AOS 2

Recursion and financial modelling

CALCULATOR QUICK LOOK-UP GUIDE

5

CHAPTER 5

Recurrence relations and basic financial applications

LESSONS

KEY KNOWLEDGE

- use of a first-order linear recurrence relation of the form: $u_0 = a$, $u_{n+1} = Ru_n + d$ where a, R and d are constants to generate the terms of a sequence
- use of a recurrence relation to model and compare (numerically and graphically) flat rate, unit cost and reducing balance depreciation of the value of an asset with time, including the use of a recurrence relation to determine the depreciating value of an asset after n depreciation periods for the initial sequence
- use of the rules for the future value of an asset after n depreciation periods for flat rate, unit cost and reducing balance depreciation and their application
- the concepts of simple and compound interest
- use of a recurrence relation to model and analyse (numerically and graphically) a compound interest investment or loan, including the use of a recurrence relation to determine the value of the compound interest loan or investment after n compounding periods for an initial sequence from first principles
- the difference between nominal and effective interest rates and the use of effective interest rates to compare investment returns and the cost of loans when interest is paid or charged, for example, daily, monthly, quarterly
- the future value of a compound interest investment or loan after n compounding periods and its use to solve practical problems.

5A Recurrence relations and their graphs

STUDY DESIGN DOT POINT

- use of a first-order linear recurrence relation of the form: $u_0 = a$, $u_{n+1} = Ru_n + d$ where a, R and d are constants to generate the terms of a sequence

5A 5B 5C 5D 5E 5F 5G

KEY SKILLS

During this lesson, you will be:

- interpreting a recurrence relation
- constructing a recurrence relation
- generating a sequence of terms using a recurrence relation.

KEY TERMS

- Sequence
- Term
- Recurrence relation
- Common difference
- Common ratio
- Linear growth/decay
- Geometric growth/decay
- Iteration
- Arithmetic
- Geometric

Recurrence relations can be used to model different number patterns. They show how the value of a number in a sequence can be dependent on the previous number. Recurrence relations are useful for generating sequences of numbers and predicting future values.

Interpreting a recurrence relation

A **sequence** is a list of numbers written in succession. Each number in a sequence is a **term**. For example, a sequence might be described by u_n where n describes that term's position in the sequence. Often, an ellipsis (...) is used to show a sequence continues indefinitely.

For example, in the following sequence, the third term, u_2, is 2.

$u_n = 0, 1, 2, 3, \ldots$

A pattern-based sequence is where each successive term follows the same change. For example, in the previous sequence, each term increases by 1 so it is pattern-based. If there is no pattern determining the sequence, it is called random.

A **recurrence relation** is a formula that links each term in a pattern-based sequence to the next. It allows predictions to be made about the value of any term in a sequence. Recurrence relations comprise two parts: the initial value (u_0) and the pattern.

The pattern is written as $u_{n+1} = \textit{an expression involving } u_n$, where u is the sequence, u_n is the current term, and u_{n+1} is the next term.

The general form of a recurrence relation is:

$u_0 = a, \quad u_{n+1} = R \times u_n + d$, where

- a is the initial value of the recurrence relation
- d is the **common difference** which shows linear growth/decay
- R is the **common ratio** which shows geometric growth/decay

Linear growth/decay is when each term in a sequence increases or decreases by a constant amount. If $d > 0$ there is linear growth and if $d < 0$ there is linear decay.

Geometric growth/decay is when each term in a sequence increases or decreases by a constant ratio. When $R > 1$ there is geometric growth and when $R < 1$ there is geometric decay.

Worked example 1

Determine the initial value for the sequence from the following recurrence relations and describe the pattern in words.

a. $R_0 = 7, \quad R_{n+1} = R_n + 1$

Explanation

Step 1: Identify the initial value.

The subscript zero denotes the initial value, R_0, which is 7.

Step 2: Identify the pattern.

The pattern is given by $R_{n+1} = R_n + 1$.

Step 3: Describe the pattern.

In the recurrence relation, $d = 1$ which shows there is linear growth.

Each term increases by 1.

Answer

The initial value is 7. The next term is equal to the current term plus one.

b. $u_0 = 2, \quad u_{n+1} = 0.6 \times u_n$

Explanation

Step 1: Identify the initial value.

The subscript zero denotes the initial value, u_0, which is 2.

Step 2: Identify the pattern.

The pattern is given by $u_{n+1} = 0.6 \times u_n$.

Step 3: Describe the pattern.

In the recurrence relation, $R = 0.6$ which shows there is geometric decay.

Each term is 60% of the previous term or 40% less than the previous term.

Answer

The initial value is 2. The next term is 40% less than the current term.

c. $S_0 = 6, \quad S_{n+1} = 2 \times S_n - 4$

Explanation

Step 1: Identify the initial value.

The subscript zero denotes the initial value.

The initial value is S_0, which is 6.

Step 2: Identify the pattern.

The pattern is given by $S_{n+1} = 2 \times S_n - 4$.

Step 3: Describe the pattern.

In the recurrence relation, $R = 2$ and $d = -4$ so there is both geometric growth and linear decay.

Answer

The initial value is 6. The next term is found by multiplying the current value by two and subtracting four.

Constructing a recurrence relation

See worked example 2

See worked example 3

Recurrence relations can be constructed from a description of the pattern found in a sequence.

They can also be constructed from a number sequence by identifying whether the terms have a common difference, a common ratio or a combination of both.

A recurrence relation has the general form:

$u_0 = a, \quad u_{n+1} = R \times u_n + d$

If the terms increase or decrease by a constant amount, the sequence has a common difference (d), and $R = 1$.

Each term in the following sequence increases by 3, so there is a common difference, d, of 3.

$u_n = 4, 7, 10, 13, \ldots$

If the terms increase or decrease by a different amount but by a constant ratio, the sequence has a common ratio (R), and $d = 0$.

Each term in the sequence decreases by half, so there is a common ratio, R, of 0.5.

$v_n = 24, 12, 6, 3, \ldots$

If the sequence is pattern-based and has neither a common difference nor a common ratio, the recurrence relation will include both R and d ($R \neq 1$ and $d \neq 0$).

Worked example 2

Write down the recurrence relation for the following patterns in terms of u.

a. Each successive term is three times as big as the previous term, starting with 1.5.

Explanation

Step 1: Identify the initial value.

The recurrence relation starts with 1.5 so the initial value is 1.5.

$u_0 = 1.5$

Step 2: Identify the pattern.

The next term is three times as big as the current term. This means the next term is the current term multiplied by 3.

$u_{n+1} = 3 \times u_n$

Answer

$u_0 = 1.5, \quad u_{n+1} = 3 \times u_n$

b. Each successive term decreases by seven, starting with 34.

Explanation

Step 1: Identify the initial value.

The recurrence relation starts with 34 so the initial value is 34.

$u_0 = 34$

Step 2: Identify the pattern.

The next term is seven less than the current term. This means the next term is found by subtracting 7 from the current term.

$u_{n+1} = u_n - 7$

Answer

$u_0 = 34, \quad u_{n+1} = u_n - 7$

Worked example 3

Construct a recurrence relation to represent the following sequences in terms of u.

a. 23, 17, 11, 5, −1, ...

Explanation

Step 1: Identify the initial value (a).

The first term in the sequence is 23.

$a = 23$

Step 2: Identify if there is a common difference (d) or a common ratio (R).

Each successive term decreases by 6.

Therefore there is a common difference of −6, so $d = -6$.

Step 3: Determine the recurrence relation.

Substitute $a = 23$ and $d = -6$ into the recurrence relation: $u_0 = a$, $u_{n+1} = u_n + d$.

Answer

$u_0 = 23, \quad u_{n+1} = u_n - 6$

b. 5, 6, 7.2, 8.64, ...

Explanation

Step 1: Identify the initial value (a).

The first term in the sequence is 5.

$a = 5$

Step 2: Identify if there is a common difference (d) or a common ratio (R).

The initial term increases by 1, the second term increases by 1.2, and the third term increases by 1.44. Each term increases by a different amount so there is not a common difference.

To identify the common ratio, calculate the ratio of each term to the previous term.

$\frac{6}{5} = 1.2$

$\frac{7.2}{6} = 1.2$

$\frac{8.64}{7.2} = 1.2$

The common ratio is 1.2, so $R = 1.2$.

Step 3: Determine the recurrence relation.

Substitute $a = 5$ and $R = 1.2$ into the recurrence relation: $u_0 = a, \quad u_{n+1} = R \times u_n$.

Answer

$u_0 = 5, \quad u_{n+1} = 1.2 \times u_n$

c. 2, 5, 11, 23, 47

Explanation

Step 1: Identify the initial value (a).

The first term in the sequence is 2.

$a = 2$

Continues →

Step 2: Identify if there is a common difference (d) or a common ratio (R).

Each term increases by a different amount so there is not a common difference.

The ratio of the second term to the first term is 2.5 and the ratio of the third term to the second term is 2.2, so there is no common ratio.

Step 3: Determine the recurrence relation that generates the sequence.

Since there is no common difference or common ratio, check to see if there is a combination of both R and d.

Each successive term is one more than double the previous term. Therefore each term is multiplied by 2 and then added by 1.

This pattern is expressed as: $u_{n+1} = 2 \times u_n + 1$.

Answer

$u_0 = 2, \quad u_{n+1} = 2 \times u_n + 1$

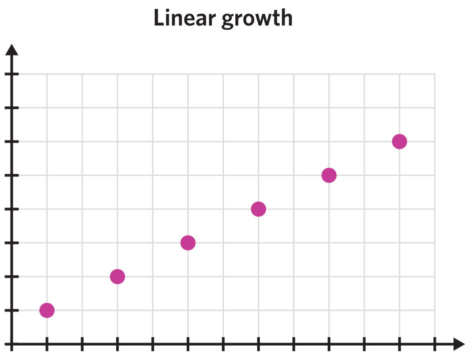
Generating a sequence of terms using a recurrence relation

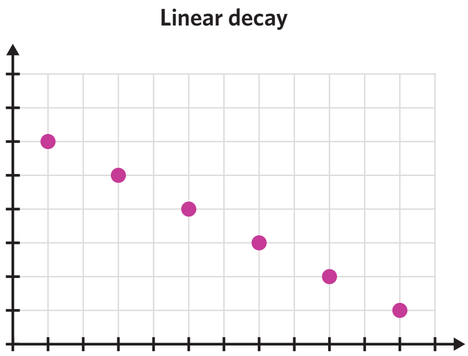
See worked example 4

A recurrence relation of the form: $u_0 = a, \quad u_{n+1} = R \times u_n + d$ can be used to generate a sequence of numbers. A sequence is generated by substituting the initial value into the recurrence relation, and then repeating this for each term to determine the next term in the sequence. Each term in a repeated process is known as an **iteration**, where u_n is the value of the term in the sequence after n iterations.

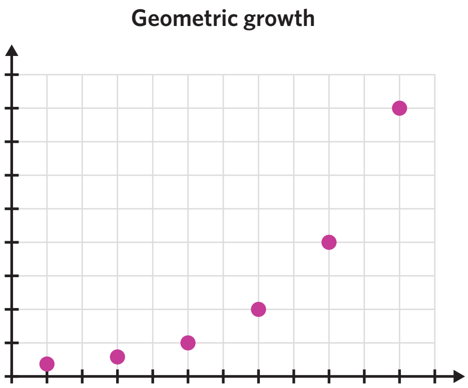
There are two types of sequences.

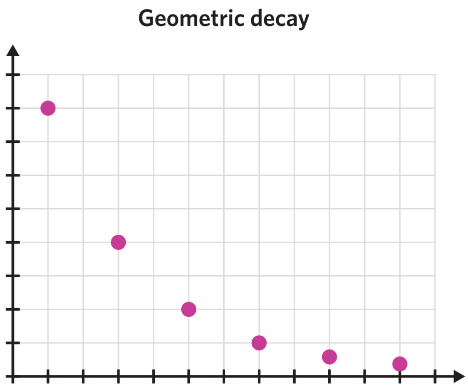
A recurrence relation that only has linear growth or decay, i.e. where $R = 1$, is **arithmetic**. It has the form: $u_0 = a$, $u_{n+1} = u_n + d$. It generates a sequence with a common difference.





A recurrence relation that only has geometric growth or decay, i.e. where $d = 0$, is **geometric**. It has the form: $u_0 = a, \quad u_{n+1} = R \times u_n$. It generates a sequence with a common ratio.





A sequence can be generated from a recurrence relation either by hand or using a calculator.

See worked example 5

Worked example 4

Find the first five terms of the sequence given by the following recurrence relations.

a. $H_0 = 42, \quad H_{n+1} = H_n - 3.5$

Explanation

Step 1: Identify the initial value.

The initial value is denoted by the subscript zero, H_0, which is 42.

Step 2: Use the recurrence relation to find the second term, H_1.

$$\begin{aligned} H_1 &= H_0 - 3.5 \\ &= 42 - 3.5 \\ &= 38.5 \end{aligned}$$

Step 3: Use the recurrence relation to find the third term, H_2.

$$\begin{aligned} H_2 &= H_1 - 3.5 \\ &= 38.5 - 3.5 \\ &= 35 \end{aligned}$$

Step 4: Repeat for the remaining terms.

Step 5: Write the values in sequential order, separated by commas.

Answer

42, 38.5, 35, 31.5, 28

b. $Y_0 = 2, \quad Y_{n+1} = 4Y_n - 3$

Explanation

Step 1: Identify the initial value.

The initial value is denoted by the subscript zero, Y_0, which is 2.

Step 2: Use the recurrence relation to find the second term, Y_1.

$$\begin{aligned} Y_1 &= 4 \times Y_0 - 3 \\ &= 4 \times 2 - 3 \\ &= 5 \end{aligned}$$

Step 3: Use the recurrence relation to find the third term, Y_2.

$$\begin{aligned} Y_2 &= 4 \times Y_1 - 3 \\ &= 4 \times 5 - 3 \\ &= 17 \end{aligned}$$

Step 4: Repeat for the remaining terms.

Step 5: Write the values in sequential order, separated by commas.

Answer

2, 5, 17, 65, 257

Worked example 5

Use a calculator to find the first five terms of the sequence given by the following geometric recurrence relation, correct to two decimal places.

$B_0 = 2.12, \quad B_{n+1} = -3.76 \times B_n$

Explanation – Method 1: TI-Nspire

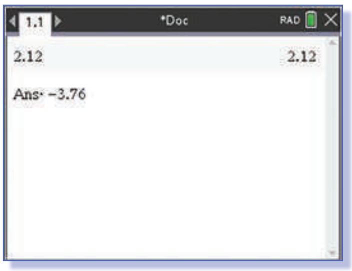
Step 1: Open a new page by pressing ctrl + doc▾ and select '1: Add Calculator'.

Step 2: Enter the initial value by typing '2.12'. Press enter.

Continues →

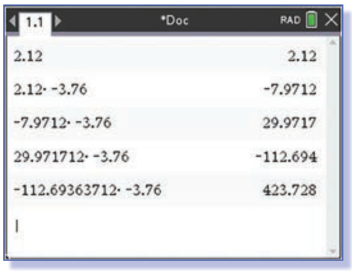
Step 3: Press '×'.

'Ans·' will appear to show that it is multiplying by the previous answer. Type '− 3.76'. Press enter.



Step 4: To find the next value in the sequence, press enter.

Step 5: Repeat step 4 for each of the remaining first five terms.



Explanation – Method 2: Casio ClassPad

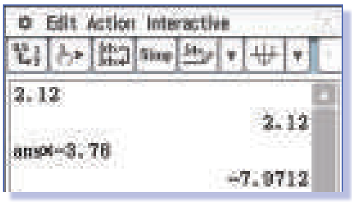
Step 1: From menu, tap √α Main.

Step 2: Enter the initial value by typing '2.12'. Press EXE.

Step 3: To calculate the second term B_1, type '×'.

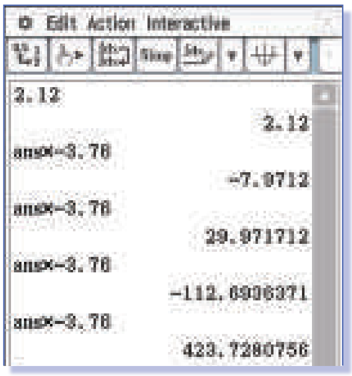
'ans' will appear to show that the calculator is using the previous answer.

Type '− 3.76'. Press EXE.



Step 4: To find the next value in the sequence, press EXE.

Step 5: Repeat step 4 for each of the remaining first five terms.



Answer – Method 1 and 2

2.12, −7.97, 29.97, −112.69, 423.73

Exam question breakdown

VCAA 2016 Exam 1 Recursion and financial modelling Q17

Consider the recurrence relation

$A_0 = 2, \quad A_{n+1} = 3A_n + 1$

The first four terms of this recurrence relation are

A. 0, 2, 7, 22 …
B. 1, 2, 7, 22 …
C. 2, 5, 16, 49 …
D. 2, 7, 18, 54 …
E. 2, 7, 22, 67 …

Explanation

Step 1: Identify the initial value.

The initial value is denoted by the subscript zero.
The initial value is 2.

Step 2: Apply the pattern to find the second term, A_1.

$A_1 = 3 \times A_0 + 1$

$A_1 = 3 \times 2 + 1$

$A_1 = 7$

Step 3: Repeat for the remaining terms.

