considered
- $s_i$, the survival rate, is the proportion of the population in age group $i$ that progress to age group $i + 1$
- $b_i$, the birth rate, is the average number of female offspring from a mother in age group $i$ during one time period.

### Leslie matrix and its interpretation

From age group

$$\begin{array}{cll} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ L = & \left[\begin{array}{cccc} 0 & 1.4 & 1.2 & 0.3 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{array}\right] \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} & \text{To age group} \end{array}$$

This is a Leslie matrix with 4 age groups. The corresponding life-cycle transition diagram is shown here.



### Recursive rules

The population matrix $S_n$ is an $m \times 1$ matrix representing the size of each age group after $n$ time periods. This is calculated using a recursive formula

$S_0$ is the initial state matrix, $S_{n+1} = LS_n$

or the explicit rule

$S_n = L^n S_0$

### Limiting behaviour of Leslie matrices

Often we will find that, after a long enough time, the proportion of the population in each age group does not change from one time period to the next. This happens if we can find a real number $k$ such that $LS_{n+1} = kS_n$ for some sufficiently large $n$. This does not happen with every Leslie matrix as we see in Example 13.