Answer

E

89% of students answered this question correctly.

5A Questions

Interpreting a recurrence relation

1. What is the initial term of the following recurrence relation?

$M_0 = 2, \quad M_{n+1} = M_n - 4$

A. -4 **B.** 0 **C.** 2 **D.** 4

2. Which statement is true of the following recurrence relation?

$W_0 = 15, \quad W_{n+1} = W_n + 2$

A. Each term is twice the previous term.
B. Each term is equal to the previous term plus 15.
C. Each term is 15.
D. Each term is equal to the previous term plus 2.

3. Determine the initial value for the sequence from the following recurrence relations. State whether the sequence has linear growth or decay, geometric growth or decay, or both.

a. $S_0 = 4, \quad S_{n+1} = 0.4 \times S_n$

b. $V_0 = 3.2, \quad V_{n+1} = 1.34 \times V_n - 6$

c. $u_0 = 14, \quad u_{n+1} = u_n + 1.3$

Constructing a recurrence relation

4. The first five terms of a sequence are 3, 7, 15, 31, 63 ...

The recurrence relation that generates this sequence is

A. $H_0 = 3, \quad H_{n+1} = H_n + 4$
B. $H_0 = 3, \quad H_{n+1} = 2 \times H_n + 1$
C. $H_0 = 3, \quad H_{n+1} = 2.333 \times H_n$
D. $H_0 = 3, \quad H_{n+1} = 4 \times H_n - 5$

5. Identify the common difference or common ratio in each of the following sequences.

a. 2, 4, 8, 16, 32, 64, ...

b. −17, −10, −3, 4, 11, 18, ...

c. 10, 9, 8.1, 7.29, 6.561, ...

d. 3, −3.6, 4.32, −5.184, 6.2208, ...

6. Construct a recurrence relation for each of the following patterns.

a. Each successive term increases by four, starting with 22.

b. Each successive term is five times smaller than the previous term, starting with 100.

c. Each successive term is three less than double the previous term, starting with 14.

d. Each successive term is six more than 40% of the previous term, starting with 3.

7. Construct a recurrence relation for each of the following sequences in terms of u.

   a. 4, 6.5, 9, 11.5, 14, 16.5, 19

   b. 768, 192, 48, 12, 3

   c. $\frac{1}{4}, \frac{5}{4}, \frac{25}{4}, \frac{125}{4}, \frac{625}{4}$

   d. 3, 11, 35, 107, 323

8. The number of people, P_n, on day n eating at a restaurant as it gets more and more popular follows the following sequence. Note that this sequence starts at $n = 1$.

day (n)	1	2	3	4	5
people (P_n)	3	6	12	24	48

   a. Assuming the sequence continues, how many people will eat at the restaurant on day 6?

   b. Write a recurrence relation for this sequence.

Generating a sequence of terms using a recurrence relation

9. Using the following recurrence relation, what is the value of S_3?

   $S_0 = 15, \quad S_{n+1} = S_n + 2$

   A. 15 B. 19 C. 21 D. 23

10. Find the terms in each recurrence relation when $n = 2$, 4 and 7. Round to three decimal places where necessary.

   a. $U_0 = 11, \quad U_{n+1} = U_n + 3$

   b. $V_0 = \frac{1}{27}, \quad V_{n+1} = 3 \times V_n$

   c. $W_0 = 3, \quad W_{n+1} = 2 \times W_n - 4$

   d. $X_0 = 21, \quad X_{n+1} = -0.8 \times X_n + 2$

11. Generate the first five terms for the following recurrence relations.

   a. $A_0 = 4.5, \quad A_{n+1} = A_n - 0.25$

   b. $B_0 = 10, \quad B_{n+1} = 1.1 \times B_n$

   c. $C_0 = 17, \quad C_{n+1} = 3 \times C_n - 22$

   d. $D_0 = 12.5, \quad D_{n+1} = 0.8 \times D_n + 5$

12. Use a calculator to find the term G_{15} in the following sequence. Round to the nearest whole number if necessary.

   $G_0 = 5, \quad G_{n+1} = 1.2 \times G_n + 3$

13. A sequence follows a pattern given by $M_{n+1} = M_n + 3$, where $M_4 = 23$. What is the initial value of the sequence?

Joining it all together

14. Consider the following recurrence relation:

   $u_0 = 64, \quad u_{n+1} = 0.5 \times u_n + 4$

   a. What is the initial value of the sequence generated by this recurrence relation?

   b. Calculate the value of u_5.

15. Theo bought his first car when he got his learner permit two years ago. He bought it for $3500. After a year, the car was revalued at $2950. His insurance company inspected the car today and valued it at $2400.

a. Assuming the arithmetic sequence continues, how many years after Theo purchased the car will it be worth $750?

b. Let V_n be the value of the car n years after it was purchased. Write a recurrence relation to show the arithmetic sequence of the value of the car.

16. The height, h, in centimetres, of a ripple in a pond can be measured. The height of the ripple depends on the distance, d, in whole number metres, from the centre of the ripple. This relationship can be modelled using the geometric sequence $h_{d+1} = 0.9h_d$.

If the ripple was 1.458 cm high three metres from the centre of the ripple, what was the height of the ripple at the centre?

Exam practice

17. The first five terms of a sequence are 2, 6, 22, 86, 342 ...

The recurrence relation that generates this sequence could be

A. $P_0 = 2, \quad P_{n+1} = P_n + 4$

B. $P_0 = 2, \quad P_{n+1} = 2P_n + 2$

C. $P_0 = 2, \quad P_{n+1} = 3P_n$

D. $P_0 = 2, \quad P_{n+1} = 4P_n - 2$

E. $P_0 = 2, \quad P_{n+1} = 5P_n - 4$

VCAA 2017 Exam 1 Recursion and financial modelling Q18

92% of students answered this question correctly.

18. The following recurrence relation can generate a sequence of numbers.

$T_0 = 10, \quad T_{n+1} = T_n + 3$

The number 13 appears in this sequence as

A. T_1 **B.** T_2 **C.** T_3

D. T_{10} **E.** T_{13}

VCAA 2020 Exam 1 Recursion and financial modelling Q21

85% of students answered this question correctly.

19. The following recurrence relation can generate a sequence of numbers.

$L_0 = 37, \quad L_{n+1} = L_n + C$

The value of L_2 is 25.

The value of C is

A. -6 **B.** -4 **C.** 4

D. 6 **E.** 37

VCAA 2021 Exam 1 Recursion and financial modelling Q17

83% of students answered this question correctly.

20. Consider the following recurrence relation.

$A_0 = 3, \quad A_{n+1} = 2A_n + 4$

The value of A_3 in the sequence generated by this recurrence relation is given by

A. $2 \times 3 + 4$ **B.** $2 \times 4 + 4$ **C.** $2 \times 10 + 4$

D. $2 \times 24 + 4$ **E.** $2 \times 52 + 4$

VCAA 2019 Exam 1 Recursion and financial modelling Q17

68% of students answered this question correctly.

Questions from multiple lessons

Data analysis

21. A reciprocal transformation was used to linearise a set of non-linear bivariate data. Subsequently, a least squares line was fitted to the transformed data.

The equation is this least squares line is

$y = 7.25 + 1.27 \times \frac{1}{x}$

The equation was used to predict the value of y when x was set to a certain value. This gave a y value of 9.24.

The value of x is closest to

A. 1.59 **B.** 1.57 **C.** 0.64 **D.** 7.39 **E.** 18.98

Adapted from VCAA 2018 Exam 1 Data analysis Q12

Recursion and financial modelling *Year 11 content*

22. The following sequences are either five consecutive terms of an arithmetic sequence or five consecutive terms of a geometric sequence.

Which one of these sequences could **not** include 15 as a term?

A. 99, 82, 65, 48, 31...

B. −12, −9, −6, −3, 0...

C. 0.46875, 0.9375, 1.875, 3.75, 7.5...

D. −15, −10, −5, 0, 5...

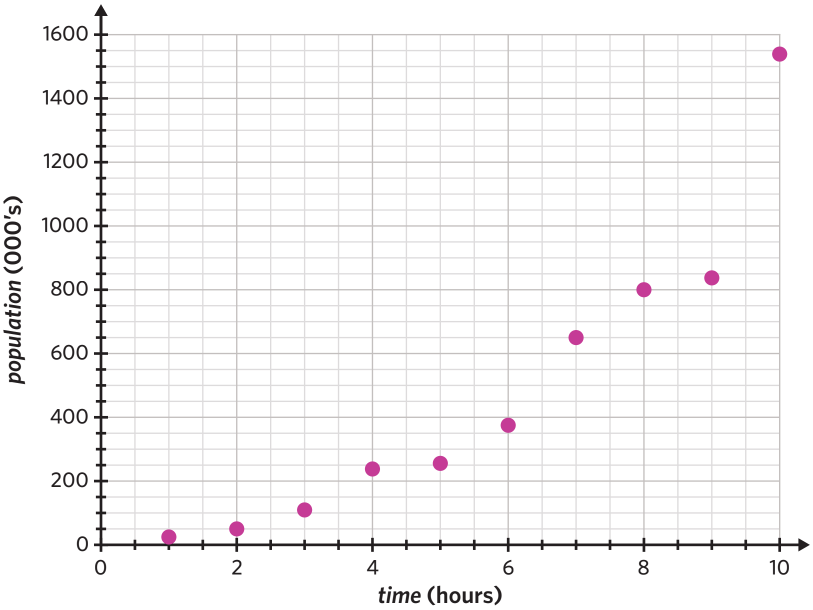
E. None of these.

Adapted from VCAA 2013 Exam 1 Number patterns Q7

Data analysis

23. Bacteria replicate and multiply over time.

The following time series plot shows the *population* of bacteria in a culture dish every hour for 10 hours. The data used to generate this plot is also given.



time (hours)	1	2	3	4	5	6	7	8	9	10
population (000's)	14.51	42.68	107.33	235.95	251.92	371.9	645.77	800.9	832.42	1533.93

The association between the *population* of the bacteria in the culture dish and *time* is non-linear.

A square transformation was performed on the variable *time* to linearise the data.

a. When the equation of the least squares regression line is calculated, the slope of this line is approximately 13.519406. Round this number to four significant figures. (1 MARK)

b. Perform the square transformation to the variable *time* to determine the equation of the least squares regression line that can be used to predict *population* from *time*^{2}. Write down the values of the intercept and slope of this equation. Round to four significant figures. (2 MARKS)

Adapted from VCAA 2017 Exam 2 Data analysis Q4a,b

5B Flat rate and unit cost depreciation – recurrence relations

STUDY DESIGN DOT POINT

- use of a recurrence relation to model and compare (numerically and graphically) flat rate, unit cost and reducing balance depreciation of the value of an asset with time, including the use of a recurrence relation to determine the depreciating value of an asset after n depreciation periods for the initial sequence

5A 5B 5C 5D 5E 5F 5G

KEY SKILLS

During this lesson, you will be:

- modelling flat rate and unit cost depreciation using recurrence relations
- using an arithmetic recurrence relation to determine the value of an asset after n depreciation periods.

KEY TERMS

- Depreciation
- Flat rate depreciation
- Principal
- Unit cost depreciation

Recurrence relations can be applied to real-life cases such as the loss in value of an asset. That is, they can be used to model depreciation. The value of an asset after a certain number of periods, or units of use, can then be determined using a process of iteration on the recurrence relation.

Modelling flat rate and unit cost depreciation using recurrence relations

The loss of value of an asset is known as **depreciation**. Depreciation can either be arithmetic (linear) or geometric. There are two types of depreciation that can be represented by arithmetic relations; flat rate and unit cost.

See worked example 1

Flat rate depreciation is used when the value of an asset decreases by a constant amount each period. This amount is a percentage of the **principal**, or initial value, of the asset. For example, flat rate depreciation could describe the value of a computer as it gets outdated each year. A computer that had an initial value of \$2000 and depreciates at a flat rate of 20% per year, will depreciate by \$400 per year. It will take 5 years before it has no value.

A recurrence relation can be used to model flat rate depreciation and calculate the value of the asset after each period.

$V_0 = principal, \quad V_{n+1} = V_n - d$, where

- d is the depreciation per period, calculated by $d = \frac{r}{100} \times V_0$
- r is the depreciation rate (%) per period
- V_n is the value after n periods

See worked example 2

Unit cost depreciation is used when an asset loses value after each unit of use. For example, an air fryer might lose 5 cents in value for every hour of use. The unit of use is often different between cases. The unit of use for an air fryer could be hours of use, while the unit of use for a car could be kilometres driven.

A recurrence relation can be used to model unit cost depreciation and calculate the value of the asset after each unit of use.

$V_0 = principal, \quad V_{n+1} = V_n - d$, where

- d is the depreciation amount per unit of use
- V_n is the value after n uses

Worked example 1

Determine the recurrence relation that models the value of the assets, V_n, after n years, in each of the scenarios.

a. Karl does not believe in banks. Instead of investing his money, he buries his \$20 000 in his backyard. Unfortunately for Karl, inflation causes his buried cash to depreciate at a constant amount per year, equal to 3% of the amount of money that he buried.

Explanation

Step 1: Determine the initial value, V_0.

Initially Karl buries \$20 000.

$V_0 = 20\ 000$

Step 2: Calculate the depreciation amount, d.

$d = \frac{r}{100} \times V_0$

The rate of depreciation is 3%, so $r = 3$.

$$d = \frac{3}{100} \times 20\ 000$$
$$= 600$$

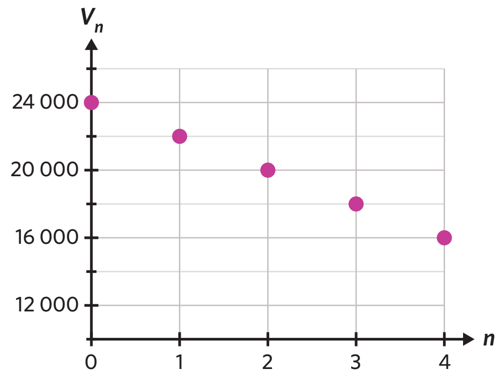
Step 3: Substitute these values into the recurrence relation.

$V_0 = principal, \quad V_{n+1} = V_n - d$

Answer

$V_0 = 20\ 000, \quad V_{n+1} = V_n - 600$

b. The graph shows the value of a car, purchased for \$24 000, over a period of four years.



Explanation

Step 1: Determine the initial value, V_0.

The value of the car when purchased was \$24 000. This is the vertical axis intercept of the graph.

$V_0 = 24\ 000$

Step 2: Calculate the depreciation amount, d.

$V_1 = 22\ 000$

This means that the value of the car depreciates by \$2000 each year.

$d = 2000$

Step 3: Substitute these values into the recurrence relation.

$V_0 = principal, \quad V_{n+1} = V_n - d$

Answer

$V_0 = 24\ 000, \quad V_{n+1} = V_n - 2000$

Worked example 2

Determine the recurrence relation that models the value of the assets, V_n after n units of use, in each of the following scenarios.

a. David spends $33 on a disposable underwater camera and estimates that it will depreciate by $1.50 for each photo that he takes.

Explanation

Step 1: Determine the initial value, V_0.

The camera was purchased for $33.

$V_0 = 33$

Step 2: Determine the depreciation amount, d.

The value of the camera depreciates by $1.50 per unit of use.

$d = 1.5$

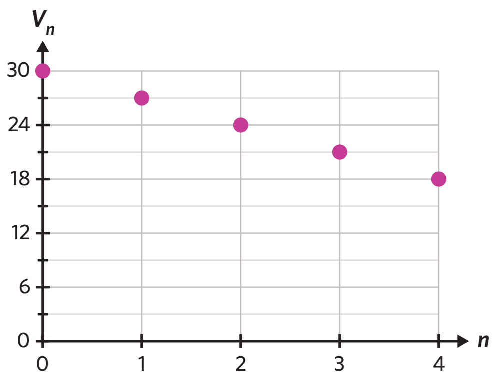
Step 3: Substitute these values into the recurrence relation.

$V_0 = principal, \quad V_{n+1} = V_n - d$

Answer

$V_0 = 33, \quad V_{n+1} = V_n - 1.5$

b. Chloe purchases a book for $30. The book depreciates each time it is read. The value of the book is shown in the graph.



Explanation

Step 1: Determine the initial value, V_0.

The book is purchased for $30.

$V_0 = 30$

Step 2: Determine the depreciation amount, d.

$V_1 = 27$

This means that the value of the book depreciates by $3 each time it is read.

$d = 3$

Step 3: Substitute these values into the recurrence relation.

$V_0 = principal, \quad V_{n+1} = V_n - d$

Answer

$V_0 = 30, \quad V_{n+1} = V_n - 3$

Using an arithmetic recurrence relation to determine the value of an asset after n depreciation periods

A recurrence relation that models depreciation can be used to determine the value of an asset after a given number of periods, or a given number of units of use. The value is found by the process of iteration.

When modelling flat rate depreciation, the initial value, V_0, is substituted into the recurrence relation to find the value of the asset after one period, V_1. V_1 is then substituted into the relation to find the value after two periods, V_2, and so on.

See worked example 3

When modelling unit cost depreciation, the initial value, V_0, is substituted into the recurrence relation to find the value of the asset after one unit of use, V_1. V_1 is then substituted into the relation to find the value after two units of use, V_2, and so on.

See worked example 4

Worked example 3

The value of a brand new laptop, V_n after n years, is depreciated using flat rate depreciation. This is represented by the following recurrence relation.

$V_0 = 2100, \quad V_{n+1} = V_n - 350$

a. What is the value of the laptop two years after it is purchased?

Explanation

Step 1: Identify the initial value, V_0.

$V_0 = 2100$

Step 2: Calculate V_1, the value after one year.

$$\begin{aligned} V_1 &= V_0 - 350 \\ &= 2100 - 350 \\ &= 1750 \end{aligned}$$

Step 3: Calculate V_2, the value after two years.

$$\begin{aligned} V_2 &= V_1 - 350 \\ &= 1750 - 350 \\ &= 1400 \end{aligned}$$

Note: These calculations can also be performed using a calculator as shown in 5A Worked example 5.

Answer

\$1400

b. Graph the value of the laptop over the first five years.

Explanation

Step 1: Calculate V_1 to V_5, the values over the first five years.

$V_0 = 2100$

$V_1 = V_0 - 350 = 2100 - 350 = 1750$

$V_2 = V_1 - 350 = 1750 - 350 = 1400$

$V_3 = V_2 - 350 = 1400 - 350 = 1050$

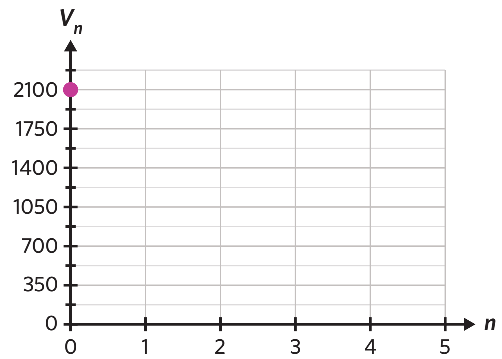
$V_4 = V_3 - 350 = 1050 - 350 = 700$

$V_5 = V_4 - 350 = 700 - 350 = 350$

Note: These calculations can also be performed using a calculator as shown in 5A Worked example 5.

Step 2: Plot V_0 on a graph.

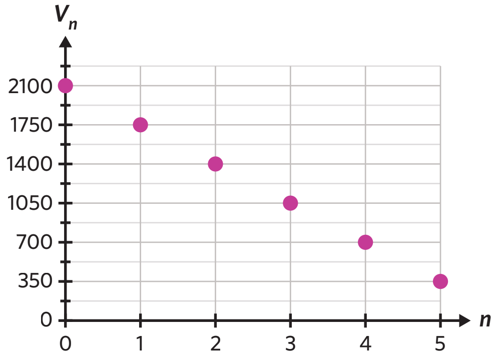
V_0 will be the vertical axis intercept.



Step 3: Plot the remaining values on the graph.

Continues →

Answer



Worked example 4

A new piece of equipment was purchased for \$300 000. The equipment depreciates by \$10 000 for every 10 uses, which means that it depreciates by \$1000 for each use.

The value, V_n, of the equipment after n uses can be modelled by the following recurrence relation.

$V_0 = 300\ 000, \quad V_{n+1} = V_n - 1000$

a. After how many uses will the value of the equipment be \$297 000?

Explanation

Step 1: Determine the initial value, V_0.

$V_0 = 300\ 000$

Step 2: Calculate the value after each use until it reaches \$297 000.

Value after one use:

$$\begin{aligned} V_1 &= V_0 - 1000 \\ &= 300\ 000 - 1000 \\ &= 299\ 000 \end{aligned}$$

Value after two uses:

$$\begin{aligned} V_2 &= V_1 - 1000 \\ &= 299\ 000 - 1000 \\ &= 298\ 000 \end{aligned}$$

Value after three uses:

$$\begin{aligned} V_3 &= V_2 - 1000 \\ &= 298\ 000 - 1000 \\ &= 297\ 000 \end{aligned}$$

Note: These calculations can also be performed using a calculator as shown in 5A Worked example 5.

Answer

3 uses

Continues →

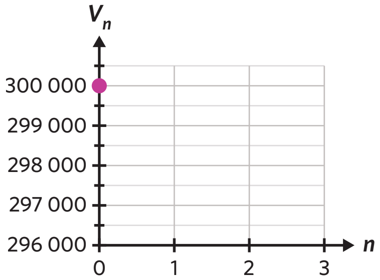
b. Using the values found in part **a**, graph the unit cost depreciation of the equipment after three uses.

Explanation

Step 1: Determine the initial value, V_0, and plot this on a graph.

$V_0 = 300\,000$

This will be the vertical axis intercept.



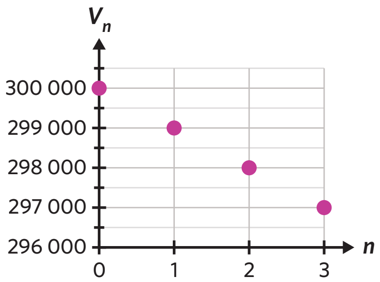
Step 2: Plot the values found in part **a** on the graph.

$V_1 = 299\,000$

$V_2 = 298\,000$

$V_3 = 297\,000$

Answer



Exam question breakdown

VCAA 2017 Exam 1 Recursion and financial modelling Q21

A printer was purchased for \$680.

After four years the printer has a value of \$125.

On average, 1920 pages were printed every year during those four years.

The value of the printer was depreciated using a unit cost method of depreciation.

The depreciation in the value of the printer, per page printed, is closest to

A. 3 cents. **B.** 4 cents. **C.** 5 cents. **D.** 6 cents. **E.** 7 cents.

Explanation

Step 1: Identify the initial value.

The printer was purchased for \$680.

Step 2: Calculate the depreciation per year.

After four years, the printer is worth \$125.

$$\textit{depreciation over four years} = 680 - 125$$
$$= 555$$

$$\textit{depreciation per year} = \frac{555}{4}$$
$$= 138.75$$

Step 3: Calculate the depreciation per page.

On average, 1920 pages were printed each year.

$$\textit{depreciation per page} = \frac{138.75}{1920}$$
$$= 0.0722...$$

Step 4: Convert the depreciation to cents.

\$0.0722 is 7.22 cents.

Answer

E

53% of students answered this question correctly.

22% of students incorrectly chose option A. This is likely because the students divided the total depreciation over 4 years by the annual amount of pages that were printed.

5B Questions

Modelling flat rate and unit cost depreciation using recurrence relations

1. Courtney's yacht loses its value each time she uses it. The following recurrence relation models the depreciation of her yacht using the unit cost method.

$V_0 = 100\,000, \quad V_{n+1} = V_n - 1500$

Fill in the following gaps.

Courtney's yacht was worth __________ when brand new. The value of her yacht decreases by __________ every time she uses it. This amount can be expressed as __________% of the principal.

2. Write a recurrence relation to model each of the following scenarios.

a. A car is depreciating at \$2000 per year. It was bought for \$18 000.

b. A laptop was bought for \$800, and depreciates at a rate of 16% of its sale price per year.

c. A bike bought for \$575 depreciates at a rate of 10% of its sale price per year.

3. Write a recurrence relation to model each of the following scenarios.

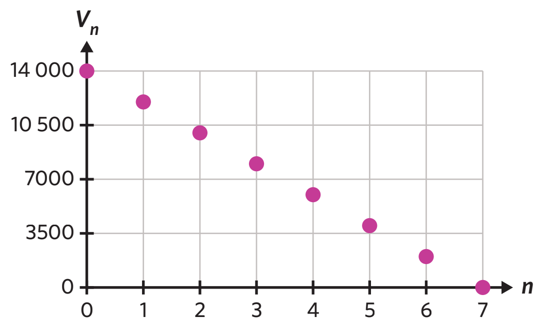
a. A cricket bat worth \$300 depreciates at \$2 per innings played with it.

b. A \$20 000 car depreciates at five cents per kilometre driven.

c. A printer worth \$500 depreciates at 0.002% of its sale price for every page printed.

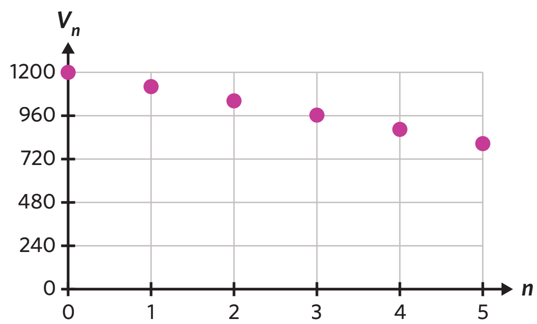
4. A brand new tractor was purchased for \$14 000. The tractor will be worthless after it is used 700 times. The graph shown models the depreciation of the tractor using the unit cost method.

Write a recurrence relation that models the value, V_n, of the tractor where n represents every 100 uses.



5. The following graph models the flat rate depreciation of a mobile phone after n years.

Determine the recurrence relation that models the depreciation of the mobile phone.



Using an arithmetic recurrence relation to determine the value of an asset after *n* depreciation periods

6. Isaac bought new DJ decks for \$740 which depreciate by 20% of the initial value each year. This is represented by the following recurrence relation.

$V_0 = 740, \quad V_{n+1} = V_n - 148$

How much will the DJ decks be worth after 2 years?

A. \$296 **B.** \$444 **C.** \$592 **D.** \$740

7. The following recurrence relation models flat rate depreciation.

$V_0 = 10, \quad V_{n+1} = V_n - 2$

a. Find the values of V_1, V_2, V_3 and V_4.

b. Plot the depreciation over four periods.

---

8. Let V_n represent the value of an item after n uses, in dollars. The value of the item depreciating using the unit cost method could be modelled using the following recurrence relation.

$V_0 = 184, \quad V_{n+1} = V_n - 14$

a. What is the initial value of the item?

b. What will the value of the item be after four uses?

c. After how many uses will the item be worth \$100?

---

9. A brand new refrigerator costs \$930. The refrigerator depreciates by \$132.86 every 10 000 times it is opened and closed. Let V_n represent the value of the refrigerator after n units of use, in dollars, where n represents every 10 000 uses.

$V_0 = 930, \quad V_{n+1} = V_n - 132.86$

a. How much is the refrigerator worth after it is opened and closed 20 000 times?

b. How many times will the refrigerator need to be opened and closed to be worthless? Round to the nearest ten thousand.

Joining it all together

10. A microwave bought for \$600 depreciates by 5.5% of its sale price each year.

a. Write a recurrence relation that describes its value, V_n, after n years.

b. How much will it be worth in five years?

---

11. Tristan's mobile tablet depreciates according to the following recurrence relation, where V_n is its value after it has been dropped n times.

$V_0 = 1200, \quad V_{n+1} = V_n - 210$

a. What is the tablet worth after it has been dropped twice?

b. After how many times being dropped will the tablet be worth less than half of its original value?

c. Express the loss in value each time the tablet is dropped as a percentage of the original sale price.

---

12. A brand new sedan made by Luxury Motors, that was bought for \$84 000, is worth \$44 000 four years later. Its value depreciates over time using the flat rate method.

a. Determine the recurrence relation that models this scenario, where V_n represents the value of the sedan after n years.

b. Plot the value of the sedan on a graph, from the time of purchase up to $n = 5$.

c. After how many years will the value drop below \$56 000?

---

13. Rachel bought a new watch for \$280. The watch depreciates by \$22 for every 1000 rotations of the hour hand.

a. Write a recurrence relation that describes the unit cost depreciation of the watch, where n represents 1000 rotations of the hour hand.

b. What is the value of the watch after 4000 rotations of the hour hand?

c. The hour hand rotates twice in one day. After how many days will the watch be worth \$236?

Exam practice

14. Geoff purchased a computer for \$4500. He will depreciate the value of his computer by a flat rate of 10% of the purchase price per annum.

A recurrence relation that Geoff can use to determine the value of the computer after n years, V_n, is

A. $V_0 = 4500, \quad V_{n+1} = V_n - 450$

B. $V_0 = 4500, \quad V_{n+1} = V_n + 450$

C. $V_0 = 4500, \quad V_{n+1} = 0.9V_n$

D. $V_0 = 4500, \quad V_{n+1} = 1.1V_n$

E. $V_0 = 4500, \quad V_{n+1} = 0.1(V_n - 450)$

VCAA 2019 Exam 1 Recursion and financial modelling Q19

76% of students answered this question correctly.

---

15. Samuel owns a printing machine.

The printing machine, in dollars, after n years, V_n, can be modelled by the recurrence relation

$V_0 = 120\,000, \quad V_{n+1} = V_n - 15\,000$

Showing recursive calculations, determine the value of the machine, in dollars, after two years. (1 MARK)

VCAA 2020 Exam 2 Recursion and financial modelling Q7b

72% of students answered this question correctly.

---

16. Phil is a builder who has purchased a large set of tools.

The initial value of the tools was \$60 000.

He can depreciate the value of the tools by a flat rate of 8% of the purchase price per annum.

Let V_n be the value of the tools after n years, in dollars.

Write down a recurrence relation, in terms of V_0, V_{n+1} and V_n, that could be used to model the value of the tools using this flat rate depreciation. (1 MARK)

Adapted from VCAA 2019 Exam 2 Recursion and financial modelling Q7d

42% of students answered this type of question correctly.

---

17. Ken's first caravan had a purchase price of \$38 000.

After eight years, the value of the caravan was \$16 000.

The caravan has travelled an average of 5000 km in each of the eight years since it was purchased.

Assume that the value of the caravan has been depreciated using the unit cost method of depreciation.

By how much is the value of the caravan reduced per kilometre travelled? (1 MARK)

Adapted from VCAA 2016 Exam 2 Recursion and financial modelling Q6c

29% of students answered this type of question correctly.

Questions from multiple lessons

Data analysis

18. The number of passengers on a particular 7:30 am Melbourne train was continuously monitored.

The following table displays the long-term average number of *passengers* and seasonal index for every day of the week.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
long-term average *passengers*	175	163	158	166	150	83	64
seasonal index	1.28	1.19	1.15	1.21	1.09	0.61	0.47

On Sunday last week, there was a second-hand market in the morning. As a result, the train contained 125 *passengers*. The deseasonalised number of *passengers* last Sunday was closest to

A. 184 **B.** 191 **C.** 205 **D.** 236 **E.** 266

Adapted from VCAA 2016 Exam 1 Data analysis Q15

Recursion and financial modelling *Year 11 content*

19. The following graph displays the growth in the value of a $200 investment over five years.

Ferdinand invested a different sum of money under the same investment conditions and received a total of $450 in interest only over a period of nine years.

How much money did Ferdinand invest?

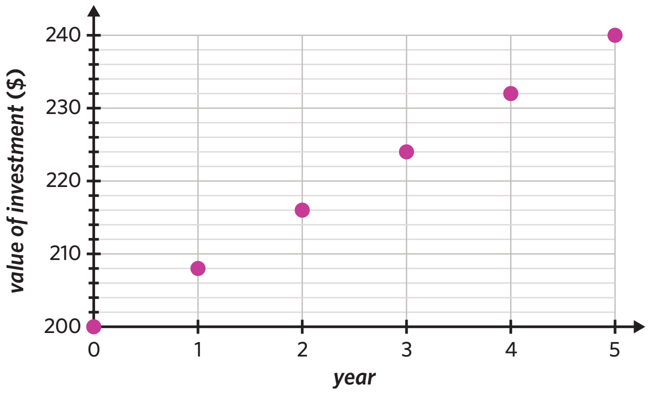
A. $400

B. $612

C. $1125

D. $1250

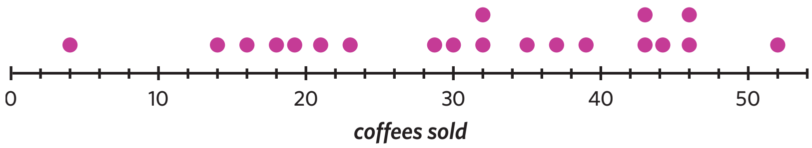
E. $2250

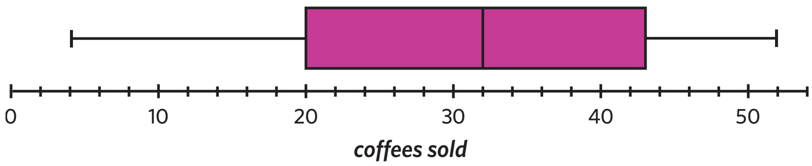


Adapted from VCAA 2013 Exam 1 Business-related mathematics Q7

Data analysis

20. Advertising for a cafe can lead to more coffees being sold. The following dot plot and boxplot both show the increase of *coffees sold*, per day for 20 days, due to an increased number of advertisements displayed around the city.





a. Describe the shape of the distribution of the increase in coffees sold for the 20 days. (1 MARK)

b. The value of 4 is above the lower fence and is not an outlier.

Determine the value of the lower fence. (1 MARK)

Adapted from VCAA 2018 Exam 2 Data analysis Q1f,g

5C Reducing balance depreciation – recurrence relations

STUDY DESIGN DOT POINT

- use of a recurrence relation to model and compare (numerically and graphically) flat rate, unit cost and reducing balance depreciation of the value of an asset with time, including the use of a recurrence relation to determine the depreciating value of an asset after n depreciation periods for the initial sequence

5A 5B 5C 5D 5E 5F 5G

KEY SKILLS

During this lesson, you will be:

- modelling reducing balance depreciation using recurrence relations
- using a geometric recurrence relation to determine the value of an asset after n depreciation periods.

KEY TERM

- Reducing balance depreciation

Recurrence relations can be used to model the loss in the value of an asset over time. As seen previously, arithmetic depreciation is where the value of an asset decreases by a constant amount. However, some assets instead depreciate by a fixed ratio each period, and are represented by geometric recurrence relations.

Modelling reducing balance depreciation using recurrence relations

Reducing balance depreciation occurs when the value of an asset depreciates by a constant ratio, rather than a constant amount, each time period. Rather than the depreciation amount being a set value, it will be a percentage of the previous value. This amount changes each period.

For example, a computer that had an initial value of \$2000 and depreciates by 5% per year according to the reducing balance method, will depreciate by $0.05 \times 2000 = \$100$ in the first year, $0.05 \times 1900 = \$95$ in the second year, and so on.

A geometric recurrence relation can be used to model reducing balance depreciation and calculate the value of the asset from one period to the next.

$V_0 = principal, \quad V_{n+1} = R \times V_n$, where

- $R = 1 - \frac{r}{100}$
- r is the depreciation rate (%) per period
- V_n is the value after n periods

Worked example 1

Determine the recurrence relation that models the depreciation in each of the following scenarios.

a. A tablet is worth \$550 and is depreciated using a reducing balance depreciation method with a rate of 4.3% per annum.

Explanation

Step 1: Determine the initial value, V_0.

The tablet is initially worth \$550.

$V_0 = 550$

Step 2: Calculate the value of R.

$R = 1 - \frac{r}{100}$

The reducing balance rate is 4.3%, so $r = 4.3$.

$R = 1 - \frac{4.3}{100}$

$= 0.957$

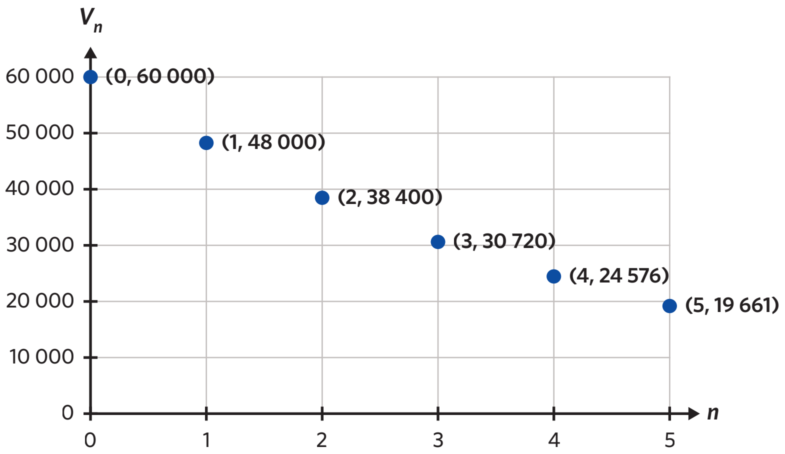
Step 3: Substitute these values into the recurrence relation.

$V_0 = principal, \quad V_{n+1} = R \times V_n$

Answer

$V_0 = 550, \quad V_{n+1} = 0.957 \times V_n$

b. The graph shows the value of a boat, purchased for \$60 000, over a period of four years.



Explanation

Step 1: Determine the initial value, V_0.

The boat was purchased for \$60 000.

$V_0 = 60\ 000$

Step 2: Calculate the ratio, R, of the value of the boat from each period to the next.

Initial value to the end of the first year:

$\frac{48\ 000}{60\ 000} = 0.8$

First year to second year:

$\frac{38\ 000}{48\ 000} = 0.8$

By repeating this for the remaining years, it can be seen that there is a constant ratio, R, of 0.8.

Step 3: Substitute these values into the recurrence relation.

$V_0 = principal, \quad V_{n+1} = R \times V_n$

Answer

$V_0 = 60\ 000, \quad V_{n+1} = 0.8 \times V_n$

Using a geometric recurrence relation to determine the value of an asset after *n* depreciation periods

A recurrence relation that models depreciation can be used to determine the value of an asset after a given number of periods. The value is found by the process of iteration.

When modelling reducing balance depreciation, the initial value, V_0, is substituted into the recurrence relation to find the value of the asset after one period, V_1. V_1 is then substituted into the relation to find the value after two periods, V_2, and so on.

Worked example 2

A tablet is worth \$550 and is depreciated using a reducing balance depreciation method with a rate of 4.3% per annum.

A recurrence relation is constructed to represent the depreciation of the tablet:

$V_0 = 550, \quad V_{n+1} = 0.957 \times V_n$

a. After how many years will the value of the tablet drop below \$505?

Explanation

Step 1: Determine the initial value, V_0.

$V_0 = 550$

Step 2: Calculate the value at the end of each year until the tablet is less than \$505.

Value after 1 year:

$$\begin{aligned} V_1 &= 0.957 \times V_0 \\ &= 0.957 \times 550 \\ &= 526.35 \end{aligned}$$

Value after 2 years:

$$\begin{aligned} V_2 &= 0.957 \times V_1 \\ &= 0.957 \times 526.35 \\ &= 503.716... \end{aligned}$$

Note: These calculations can also be performed using a calculator as shown in 5A Worked example 5.

Answer

2 years

b. After three years, what is the value of the tablet?

Explanation

Calculate the value after 3 years.

The value after 2 years, V_2, was found in part **a**.

$$\begin{aligned} V_2 &= 503.716... \\ V_3 &= 0.957 \times V_2 \\ &= 0.957 \times 503.716... \\ &= 482.057... \end{aligned}$$

Answer

\$482.06

Continues →

c. What is the amount of depreciation in the third year?

Explanation

Calculate the amount of depreciation in year 3.

Use the value of the tablet after year 2 and year 3.

$V_2 = 503.72$

$V_3 = 482.06$

The amount of depreciation is the difference between the value after year 2 and 3.

$503.72 - 482.06 = 21.66$

Answer

$21.66

---

d. Graph the value of the tablet over the first four years.

Explanation

Step 1: Identity the values over the first four years, V_1 to V_4.

$V_0 = 550$

$V_1 = 526.35$

$V_2 = 503.72$

$V_3 = 482.06$

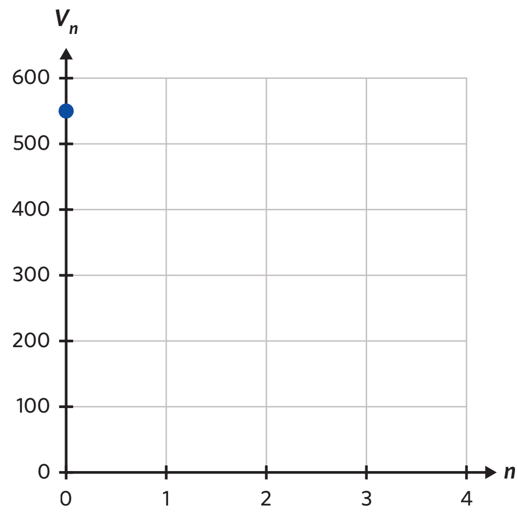
$V_4 = 0.957 \times V_3$

$= 0.957 \times 482.06$

≈ 461.33

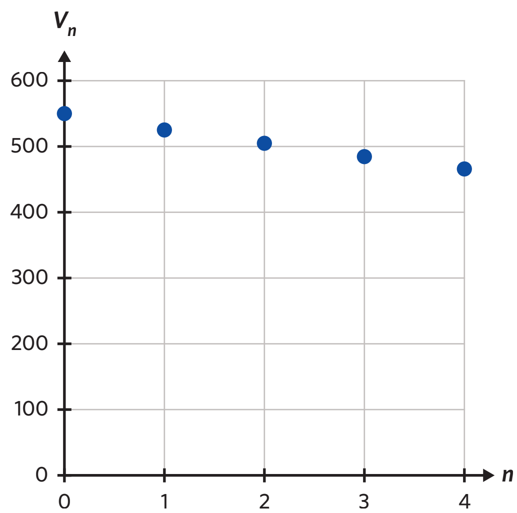
Step 2: Plot V_0 on a graph.

V_0 will be the vertical axis intercept.



Step 3: Plot the remaining values on the graph.

Answer



Exam question breakdown *VCAA 2016 Exam 1 Recursion and financial modelling Q19*

The purchase price of a car was \$26 000.

Using the reducing balance method, the value of the car is depreciated by 8% each year.

A recurrence relation that can be used to determine the value of the car after n years, C_n, is

A. $C_0 = 26\,000, \quad C_{n+1} = 0.92C_n$
B. $C_0 = 26\,000, \quad C_{n+1} = 1.08C_n$
C. $C_0 = 26\,000, \quad C_{n+1} = C_n + 8$
D. $C_0 = 26\,000, \quad C_{n+1} = C_n - 8$
E. $C_0 = 26\,000, \quad C_{n+1} = 0.92C_n - 8$

Explanation

Step 1: Determine the initial value, C_0.

The purchase price of the car was \$26 000.

$C_0 = 26\,000$

Step 2: Calculate the value of R.

$R = 1 - \frac{r}{100}$

The reducing balance rate is 8%, so $r = 8$.

$R = 1 - \frac{8}{100}$
$= 0.92$

Step 3: Substitute these values into the recurrence relation.

$C_0 = principal, \quad C_{n+1} = R \times C_n$

$C_0 = 26\,000, \quad C_{n+1} = 0.92 \times C_n$

Answer

A

74% of students answered this question correctly.

14% of students incorrectly answered option B. The recurrence relation in option B is modelling geometric growth, since $R > 1$. Depreciation is a decrease in an asset's value, therefore R must be less than 1.

5C Questions

Modelling reducing balance depreciation using recurrence relations

1. Alex recently purchased a new tractor to use on his farm for \$13 000. The value of the tractor depreciates on a reducing balance basis at a rate of 10% per annum. Which of the following recurrence relations models the depreciation of Alex's tractor?

A. $V_0 = 13\,000, \quad V_{n+1} = V_n - 1300$
B. $V_0 = 13\,000, \quad V_{n+1} = V_n + 1300$
C. $V_0 = 13\,000, \quad V_{n+1} = 1.1V_n$
D. $V_0 = 13\,000, \quad V_{n+1} = 0.9V_n$

2. The value of Ella's car, V_n, at year n, is depreciating according to the following recurrence relation.

$V_0 = 2000, \quad V_{n+1} = \left(1 - \frac{6}{100}\right) \times V_n$

a. How much did Ella pay for her car?

b. What is the depreciation rate per period of Ella's car, as a percentage?

3. Write the recurrence relations, O_n and A_n, to model the reducing balance depreciation of the following items, n years after purchase.

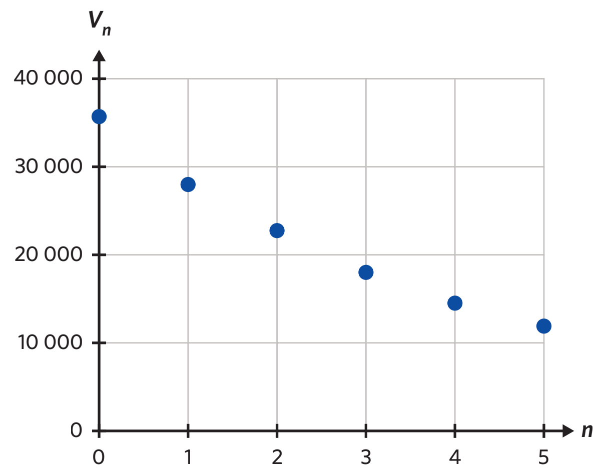
a. Oliver's camera was bought for \$600, and depreciates at a rate of 8% per year.

b. Ava's tablet was bought for \$500, and depreciates at a rate of 14% per year.

4. Grace purchased a brand new grand piano for \$36 000. The piano loses value according to reducing balance depreciation, as shown in the graph.

The value of the grand piano after year 1 is \$28 800.

Construct a recurrence relation to represent the depreciation of the piano.



Using a geometric recurrence relation to determine the value of an asset after *n* depreciation periods

5. The value of Caitlyn's new TV, V_n, at year n, is given by the following recurrence relation.

$V_0 = 2000, \quad V_{n+1} = 0.98V_n$

What is the value of the TV, to the nearest dollar, after two years?

A. \$1882 **B.** \$1920 **C.** \$1921 **D.** \$1960

6. Maddy has saved up to buy a second hand car from her cousin. The car is worth \$11 800 and is depreciated by 9% per annum according to the reducing balance method.

The following recurrence relation models the depreciation of Maddy's car.

$M_0 = 11\,800, \quad M_{n+1} = 0.91M_n$

a. What is the value of the car after Maddy has owned it for three years?

b. What is the total amount of depreciation for the three years after Maddy buys the car?

7. The following recurrence relation models the monthly depreciation of a brand new jet ski that was purchased for \$28 750.

$J_0 = 28\,750, \quad J_{n+1} = 0.985J_n$

a. How much is the jet ski worth after one year?

b. How long will it be until the jet ski is worth \$26 657.47?

c. What is the amount of depreciation in the sixth month to the nearest dollar?

8. The following recurrence relation models the depreciation per year of a computer that was purchased for \$3250.

$V_0 = 3250, \quad V_{n+1} = 0.9V_n$

a. Determine the values of V_1, V_2, V_3 and V_4.

b. Plot the depreciation of the computer over four years.

c. After how many years does the value of the computer fall below \$2500?

9. Suppose a car is depreciated by the reducing balance method at a rate of 17% per annum. The value of the car, V_n, after n years, is given by the following recurrence relation.

$V_0 = 54\,000, \quad V_{n+1} = 0.83V_n$

a. Find the value of the car, to the nearest dollar, after three years.

b. Suppose that the car depreciates by 17% each year for the first three years of its life, but this rate then reduces to 11% for the remainder of the car's life. What is the value of the car five years after it is purchased? Round to the nearest dollar.

Joining it all together

10. Tracy and her brother Dean both bought a laptop. Tracy's laptop is depreciated according to the recurrence relation

$T_0 = 800, \quad T_{n+1} = 0.98T_n,$

where n represents the number of months since purchase.

a. What is the reducing balance rate of depreciation per period for Tracy's laptop?

b. How much value will Tracy's laptop lose over the first three months? Round to the nearest cent.

Dean's laptop was bought for $900 and lost $22.50 in value in the first month. His laptop depreciates according to the reducing balance method.

c. Find the monthly rate of depreciation.

d. Write a recurrence relation for the value of Dean's laptop, D_n, in terms of months, n.

---

11. Sam recently bought a new phone for $1700. It will be depreciated at a reducing balance rate of 16% per annum.

a. Write a recurrence relation to model the value of Sam's phone, S_n, where n is the number of years after it was purchased.

b. What will the value of Sam's phone be after four years?

c. Plot the future value of Sam's phone on a graph over the first four years.

d. After how many years will the value of the phone fall below $1200?

---

12. Courtney was given a designer handbag for her birthday which cost $3250. The handbag loses its value each year as the style becomes more outdated. It is depreciated at a reducing balance rate of 10% per annum.

a. Write a recurrence relation that represents the value of Courtney's handbag, C_n, over n years.

b. How many years will it take for the handbag to lose half its value?

c. Suppose that once the handbag is worth less than half of the initial purchase price, the rate of depreciation reduces to 5% per annum. Calculate the value of the handbag, to the nearest dollar, two years after it loses half its value.

Exam practice

13. Phil is a builder who has purchased a large set of tools.

The value of Phil's tools is depreciated using the reducing balance method.

The value of the tools, in dollars, after n years, V_n, can be modelled by the following recurrence relation.

$V_0 = 60\,000, \quad V_{n+1} = 0.9V_n$

Use recursion to show that the value of the tools after two years, V_2, is $48 600. (1 MARK)

VCAA 2019 Exam 2 Recursion and financial modelling Q7a

74% of students answered this question correctly.

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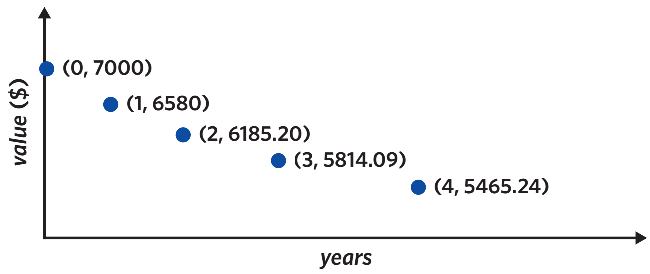
14. Consider the following graph.



This graph could show the value of

A. a piano depreciating at a flat rate of 6% per annum.

B. a car depreciating with a reducing balance rate of 6% per annum.

C. a compound interest investment earning interest at the rate of 6% per annum.

D. an item of equipment depreciating with a reducing balance rate of 10% per annum.

E. a boat depreciating at a flat rate of 10% per annum.

Adapted from VCAA 2017 Exam 1 Recursion and financial modelling Q22

65% of students answered this type of question correctly.

15. Julie withdraws $14 000 from her account to purchase a car for her business. For tax purposes, she plans to depreciate the value of her car using the reducing balance method.

The value of Julie's car, in dollars, after n years, C_n, can be modelled by the following relation shown.

$C_0 = 14\,000, \quad C_{n+1} = R \times C_n$

a. For each of the first three years of reducing balance depreciation, the value of R is 0.85.

What is the annual rate of depreciation in the value of the car during these three years? (1 MARK)

b. For the next five years of reducing balance depreciation, the annual rate of depreciation in the value of the car is changed to 8.6%.

What is the value of the car eight years after it was purchased?

Round to the nearest cent. (2 MARKS)

VCAA 2018 Exam 2 Recursion and financial modelling Q5a,b

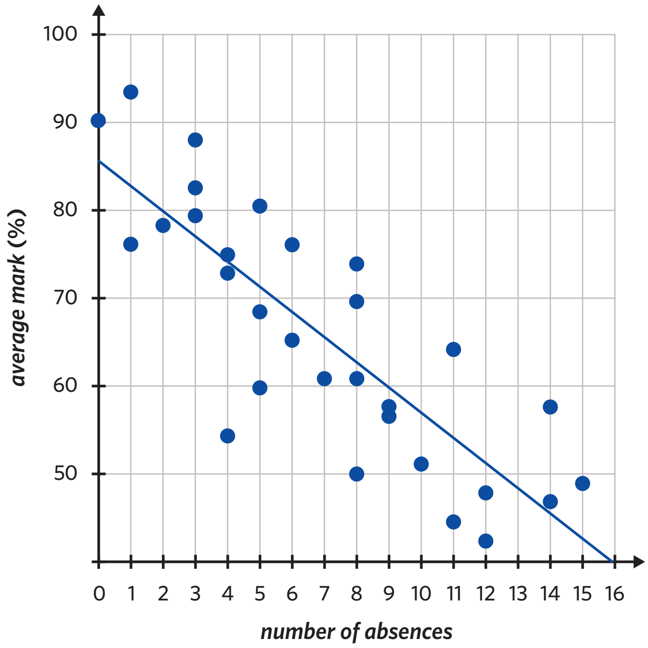
Part **a**: **55%** of students answered this question correctly.

Part **b**: The average mark on this question was **0.8**.

Questions from multiple lessons

Data analysis

16. The following scatterplot displays the distribution of the *number of absences* in a semester and *average mark* (%) of 30 university students. A least squares regression line has been fitted to the data.



The correlation coefficient of the distribution is $r = -0.8259$.

Which of the following statements is false?

A. There is a strong negative association between *number of absences* and *average mark* for these students.

B. Approximately 68% of the variation in *average mark* can be explained by the variation in *number of absences*.

C. Using the regression line to predict the *average mark* of a student with 16 absences is an example of interpolation.

D. The least squares line has a negative slope.

E. Students with higher numbers of absences tend to have lower average marks.

Adapted from VCAA 2017NH Exam 1 Data analysis Q11

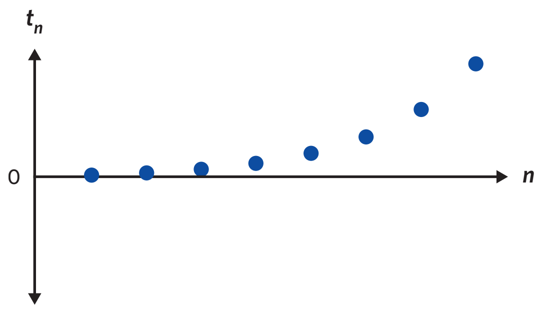
Recursion and financial modelling *Year 11 content*

17. The first term in a geometric sequence is x, where $x > 0$.

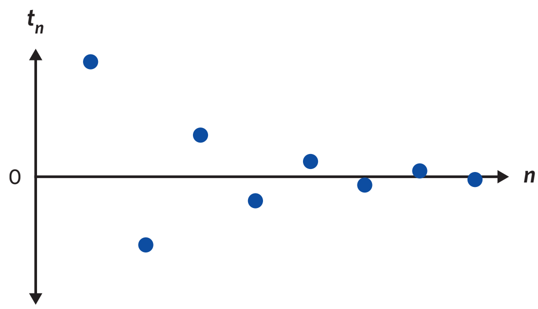
The common ratio of the sequence is R, where $-1 < R < 0$.

Which of the following graphs could show the first eight terms of the sequence?

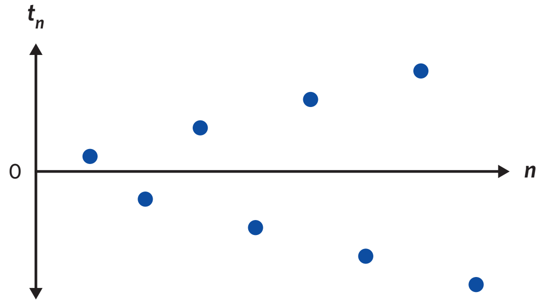
A.



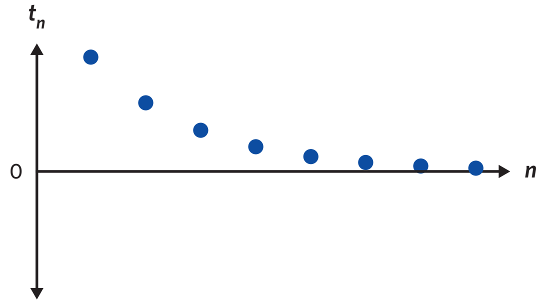
B.



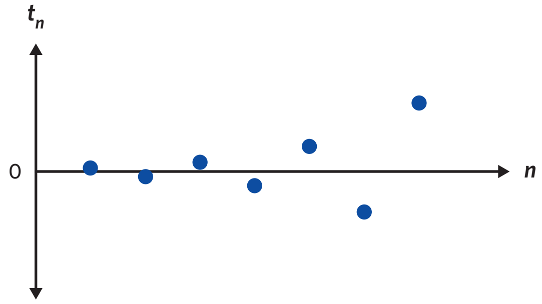
C.



D.



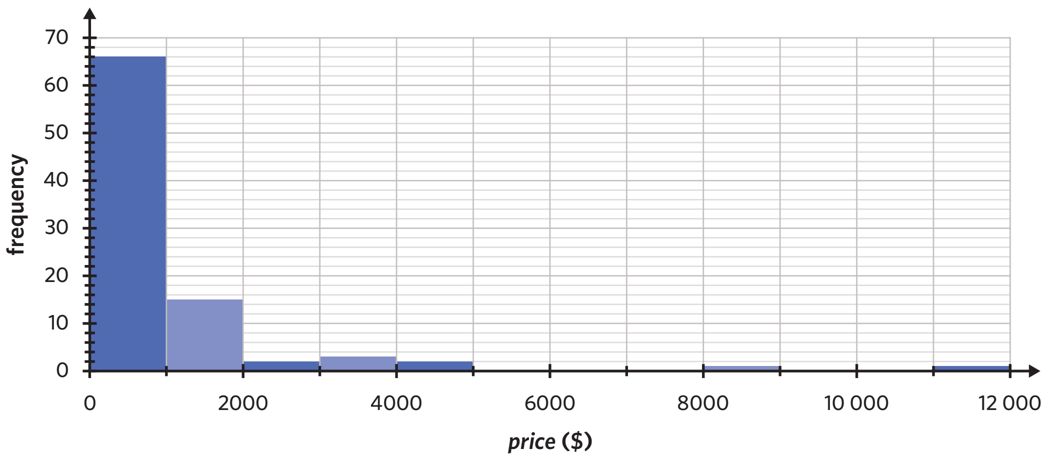
E.



Adapted from VCAA 2014 Exam 1 Number patterns Q8

Data analysis

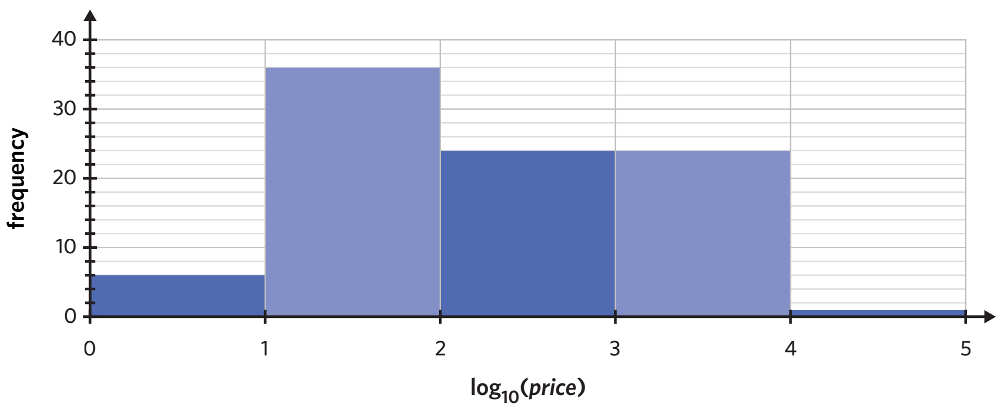
18. The following histogram shows the price of 90 handbags from various shops in Melbourne.



a. Describe the shape of the distribution, including any possible outliers. (1 MARK)

b. What percentage of the handbags cost under $2000? (1 MARK)

c. The following histogram displays the distribution of $\log_{10}(price)$ for the same sample of 90 handbags.



How many of the handbags cost between $100 and $1000? (1 MARK)

Adapted from VCAA 2018NH Exam 2 Data analysis Q3

5D Depreciation – finding the rule for the *n*th term

STUDY DESIGN DOT POINT

- use of the rules for the future value of an asset after *n* depreciation periods for flat rate, unit cost and reducing balance depreciation and their application

5A 5B 5C **5D** 5E 5F 5G

KEY SKILLS

During this lesson, you will be:

- creating and using a rule for flat rate depreciation
- creating and using a rule for unit cost depreciation
- creating and using a rule for reducing balance depreciation.

Depreciation can be modelled accurately using recurrence relations. However, a recurrence relation can only be used to calculate the next term in the sequence. It is useful to create rules that model depreciation so that any term in the sequence can be determined from one calculation, instead of multiple iterations.

Creating and using a rule for flat rate depreciation

Recall that flat rate depreciation is used when the value of an asset decreases by a constant amount each period, calculated from the initial value.

The following rule can be used to model V_n, the value of an asset after n periods, depreciated by the flat rate method.

$V_n = V_0 - nd$, where

- V_n is the value after n periods
- V_0 is the principal
- d is the depreciation amount per period, calculated by $d = \frac{r}{100} \times V_0$
- r is the depreciation rate (%) per period

Once the rule has been created, the value of the asset after n periods can be found by substituting the value of n into the rule.

Worked example 1

Kwyjibo bought an apartment for $670 000 in what he thought was a high-growth suburb. Unfortunately, the growth stopped once he bought the property and it is now depreciating yearly by 1.5% of its initial value.

a. Write the rule to determine V_n, the value of the apartment after n years.

Explanation

Step 1: Determine the initial value, V_0.

Kwyjibo bought the real estate for $670 000.

$V_0 = 670\,000$

Continues →

Step 2: Calculate the depreciation amount, d.

$d = V_0 \times \frac{r}{100}$

The rate of depreciation per period is 1.5%, so $r = 1.5$.

$$d = 670\,000 \times \frac{1.5}{100}$$
$$= 10\,050$$

Step 3: Substitute these values into the rule for V_n.

$V_n = V_0 - nd$

Answer

$V_n = 670\,000 - 10\,050n$

b. Find the value of Kwyjibo's apartment after 9 years.

Explanation

Calculate V_9, the value of the apartment after 9 years.

Substitute $n = 9$ into the rule for V_n.

$$V_9 = 670\,000 - 10\,050 \times 9$$
$$= 670\,000 - 90\,450$$
$$= 579\,550$$

Answer

$579 550

Creating and using a rule for unit cost depreciation

Recall that unit cost depreciation is used when the value of an asset decreases by a constant amount for each unit of use.

The following rule can be used to model V_n, the value of an asset after n units of use, depreciated using the unit cost method.

$V_n = V_0 - nd$

- V_n is the value after n units of use
- V_0 is the principal
- d is the depreciation amount per unit of use

Once a rule has been created, the value of the asset after n units of use can be found by substituting the value of n into the rule.

Worked example 2

Samson purchased a computer worth $2000. The research he did online told him that the model of computer he purchased depreciates by $0.32 for every hour of use.

a. Write the rule to determine V_n, the value of the computer after n hours of use.

Explanation

Step 1: Determine the initial value, V_0.

Samson purchased the computer for $2000.

$V_0 = 2000$

Step 2: Calculate the depreciation amount, d.

The computer depreciates by $0.32 for every hour of use.

$d = 0.32$

Continues →

Step 3: Substitute these values into the rule for V_n.

$V_n = V_0 - nd$

Answer

$V_n = 2000 - 0.32n$

b. What is the value of Samson's computer after 300 hours of use?

Explanation

Calculate V_{300}, the value of the computer after 300 hours of use.

Substitute $n = 300$ into the rule for V_n.

$$V_{300} = 2000 - 0.32 \times 300$$
$$= 2000 - 96$$
$$= 1904$$

Answer

$1904

Creating and using a rule for reducing balance depreciation

Recall that reducing balance depreciation occurs when the value of an asset depreciates by a percentage, calculated from the value at the previous time period.

The following rule can be used to model V_n, the value of an asset after n periods, depreciated by reducing balance depreciation.

$V_n = V_0 \times R^n$, where

- V_n is the value after n periods
- V_0 is the principal
- $R = 1 - \frac{r}{100}$
- r is the depreciation rate (%) per period

Once the rule has been created, the value of the asset after n periods can be found by substituting the value of n into the rule.

Worked example 3

Gemima purchased a $10 000 lawnmower that depreciates at a rate of 10% each year, using the reducing balance method.

a. Write the rule to determine V_n, the value of the lawnmower after n years.

Explanation

Step 1: Determine the initial value, V_0.

Gemima purchased the lawnmower for $10 000.

$V_0 = 10\,000$

Step 2: Calculate the value of R.

$R = 1 - \frac{r}{100}$

The rate of depreciation is 10% per period, so $r = 10$.

$$R = 1 - \frac{10}{100}$$
$$= 0.9$$

Continues →

Step 3: Substitute these values into the rule for V_n.

$V_n = V_0 \times R^n$

Answer

$V_n = 10\,000 \times 0.9^n$

b. Find the value of the lawnmower after 5 years.

Explanation

Calculate V_5, the value of the lawnmower after 5 years.

Substitute $n = 5$ into the rule for V_n.

$$V_5 = 10\,000 \times 0.9^5$$
$$= 5904.9$$

Answer

$5904.90

Exam question breakdown

VCAA 2019 Exam 1 Recursion and financial modelling Q22

A machine is purchased for $30 000.

It produces 24 000 items each year.

The value of the machine is depreciated using a unit cost method of depreciation.

After three years, the value of the machine is $18 480.

A rule for the value of the machine after n units are produced, V_n, is

A. $V_n = 0.872n$
B. $V_n = 24\,000n - 3840$
C. $V_n = 30\,000 - 24\,000n$
D. $V_n = 30\,000 - 0.872n$
E. $V_n = 30\,000 - 0.16n$

Explanation

Step 1: Identify the initial value, V_0.

The machine is purchased for $30 000.

$V_0 = 30\,000$

Step 2: Calculate the depreciation per year.

After three years, the machine is worth $18 480.

$$\textit{depreciation over three years} = 30\,000 - 18\,480 = 11\,520$$

$$\textit{depreciation per year} = \frac{11\,520}{3} = 3840$$

Step 3: Calculate the depreciation per unit produced, d.

24 000 items are produced each year.

$$d = \frac{3840}{24\,000} = 0.16$$

Step 4: Substitute these values into the rule for V_n.

$V_n = V_0 - nd$

$V_n = 30\,000 - 0.16n$

53% of students answered this question correctly.

17% of students incorrectly answered D. These students calculated the average yearly depreciation (11 520 ÷ 3 = 3840) as a percentage of the initial value (3840 ÷ 30 000 = 0.128 or 12.8%). They then applied the rule for reducing balance depreciation, rather than for unit cost depreciation.

Answer

E

5D Questions

Creating and using a rule for flat rate depreciation

1. A $15 000 asset depreciates by a flat rate of 9% each year. By what dollar amount does the asset depreciate each year?

   A. $900 **B.** $1250 **C.** $1350 **D.** $1500

2. Rodney purchases a $400 scientific calculator that depreciates at a flat rate. He creates the rule $C_n = 400 - 45n$ to calculate C_n, the value of the calculator after n years. The value of the calculator after two years is

   A. $310 **B.** $355 **C.** $445 **D.** $490

3. For each of the following scenarios:
   - write the rule to determine V_n, the value of the asset after n periods.
   - calculate the value of the asset after nine periods.

   a. A $1 000 000 property that depreciates by 6% of the initial value each year.

   b. A $550 designer t-shirt that depreciates by 3% of the sale price every month.

   c. A $380 fridge that depreciates by 6.5% of the initial value each year.

   d. A $1545 pair of shoes that depreciates by 4% of its sale price every month.

4. Steve buys a smartphone for $550. It depreciates by 10% of the sale price every year.

   a. Write a rule to determine S_n, the value of the smartphone after n years.

   b. What is the value of the smartphone after the third year?

   c. After how many years will the smartphone have no value?

5. Riya buys a computer for $1225. Two years later she gets it valued, and finds it is worth $955.50.

   a. What is the annual flat rate percentage depreciation for her computer?

   b. Write the flat rate depreciation rule to determine C_n,the value of her computer after n years.

   c. What will the value of the computer be after 7 years?

6. Hessa buys a car for $8600. She hopes that in three years it will still be worth at least $6000. What is the highest rate of flat rate depreciation per year, to the nearest percent, the car can have for this to happen?

Creating and using a rule for unit cost depreciation

7. Hassan bought a SodaStream for $100 that depreciates by $0.25 each time it is used. He wants to create a rule to determine S_n, the value of the SodaStream after n uses. Which of the following is correct?

   A. $S_0 = 0.25, \quad d = 100$

   B. $S_n = 100, \quad d = 0.25$

   C. $S_0 = 100, \quad d = 0.25$

   D. $S_0 = 100, \quad d = 25$

8. For each of the following scenarios:
   - write the rule to determine V_n, the value of the asset after n units of use.
   - calculate the value of the asset after five units of use.

   a. A \$65 air fryer that depreciates by \$0.08 for each hour of use.

   b. A \$315 printer depreciates by \$0.05 for each page printed.

   c. A \$20 000 car that depreciates by \$245 for every 1000 km driven.

   d. An \$825 dishwasher that depreciates by \$7.50 after each wash.

9. A parachute costs \$400 and depreciates by \$8 per jump.

   a. Write a rule to determine P_n, the value of the parachute after n jumps.

   b. What is the value of the parachute after 20 jumps?

   c. After how many jumps will the parachute be worth \$200?

10. The following table shows the value of a jet at various points during its service life.

***distance travelled* (km)**	400 000	500 000	600 000
***value* (\$)**	4 358 200	4 017 800	3 677 400

   a. What is the depreciation of the jet, per kilometre travelled, rounded to the nearest cent?

   b. Using the rounded answer from part **a**, what was the original value of the jet?

11. Anakin owns a cafe and purchases an industrial coffee machine for \$9300. The coffee machine depreciates for each coffee made. On average, his cafe sells 215 coffees per day. After 120 days, the coffee machine is worth \$8655. Assuming the cafe continues selling coffees at the same average daily rate, how much will the coffee machine be worth after 350 days?

Creating and using a rule for reducing balance depreciation

12. Max uses the following rule to calculate V_n, the value of his photocopier after n years.

   $V_n = 2500 \times 0.95^n$

   By how much does the photocopier depreciate each year?

   A. 0.05% **B.** 0.5% **C.** 0.95% **D.** 5%

13. For each of the following scenarios:
   - write the rule to determine V_n, the value of the asset after n periods.
   - calculate the value of the asset, to the nearest cent, after the specified number of periods.

   a. A car, originally valued at \$13 000, depreciating at a rate of 6% per annum for three years.

   b. A television with an initial value of \$3100, depreciating at a rate of 10% per annum for seven years.

   c. A printer with an initial value of \$2000, depreciating at a rate of 3.5% every six months for four years.

   d. A drum set purchased for \$6500, depreciating at a rate of 2% every fortnight for one year.

14. Marko wants to buy a new smartwatch, which is currently priced at \$779, but he only has \$700. After doing some market research, he finds that this exact model of watch depreciates at a rate of 1.75% every three months, using a reducing balance depreciation method. How long will Marko have to wait before he can buy the new smartwatch?

Joining it all together

15. Baxter purchases a brand new portable electric heater for $170.

After doing some research he finds out that depending on who he tries to sell the heater to, it can depreciate using one of three methods.

- The flat rate method at 2.8% each month.
- The unit cost method at $0.22 for every 10 hours of use.
- The reducing balance method of 3% each month.

a. Write a rule to determine H_n, the value of the heater after n periods or units of use for each of the depreciation methods.

b. Baxter works out that he will use the heater for an average of 240 hours each month. After two months of owning the heater, he decides to travel overseas and wants to sell the heater. Which depreciation method will allow Baxter to sell the heater for the largest amount, and how much would he be able to sell it for?

c. Due to travel restrictions, Baxter has to stay home and ends up using the heater for a total of six months before he tries to sell it. Which depreciation method will allow Baxter to sell the heater for the highest amount, and how much would he be able to sell it for?

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16. Mildred buys a high pressure washer for $1200. She is told that the washer depreciates by $2.50 for every 10 litres of water used, and estimates that she will use 15 litres of water each month.

a. Assuming Mildred uses an average of 15 litres of water each month, what is the annual rate of depreciation that will ensure its depreciated value using the flat rate method will be equivalent to the unit cost method after one year?

b. Using the reducing balance method, what is the **annual** rate of depreciation that will ensure the value of the washer will be equivalent to the unit cost method after two years? Round to two decimal places.

c. Mildred wants to sell the washer after 10 years. Calculate the depreciated value of the washer using the flat rate and reducing balance methods, using the answers calculated in parts **a** and **b**. Which method will allow Mildred to sell the washer for the highest amount?

Exam practice

17. The value of a van purchased for $45 000 is depreciated by k% per annum using the reducing balance method.

After three years of this depreciation, it is then depreciated in the fourth year under the unit cost method at the rate of 15 cents per kilometre.

The value of the van after it travels 30 000 km in this fourth year is $26 166.24.

The value of k is

A. 9 **B.** 12 **C.** 14
D. 16 **E.** 18

VCAA 2020 Exam 1 Recursion and financial modelling Q29

54% of students answered this question correctly.

---

18. Sammy purchased a boat for $72 000.

The value of the boat is depreciated each year by 10% using the reducing balance method.

In the third year, the boat will depreciate in value by 10% of

A. $47 239.20 **B.** $52 488.00 **C.** $58 320.00
D. $64 800.00 **E.** $72 000.00

VCAA 2021 Exam 1 Recursion and financial modelling Q20

44% of students answered this question correctly.

19. Sienna owns a coffee shop.

A coffee machine, purchased for \$12 000, is depreciated in value using the unit cost method.

The rate of depreciation is \$0.05 per cup of coffee made.

The recurrence relation that models the year-to-year value, in dollars, of the coffee machine is

$M_0 = 12\,000, \quad M_{n+1} = M_n - 1440$

a. The recurrence relation could also represent the value of the coffee machine depreciating at a flat rate.

What annual flat rate percentage of depreciation is represented? (1 MARK)

b. Complete the following rule that gives the value of the coffee machine, M_n, in dollars, after n cups have been produced.

$M_n = \boxed{} + \boxed{} \times n$ (1 MARK)

VCAA 2021 Exam 2 Recursion and financial modelling Q7b,c

Part **a**: **49%** of students answered this question correctly.

Part **b**: **21%** of students answered this question correctly.

Questions from multiple lessons

Data analysis

20. The seasonal indices for the number of tickets sold at a tourist attraction each month, excluding August, are shown in the following table.

month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
seasonal index	1.48	1.34	1.07	0.89	0.66	0.68	0.63		0.75	0.91	1.24	1.68

If 4592 tickets were sold last August then the deseasonalised number of tickets is closest to

A. 3076 **B.** 3077 **C.** 3168 **D.** 6655 **E.** 6854

Adapted from VCAA 2017NH Exam 1 Data analysis Q15

Recursion and financial modelling *Year 11 content*

21. Find the first five terms of the following recurrence relation.

$L_0 = -4, \quad L_{n+1} = 8 - 6 \times L_n$

A. $-4, -32, -184, -1112, -6664...$

B. $-4, -24, -144, 864, -5184...$

C. $-4, -16, -88, -520, -3112...$

D. $-4, 32, -184, 1112, -6664...$

E. $32, -184, 1112, -6664, 39\,992...$

Adapted from VCAA 2016 Exam 1 Recursion and financial modelling Q17

Recursion and financial modelling *Year 11 content*

22. Argus realises that he is allergic to his dog, so he decides to sell it to his friend Houston. In order to pay for the dog, Houston agrees to make eight payments over eight months. He will pay \$32 in the first month, January, and in each subsequent month he will pay 50% more than the previous month. This follows the geometric sequence $V_{n+1} = 1.5V_n$.

a. How much money does Houston pay Argus in the sixth month? (1 MARK)

b. In which month will Houston first pay over \$150? (1 MARK)

c. How much does Houston end up paying Argus in total? (1 MARK)

Adapted from VCAA 2013 Exam 2 Number patterns Q2

5E Simple interest

STUDY DESIGN DOT POINT

- the concepts of simple and compound interest

5A 5B 5C 5D 5E 5F 5G

KEY SKILLS

During this lesson, you will be:

- modelling simple interest using recurrence relations
- modelling simple interest using a rule.

KEY TERMS

- Interest
- Simple interest

Simple interest is the most straightforward type of interest. It is often found in the real world with short-term loans, such as car loans or small personal loans. Simple interest is often the most preferred type of interest by those paying it, as it is usually the cheapest option when borrowing money.

Modelling simple interest using recurrence relations

Interest is the cost of borrowing money. For loans, it is the fee that banks and lenders charge the borrowers for borrowing their money. On the other hand, for investments, the investor will receive interest from the investment location, such as a bank, since they are effectively borrowing money from them.

Simple interest is a constant amount of interest that is charged or earned each period. It is calculated as a percentage of the principal.

An investment or loan with simple interest can be modelled by a recurrence relation which can then be used to calculate the value of the interest or loan after any amount of time periods.

The value of these loans and investments, V_n, after n periods can be modelled by an arithmetic recurrence relation of the form

$V_0 = principal, \quad V_{n+1} = V_n + d$, where

- d is the interest paid each period, calculated by $d = \frac{r}{100} \times V_0$
- r is the interest rate (%) per period

Note: This recurrence relation is only valid if no loan repayments are made or if no additional payments are added to an investment.

Worked example 1

Sally borrows \$300 000 where simple interest is charged at a rate of 0.5% per year.

a. Construct a recurrence relation to model the value of this loan, V_n after n years.

Explanation

Step 1: Determine the principal and interest rate.

$V_0 = 300\ 000$

$r = 0.5\%$ per annum

Step 2: Calculate the interest, d.

$$d = \frac{0.5}{100} \times 300\ 000 = 1500$$

Step 3: Construct the recurrence relation.

Answer

$V_0 = 300\ 000, \quad V_{n+1} = V_n + 1500$

b. What is the value of the loan after three years?

Explanation

Step 1: Calculate V_1, the value of the loan after one year.

$$\begin{aligned} V_1 &= V_0 + 1500 \\ &= 300\ 000 + 1500 \\ &= 301\ 500 \end{aligned}$$

Step 2: Calculate V_2, the value of the loan after two years.

$$\begin{aligned} V_2 &= V_1 + 1500 \\ &= 301\ 500 + 1500 \\ &= 303\ 000 \end{aligned}$$

Step 3: Calculate V_3, the value of the loan after three years.

$$\begin{aligned} V_3 &= V_2 + 1500 \\ &= 303\ 000 + 1500 \\ &= 304\ 500 \end{aligned}$$

Answer

\$304 500.00

c. How much interest has been charged after three years?

Explanation

The interest charged is the difference between the current value and the principal.

$$\begin{aligned} interest &= V_3 - V_0 \\ &= 304\ 500 - 300\ 000 \\ &= 4500 \end{aligned}$$

Answer

\$4500.00

Modelling simple interest using a rule

In VCE General Mathematics, a rule can be used to calculate the value of an investment or loan accumulating simple interest. This rule is of the form

$V_n = V_0 + nd$, where

- V_0 is the principal
- d is the interest paid each period, calculated by $d = \frac{r}{100} \times V_0$
- r is the interest rate (%) per period

This is used to calculate V_n for any value of n.

Worked example 2

Joan invests \$5000 in an account paying a simple interest rate of 4.5% per year.

a. Construct a rule that will calculate the value of the investment, V_n, after n years.

Explanation

Step 1: Determine the principal and interest rate.

$V_0 = 5000$

$r = 4.5$

Step 2: Calculate the interest, d.

$$d = \frac{4.5}{100} \times 5000$$
$$= 225$$

Step 3: Construct the rule.

Answer

$V_n = 5000 + 225n$

b. Use the rule to calculate the value of the investment after five years.

Explanation

Calculate V_5, the value of the investment after 5 years.

$$V_5 = 5000 + 225 \times 5$$
$$= 6125$$

Answer

\$6125.00

5E Questions

Modelling simple interest using recurrence relations

1. A \$13 000 investment earns simple interest at a rate of 1.5% per period. Which of the following recurrence relations can be used to model the investment?

   A. $V_0 = 195, \quad V_{n+1} = V_n + 13\,000$

   B. $V_0 = 13\,000, \quad V_{n+1} = V_n + 19\,500$

   C. $V_0 = 13\,000, \quad V_{n+1} = V_n + 1.5 \times V_0$

   D. $V_0 = 13\,000, \quad V_{n+1} = V_n + 195$

2. Write a recurrence relation to model each of the following scenarios.
   a. Tina invests \$1500 in a savings account that pays interest at a flat rate of \$30 every month.
   b. Ahmed puts \$500 in a bank account. The account pays simple interest once a year, at a rate of 8% of the opening balance.

3. The following recurrence relations describe the balance of three savings accounts, V_n, X_n, Y_n after n months. What will the balance of each account be after five months?
   a. $V_0 = 700, \quad V_{n+1} = V_n + 15$
   b. $X_0 = 1200, \quad X_{n+1} = X_n + 60$
   c. $Y_0 = 820, \quad Y_{n+1} = Y_n + 43$

4. The balance of Mala's loan, M_n, at month n, follows the given recurrence relation. The loan charges simple interest at a monthly rate.
   $M_0 = x, \quad M_{n+1} = M_n + 43$
   a. If the balance owing when $n = 4$ is \$1772, what is x?
   b. What is the simple rate of interest per month for this account? Round the answer to two decimal places.

Modelling simple interest using a rule

5. For the following rule, what is V_{19}?
   $V_n = 223 + 19n$
   A. 223 **B.** 242 **C.** 565 **D.** 584

6. If \$9354 was invested in an account that earns simple interest at a rate of 0.7% per month, what is the total amount of simple interest paid at the end of a 22-month period, rounded to a whole number?
   A. 120 **B.** 1200 **C.** 1440 **D.** 1441

7. A \$200 loan charges simple interest at a rate of 9% per annum. What is the total amount outstanding on the loan after 8 years, assuming no repayments have been made?

8. The balance of Alonso's simple interest savings account n months after the initial deposit is given by the following rule.
   $A_n = 1200 + 50n$
   a. What was Alonso's initial deposit?
   b. How much interest does the account earn every month?
   c. Correct to two decimal places, what is the monthly interest rate of the account?

9. Manish takes out a loan of \$3000 from EasyShark Loans. The loan accumulates simple interest at a rate of 3.75% per month.
   a. How much interest is charged every month?
   b. What will the balance of the loan be after the first six months, assuming no repayments have been made?

10. Alex wants to open a simple interest bank account and has two options. An account with the Bank of Edrolo earns interest at a rate of 1% paid out at the end of each month, whereas an account with The Bank of Victoria earns interest at a rate of 8% paid each year. She has $765 to invest in the account.

 a. Which bank pays more interest every year?

 b. After 15 years, how much more money will she earn from the bank with the higher interest amount per year?

11. Janine put $400 into her savings account when she opened it. It pays simple interest at a rate of 3% per month.

 a. Construct a rule to calculate the balance of the account, J_n, at month n.

 b. How much money will be in the account after six months?

 c. After how many months will the balance exceed $500?

12. In order to save up to buy Christmas gifts for her family, Olivia requires $30 of interest to be paid to her bank account each month, for the remaining 5 months of the year. Olivia opens a savings account with The Bank of Edrolo, with a simple interest rate of 6% per month.

 a. What principal amount must she place into the account to achieve her goal?

 b. What will the balance of her bank account be at the end of the year?

Joining it all together

13. Tao borrowed $3650 from his local loan agency, and won't make any repayments for the first 6 years. For each of these 6 years, the value of this loan that must be repaid increases by $83.

 a. Given this information, find the simple interest rate charged per annum, rounded to two decimal places.

 b. Write down a rule to model this scenario, T_n at year n.

 c. If Tao wishes to repay the full loan 6 years later, how much does he owe the loan agency?

14. Charlee sets herself a savings goal of $700, so she opens a brand new savings account and moves all the money from her piggy bank into it. Each year she earns simple interest of $22 with an interest rate of 4% per annum.

 a. How much money did she initially move into the savings account from her piggy bank?

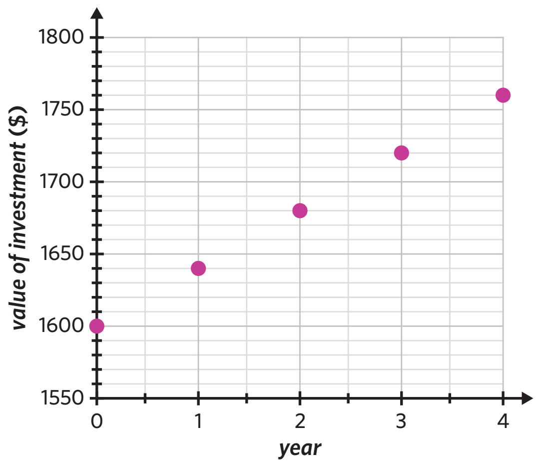
 b. Write down a recurrence relation to model this scenario.

 c. After how many years has Charlee reached her savings goal?

15. Sally invested $1600 in savings account four years ago and plotted the value of the investment every year, shown in the following graph. Three years ago, Harry invested in the same account and received $270 total in simple interest.

 a. How much did Harry invest?

 b. Construct a rule that can be used to calculate H_n, the balance of Harry's account after n years.



Exam practice

16. \$3000 is invested at a simple interest rate of 6.5% per annum.

The total interest earned in three years is

A. \$195.00 **B.** \$580.50 **C.** \$585.00
D. \$3623.85 **E.** \$3585.00

VCAA 2012 Exam 1 Business-related mathematics Q2

90% of students answered this question correctly.

17. A sum of money is invested in an account paying simple interest at a rate of 8% per annum.

The total interest earned on this investment over 6 years is \$27 000.

The sum of money invested is

A. \$12 960 **B.** \$45 000 **C.** \$56 250
D. \$202 500 **E.** \$337 500

VCAA 2007 Exam 1 Business-related mathematics Q3

68% of students answered this question correctly.

Questions from multiple lessons

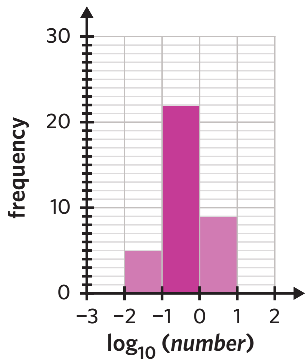
Data analysis

18. The following histogram shows the distribution of the average *number* of pets owned per household for 36 different suburbs plotted on a $\log_{10}$ scale.

The number of suburbs where the average *number* of pets is one or more per household is

A. 0
B. 5
C. 9
D. 22
E. 31

Adapted from VCAA 2016 Exam 1 Data analysis Q7



Recursion and financial modelling *Year 11 content*

19. Consider the following recurrence relation.

$T_0 = 5, \quad T_{n+1} = -2T_n + 6$

The first four terms of the recurrence relation are

A. 5, −10, 4, −8 **B.** 5, −4, 14, −22 **C.** 5, −4, 2, 10 **D.** 6, 4, 2, 0 **E.** 6, −6, 18, −30

Adapted from VCAA 2016 Exam 1 Recursion and financial modelling Q17

Data analysis *Year 11 content*

20. The humidity level in the morning and in the evening on 15 different days in Melbourne was recorded. The data is displayed on the following scatterplot.

The least squares regression line for the given data is:

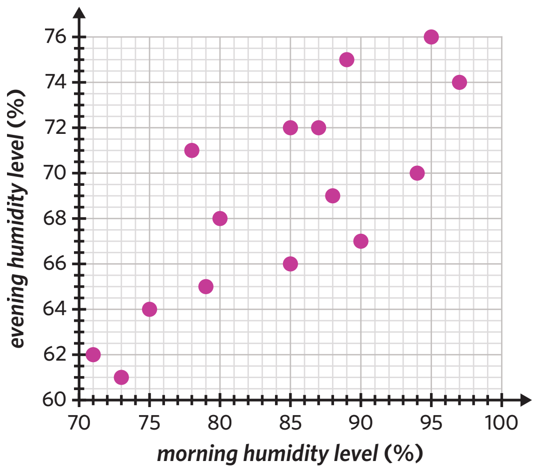
evening humidity level = 30.7 + 0.45 × *morning humidity level*

a. Use the least squares line to predict the humidity level in the evening when the morning humidity level is 76%. (1 MARK)

b. What is the residual value when the least squares line is used to predict the evening humidity level when the morning humidity level is 80%? Round to the nearest one decimal place. (2 MARKS)

c. The correlation coefficient *r* is 0.79. What percentage of the variation in the evening humidity level can be explained by the variation in the morning humidity level? Round to the nearest percent. (1 MARK)

Adapted from VCAA 2018 Exam 2 Data analysis Q2



5F Compound interest

STUDY DESIGN DOT POINTS

- the concepts of simple and compound interest
- use of a recurrence relation to model and analyse (numerically and graphically) a compound interest investment or loan, including the use of a recurrence relation to determine the value of the compound interest loan or investment after n compounding periods, for an initial sequence from first principles
- the future value of a compound interest investment or loan after n compounding periods and its use to solve practical problems

5A 5B 5C 5D 5E **5F** 5G

KEY SKILLS

During this lesson, you will be:

- modelling compound interest using recurrence relations
- modelling compound interest using a rule.

KEY TERMS

- Compound interest

While simple interest increases a value at a fixed rate, compound interest is more adaptive and responsive to changes over time. For this reason, compound interest can be very practical, and is used for a wide range of real life situations, such as investments and loans. Compound interest is the most common form of interest and most people deal with it at certain points in their lives, making it an important concept to understand.

Modelling compound interest using recurrence relations

As opposed to simple interest, **compound interest** takes into account both the principal and any accumulated interest earned in previous compounding periods. This means that as the value of the investment or loan grows, its rate of growth increases, whilst keeping the same interest rate (%).

The value of these loans and investments, V_n, after n compounding periods, can be modelled by a geometric recurrence relation of the form

$V_0 = principal, \quad V_{n+1} = R \times V_n$, where

- $R = 1 + \frac{r}{100}$
- r is the interest rate (%) per compounding period

When interest compounds more than once a year, the annual interest rate must be converted to the interest rate per compounding period. This can be done by dividing the annual interest rate by the number of compounding periods per year. Some common compounding periods are shown in the following table.

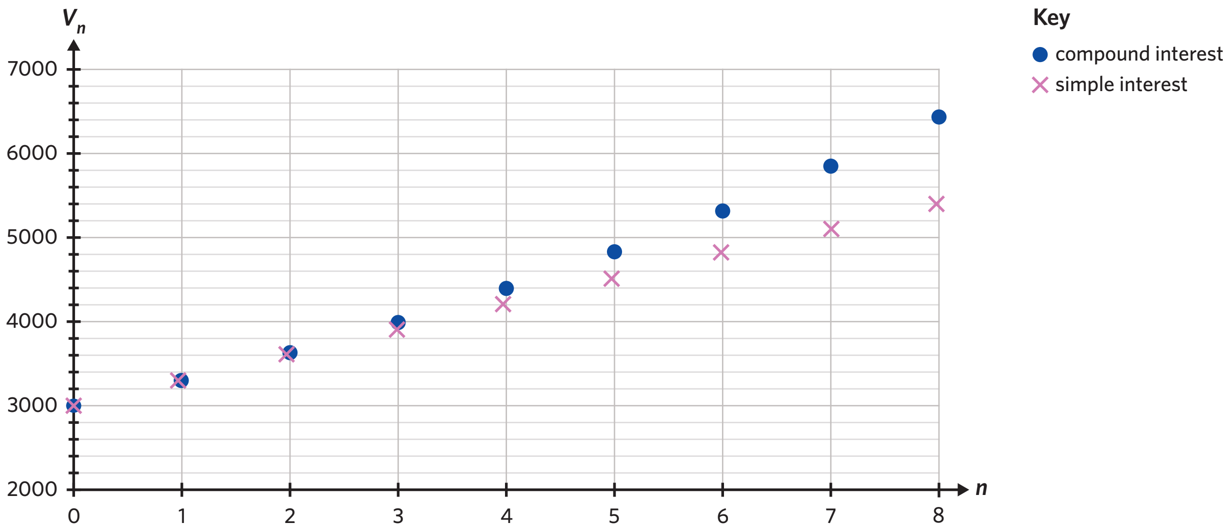
type	compounding periods per year
annual	1
biannual	2
quarterly	4
monthly	12
fortnightly	26
weekly	52

For example, if the interest rate is 6% per annum compounding monthly, the interest rate per month is given by $r = \frac{6}{12}\% = 0.5\%$.

Additionally, if the interest rate is converted to the interest rate per compounding period, the number of compounding periods, n, should be updated accordingly. For example, 5 years is equivalent to $5 \times 12 = 60$ months.

Compound interest can also be modelled graphically, by plotting each balance, V_n, for each compounding period, n. As compound interest will lead to increased growth as the balance increases, the graph will be non-linear.

The graph shown compares simple and compound interest for two investments with the same starting value of $3000 and an interest rate of 10% per annum. The compound interest investment compounds annually, whereas the simple interest investment adds the same amount every year.



Worked example 1

Duncan has an investment of $10 000 that is earning interest at a rate of 8% per annum, compounding fortnightly.

a. Construct a recurrence relation that can be used to calculate value of the investment, V_n, after n fortnights. Round to three decimal places where necessary.

Explanation

Step 1: Identify the principal and interest rate.

$V_0 = 10\,000$

As interest is compounding fortnightly, the annual interest rate must be divided by 26 to give r, the interest rate per fortnight.

$$r = \frac{8}{26}\%$$
$$= 0.307...\% \text{ per compounding period}$$

Step 2: Calculate R.

$$R = 1 + \frac{r}{100}$$
$$R = 1 + \frac{0.307...}{100}$$
$$= 1.00307...$$
$$\approx 1.003$$

Step 3: Construct the recurrence relation.

Answer

$V_0 = 10\,000, \quad V_{n+1} = 1.003 \times V_n$

b. Calculate the value of Duncan's investment after 4 fortnights.

Explanation

Step 1: Calculate V_1, the value of the investment after 1 fortnight.

$$V_1 = 1.003 \times V_0$$
$$= 1.003 \times 10\,000$$
$$= 10\,030$$

Step 2: Calculate V_2, the value of the investment after 2 fortnights.

$$V_2 = 1.003 \times V_1$$
$$= 1.003 \times 10\,030$$
$$= 10\,060.09$$

Continues →

Step 3: Calculate V_3, the value of the investment after 3 fortnights.

$$V_3 = 1.003 \times V_2$$
$$= 1.003 \times 10\,060.09$$
$$= 10\,090.270...$$

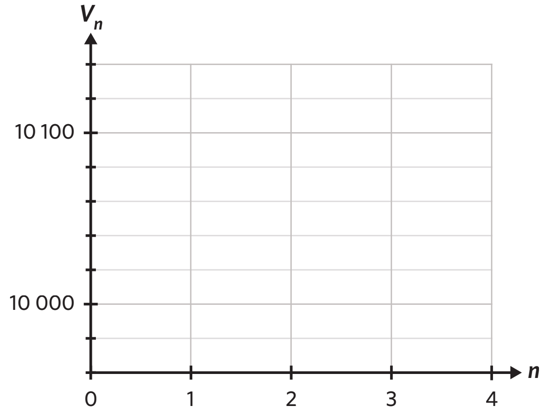
Step 4: Calculate V_4, the value of the investment after 4 fortnights.

$$V_4 = 1.003 \times V_3$$
$$= 1.003 \times 10\,090.270...$$
$$= 10\,120.541...$$

Answer

$10 120.54

c. Plot the value of the investment for the first four fortnights on the set of axes shown.



Explanation

Step 1: Determine the values over the first year.

From part **b**,

$V_0 = 10\,000$

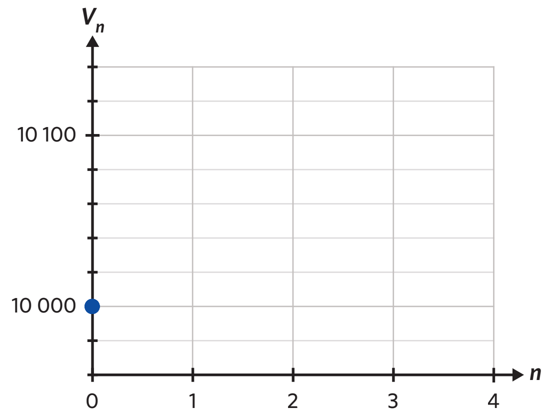
$V_1 = 10\,030$

$V_2 = 10\,060.09$

$V_3 \approx 10\,090.27$

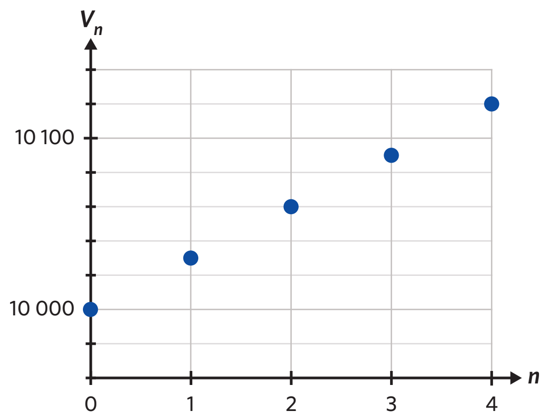
$V_4 \approx 10\,120.54$

Step 2: Plot V_0 on the graph.



Step 3: Plot the remaining values on the graph.

Answer



Modelling compound interest using a rule

A rule can be used to calculate the value of an investment or loan earning compound interest. This rule is of the form

$V_n = V_0 \times R^n$, where

- V_n is the value after n periods
- V_0 is the principal
- $R = 1 + \frac{r}{100}$
- r is the interest rate (%) per compounding period

This is useful for calculating V_n for larger values of n.

Note: This rule format is very similar to the compound interest formula, $A = P \times \left(1 + \frac{r}{100}\right)^n$. One useful aspect of recurrence relations is that the term 'V_n' can be used to keep track of the amount of time that is associated with an amount of money. For example, V_3 represents the amount of money after 3 periods, V_4 represents the amount of money after 4 periods, and so on.

Worked example 2

$20 000 is invested into an account earning compound interest at a rate of 9% per annum, compounding monthly.

a. Construct a rule that can be used to calculate the balance of the account, V_n after n months.

Explanation

Step 1: Identify the principal and interest rate.

$V_0 = 20\,000$

As interest is compounding monthly, the annual interest rate must be divided by 12 to give r, the interest rate per month.

$r = \frac{9}{12}\%$

$= 0.75\%$ per compounding period

Step 2: Calculate R.

$R = 1 + \frac{r}{100}$

$R = 1 + \frac{0.75}{100}$

$= 1.0075$

Step 3: Construct the rule.

Answer

$V_n = 20\,000 \times 1.0075^n$

b. How much interest would the account have earned after 4 years?

Explanation

Step 1: Determine the value of n by converting 4 years into months.

$n = 4 \times 12$

$= 48$

Step 2: Calculate V_{48}, the value of the account after 48 months.

$V_{48} = 20\,000 \times 1.0075^{48}$

$= 28\,628.106...$

Step 3: Calculate the interest earned after 48 months.

This is the difference between V_{48} and V_0.

$interest = 28\,628.106... - 20\,000.00$

$= 8628.106...$

Answer

$8628.11

Exam question breakdown

VCAA 2020 Exam 2 Recursion and financial modelling Q9a

Samuel opens a savings account.

Let B_n be the balance of this savings account, in dollars, n months after it was opened.

The month-to-month value of B_n can be determined using the recurrence relation shown.

$B_0 = 5000, \quad B_{n+1} = 1.003 \times B_n$

Write down the value of B_4, the balance of the savings account after four months.

Round the answer to the nearest cent. (1 MARK)

Explanation

Step 1: Calculate B_1, the balance of the account after 1 month.

$$B_1 = 1.003 \times B_0$$
$$= 1.003 \times 5000$$
$$= 5015$$

Step 2: Calculate B_2, the balance of the account after 2 months.

$$B_2 = 1.003 \times B_1$$
$$= 1.003 \times 5015$$
$$= 5030.045$$

Step 3: Calculate B_3, the balance of the account after 3 months.

$$B_3 = 1.003 \times B_2$$
$$= 1.003 \times 5030.045$$
$$= 5045.1351...$$

Step 4: Calculate B_4, the balance of the account after 4 months.

$$B_4 = 1.003 \times B_3$$
$$= 1.003 \times 5045.1351...$$
$$= 5060.2705...$$

64% of students answered this question correctly.

Some students rounded to the nearest ten cents to give an answer of \$5060.30. The question required them to round to the nearest cent instead.

Answer

\$5060.27

5F Questions

Modelling compound interest using recurrence relations

1. Sarah invests \$2400 in a bank account which pays 4% interest per annum, compounding annually. Which of the following recurrence relations could describe the growth of Sarah's account?

A. $V_0 = 0, \quad V_{n+1} = 0.04 \times V_n$

B. $V_0 = 0, \quad V_{n+1} = 1.04 \times 2400$

C. $V_0 = 2400, \quad V_{n+1} = 1.04 \times V_n$

D. $V_0 = 2400, \quad V_{n+1} = V_n + 0.04$

---

2. The amount of money in Said's bank account, V_n, after n months, is given by the following recurrence relation.

$V_0 = 3000, \quad V_{n+1} = 1.03 \times V_n$

a. What was the initial balance in Said's account?

b. What is the monthly interest rate for this account?

3. The amount of money in Nicole's bank account, V_n, after n months, is given by the following recurrence relation.

$V_0 = 1500, \quad V_{n+1} = 1.05 \times V_n$

a. What will the balance of Nicole's bank account be after four months, correct to the nearest cent?

b. When will the account reach $2000?

4. Write recurrence relations, L_n and S_n, to model the following situations.

a. Leyla deposits $2800 when opening a new savings account. The account pays 2% interest per month, compounding monthly. Use n to represent months.

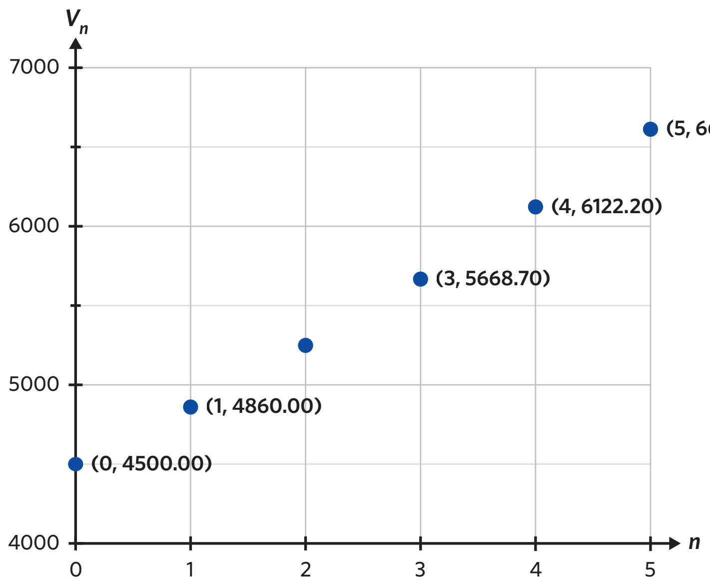
b. Shaun opens a new savings account with a deposit of $800. The account has 16% interest per annum, compounded quarterly. Use n to represent quarters.

5. Harold is saving up to build a swimming pool in his backyard. He currently has $29 000 but the pool will cost $30 000 to build. Harold decides to invest the $29 000 in a savings account that has an interest rate of 9.6% per annum, compounding monthly.

a. Write a recurrence relation to model the value of Harold's investment month to month.

b. How many months does Harold have to wait until he can afford to build the pool?

6. The graph shown displays the value, V_n, of an investment after n compounding periods. One value, V_2, is missing from the graph.



Using the graph, determine the value of V_2.

7. Elise is trying to decide whether the Super Saver account or the Save Big account is best for her. She has $1200 to invest. The recurrence relation for the Super Saver account from month to month is given.

$S_0 = 1200, \quad S_{n+1} = 1.015 \times S_n$

a. How much interest will she earn with the Super Saver account over six months? Round to the nearest cent.

b. The Save Big account has 24% interest per annum, compounding monthly, but has a $100 account start-up fee. Deduct the fee from Elise's initial deposit and write a recurrence relation for the balance of the Save Big account, B_n, after n months.

Modelling compound interest using a rule

8. The rule $V_n = 6000 \times 1.09^n$ could be used to model which of the following scenarios?
   - **A.** An investment of \$6000 with an interest rate of 0.09% p.a, compounding monthly.
   - **B.** An investment of \$6000 with an interest rate of 0.09% p.a, compounding quarterly.
   - **C.** An investment of \$6000 with an interest rate of 9% p.a, compounding monthly.
   - **D.** An investment of \$6000 with an interest rate of 9% p.a, compounding annually.

9. Find the final value of each of the following investments, compounding yearly, correct to the nearest cent.
   - **a.** \$120 invested for three years with compound interest of 5% per annum.
   - **b.** \$9 000 000 invested in a compound interest account for five years at 8.2% per annum.
   - **c.** \$230 000 invested in a compound interest account for twenty years at 2.31% per annum.

10. Andy, Bec and Charlie each invested \$3000 into compound interest savings accounts three years ago. Find the current balance of each of their savings accounts, correct to the nearest cent.
    - **a.** Andy: compound interest rate of 4.5% per annum, compounding yearly.
    - **b.** Bec: compound interest rate of 6.2% per annum, compounding quarterly.
    - **c.** Charlie: compound interest rate of 3.9% per annum, compounding weekly.

11. Find the value of the following investment accounts correct to the nearest dollar.
    - **a.** An initial deposit of \$40 000 compounding monthly at an annual interest rate of 6.4% for ten months.
    - **b.** An account with a starting balance of \$1200 compounding quarterly at an annual interest rate of 8.1% for one and a half years.
    - **c.** An investment of \$650 was made for two years at 3.3% annual interest, compounding half-yearly.

12. Eliza and Frankie win \$1000 each and invest their winnings into savings accounts with an interest rate of 8% per annum. Eliza's account compounds monthly whereas Frankie's account compounds quarterly. After five years, who will have more money in their account and how much more will they have, correct to the nearest cent?

Joining it all together

13. Which recurrence relation, corresponding to the rule $V_n = 2500 \times 1.082^n$, is written correctly?
    - **A.** $V_0 = 2500, \quad V_{n+1} = 1.082 \times V_0$
    - **B.** $V_0 = 2500, \quad V_{n+1} = 1.082 \times V_n$
    - **C.** $V_n = 2500, \quad V_{n+1} = 1.082 \times V_n$
    - **D.** $V_n = 2500, \quad V_{n+1} = 1.082 \times V_0$

14. Which of the following rules corresponds to the recurrence relation
    $S_n = 10\,000, \quad S_{n+1} = 1.002 \times S_0$?
    - **A.** $S_n = 10\,000 \times 0.2$
    - **B.** $S_n = 10\,000 \times 1.02$
    - **C.** $S_n = 10\,000 \times 1.002$
    - **D.** $S_n = 10\,000 \times 1.002^n$

15. For each of the following scenarios, model the scenario by
    - generating a recurrence relation
    - generating a rule.

    a. \$1000 invested in an account that earns interest at a rate of 1.8% p.a., compounding annually (let V_n be the account's value after n years).

    b. \$3500 invested in an account that earns interest at a rate of 2.4% p.a., compounding monthly (let V_n be the account's value after n months).

    c. \$5000 invested in an account that earns interest at a rate of 6% p.a., compounding quarterly (let V_n be the account's value after n quarters).

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16. Rachel and Naomi are arguing about which bank offers a better savings account, regarding the amount of interest it pays in 3 years, from an initial deposit of \$5000.
    - Rachel believes that account A is more profitable, as it provides 6.8% interest per annum, compounding annually.
    - Naomi believes that account B is better, because it provides 6.6% interest per annum, but compounds monthly.

    Which account is more profitable? Justify your answer using relevant calculations.

Exam practice

17. Manu invests \$3000 in an account that pays interest compounding monthly.

    The balance of his investment after n months, B_n , can be determined using the recurrence relation

    $B_0 = 3000, \quad B_{n+1} = 1.0048 \times B_n$

    The total interest earned by Manu's investment after the first five months is closest to

    A. \$57.60 **B.** \$58.02 **C.** \$72.00
    D. \$72.69 **E.** \$87.44

    VCAA 2020 Exam 1 Recursion and financial modelling Q24

80% of students answered this question correctly.

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18. Alex sends a bill to his customers after repairs are completed.

    If a customer does not pay the bill by the due date, interest is charged.

    Alex charges interest after the due date at the rate of 1.5% per month on the amount of an unpaid bill.

    The interest on this amount will compound monthly.

    a. Alex sent Marcus a bill of \$200 for repairs to his car.

       Marcus paid the full amount one month after the due date.

       How much did Marcus pay? (1 MARK)

    Alex sent Lily a bill of \$428 for repairs to her car.

    Lily did not pay the bill by the due date.

    Let A_n be the amount of this bill n months after the due date.

    b. Write down a recurrence relation, in terms of A_0 , A_{n+1} and A_n, that models the amount of the bill. (2 MARKS)

    c. Lily paid the full amount of her bill four months after the due date.

       How much interest was Lily charged?

       Round your answer to the nearest cent. (1 MARK)

    VCAA 2017 Exam 2 Recursion and financial modelling Q6

Part **a**: **61%** of students answered this question correctly.
Part **b**: The average mark on this question was **1**.
Part **c**: **26%** of students answered this question correctly.

Questions from multiple lessons

Data analysis

19. A least squares regression line

A. maximises the number of data points that lie on the line.

B. minimises the sum of the vertical distances from the data points to the line.

C. minimises the sum of the squares of the vertical distances from the data points to the line.

D. maximises the sum of the vertical distances from the data points to the line.

E. maximises the sum of the squares of the vertical distances from the data points to the line.

Adapted from VCAA 2017NH Exam 1 Data analysis Q7

Recursion and financial modelling *Year 11 content*

20. A family recently planted an apple tree. The number of apples grown each year follows a geometric sequence.

During the first year, the tree grew 3 apples.

During the second year, the tree grew 6 apples.

During the third year, the tree grew 12 apples.

How many apples are expected to grow in the sixth year?

A. 48 **B.** 96 **C.** 192 **D.** 384 **E.** 768

Adapted from VCAA 2015 Exam 1 Number patterns Q5

Recursion and financial modelling *Year 11 content*

21. A family living in the country decide to start a hobby farm.

Each year, a number of new animals are added to the farm.

In the third year, 11 new animals are added to the farm.

The number of new animals, A_n, added to the farm after n years, is modelled by the recurrence relation

$A_0 = x, \quad A_{n+1} = A_n + 3$

a. What is the value of x? (1 MARK)

b. What type of sequence does the recurrence relation generate? (1 MARK)

The number of new animals, A_n, added to the farm after n years can also be modelled by the rule $A_n = b + c \times n$.

c. Determine the values of b and c. (2 MARKS)

Adapted from VCAA 2013 Exam 2 Number patterns Q3

5G Nominal and effective interest rates

STUDY DESIGN DOT POINT

- the difference between nominal and effective interest rates and the use of effective interest rates to compare investment returns and the cost of loans when interest is paid or charged, for example, daily, monthly, quarterly

5A 5B 5C 5D 5E 5F **5G**

KEY SKILLS

During this lesson, you will be:

- calculating effective interest rates
- comparing nominal and effective interest rates and returns.

KEY TERMS

- Nominal interest rate
- Effective interest rate

The nominal and effective interest rates of a loan or investment both describe the interest on the same loan or investment. However, interest can be earned or charged at different compounding periods such as yearly or monthly. Comparing the effective interest rate of loans or investments with different compounding periods can help determine which options are the most profitable.

Calculating effective interest rates

The **nominal interest rate** is the quoted interest rate for a loan or investment, and is also known as the annual interest rate.

The **effective interest rate** is the adjusted or 'real' rate, respective to the number of compounding periods in the year. It can be used to compare the total amount of interest earned or charged on different investment or loan options with different compounding periods.

The effective interest rate, $r_{effective}$, can be calculated by hand using the formula

$$r_{effective} = \left(\left(1 + \frac{r}{100n}\right)^n - 1\right) \times 100, \text{ where}$$

- r is the nominal interest rate, per annum
- n is the number of compounding periods per year.

However, to save time and ensure accuracy, the effective interest should be calculated on either the TI-Nspire or the Casio ClassPad calculator.

Worked example 1

Convert an interest rate of 12.7% per annum, compounding monthly, to an effective interest rate, correct to two decimal places.

Explanation – Method 1: By hand

Step 1: Determine the nominal interest rate and number of compounding periods per year.

$r = 12.7\%$ per annum

$n = 12$ compounding periods in a year, since interest compounds monthly.

Step 2: Calculate the effective interest rate.

$$r_{eff} = \left(\left(1 + \frac{12.7}{100 \times 12}\right)^{12} - 1\right) \times 100$$
$$= (1.01058...^{12} - 1) \times 100$$
$$= 13.465...\%$$

Continues →

Explanation – Method 2: TI-Nspire

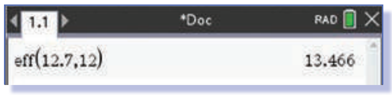
Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

Step 2: Press menu. Select '8: Finance' → '5: Interest Conversion' → '2: Effective Interest Rate'.

The function is in the format of eff(nominal interest rate, number of compounding periods per year).

Step 3: Enter the nominal interest rate, followed by the number of compounding periods per year.

The nominal interest rate is 12.7% p.a. and there are 12 compounding periods per year, so this should be '12.7, 12'. Press enter.



Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Tap 'Action' → 'Financial' → 'Interest Conversion' → 'convEff'.

The function is in the format of convEff(number of compounding periods, nominal interest rate).

Step 3: Enter the number of compounding periods per year, followed by the nominal interest rate.

There are 12 compounding periods per year and the nominal interest rate is 12.7% p.a., so this should be '12, 12.7'. Press EXE.

Answer – Method 1, 2 and 3

13.47%

Comparing nominal and effective interest rates and returns

A nominal interest rate does not give the full picture on how much interest is being calculated. For example, it is difficult to see if an interest rate of 4.5% per annum compounding monthly will be more profitable than a rate of 4.3% per annum compounding weekly.

The effective interest rate is used to compare different loan or investment options, and determine the option that is more financially beneficial.

If interest is calculated more frequently, given the same nominal interest rate, then the effective interest rate of the investment or loan will be greater. This means that the account will earn or be charged more interest over time.

For a loan, a borrower would typically prefer a lower effective interest rate so that they can pay less interest. On the other hand, for an investment, an investor would generally prefer a higher effective interest rate so that they can receive more interest.

Worked example 2

Pete is saving up money for schoolies and is considering investing his savings into either of two accounts with the following interest rates:

- Superior Saver: 4.5% per annum, compounding quarterly
- Supreme Saver: 4.4% per annum, compounding weekly

Which account should Pete choose?

Continues →

Explanation

Step 1: Calculate the effective interest rate for Superior Saver.

$r = 4.5\%$ p.a. and $n = 4$ compounding periods per year.

$r_{effective} = 4.5765...\%$

Step 2: Calculate the effective interest rate for Supreme Saver.

$r = 4.4\%$ p.a. and $n = 52$ compounding periods per year.

$r_{effective} = 4.4962...\%$

Step 3: Determine the best option.

The question involves an investment so the option with a higher effective interest rate will be the preferred choice.

Superior Saver has a greater effective interest rate.

Answer

Superior Saver

5G Questions

Calculating effective interest rates

1. For a loan with a nominal interest rate of 4.30% per annum, compounding fortnightly, the effective interest rate is closest to

   A. 4.19% **B.** 4.30% **C.** 4.39% **D.** 4.51%

2. Convert the following nominal interest rates to effective interest rates, correct to two decimal places.
   - **a.** 10% per annum, compounding monthly.
   - **b.** 8% per annum, compounding weekly.
   - **c.** 9.2% per annum, compounding daily.
   - **d.** 7.32% per annum, compounding yearly.
   - **e.** 2.01% per annum, compounding fortnightly.

3. Brigitte decides to put $550 in a savings account that has an annual interest rate of 11.75%, compounding monthly. What is the effective interest rate, correct to one decimal place?

Comparing nominal and effective interest rates and returns

4. Which of the following nominal interest rates returns the lowest effective interest rate?
   - **A.** 5.5% per annum, compounding weekly
   - **B.** 5.5% per annum, compounding fortnightly
   - **C.** 5.5% per annum, compounding monthly
   - **D.** 5.5% per annum, compounding annually

5. In each of the following parts, state which of the two interest rates is lower.
   - **a.** An effective interest rate of 8.5% or a nominal rate of 7.9% p.a., compounding monthly.
   - **b.** An effective interest rate of 11.3% or a nominal rate of 10.9% p.a., compounding quarterly.
   - **c.** An effective interest rate of 9.5% or a nominal rate of 8.7% p.a., compounding weekly.

6. Emily is looking to invest her savings into an account, in the hope that the interest she earns will support her art business. Her bank offers her two options:
   - The SuperAccount with an interest rate of 5.91% per annum, compounding weekly
   - The SmartAccount with an interest rate of 6.00% per annum, compounding yearly

   Which account should Emily choose?

7. Liam wants to buy a Hilux in time for summer. He needs to take out a loan of $30 000 to cover the costs. Liam shops around for the best loan option and narrows it to two.
   - ANB is offering a loan with an interest rate of 7.10% per annum, compounding monthly
   - Westbank is offering a loan with an interest rate 7.05% per annum, compounding weekly

   Which bank should he choose?

Joining it all together

8. If an account has an effective interest rate of 14.6% and the interest compounds monthly, then the nominal interest rate is closest to

   A. 12.9
   B. 13.5
   C. 13.7
   D. 13.9

9. Rex wins $5000 in a snooker competition and wants to put it into a savings account. VicBank is offering an interest rate of 4.12% per annum, compounding weekly, and AussieBank is offering an interest rate of 4.21% per annum, compounding quarterly. Which account would be the most profitable for Rex?

10. Jess currently has $30 000 invested in an account with an interest rate of 5.6% p.a. compounding quarterly.

    For a limited time, there is a special offer from another bank for an account with an interest rate of 5.54%, compounding daily.

    Should Jess switch banks? Explain why.

Exam practice

11. The nominal interest rate for a loan is 8% per annum.

    When rounded to two decimal places, the effective interest rate for this loan is **not**

    A. 8.33% per annum when interest is charged daily.
    B. 8.32% per annum when interest is charged weekly.
    C. 8.31% per annum when interest is charged fortnightly.
    D. 8.30% per annum when interest is charged monthly.
    E. 8.24% per annum when interest is charged quarterly.

    VCAA 2020 Exam 1 Recursion and financial modelling Q28

58% of students answered this question correctly.

12. Daniel borrows $5000, which he intends to repay fully in a lump sum after one year.

The annual interest rate and compounding period for five different compound interest loans are shown:

- Loan I – 12.6% per annum, compounding weekly
- Loan II – 12.8% per annum, compounding weekly
- Loan III – 12.9% per annum, compounding weekly
- Loan IV – 12.7% per annum, compounding quarterly
- Loan V – 13.2% per annum, compounding quarterly

When fully repaid, the loan that will cost Daniel the least amount of money is

A. Loan I. **B.** Loan II. **C.** Loan III.
D. Loan IV. **E.** Loan V.

VCAA 2018 Exam 1 Recursion and financial modelling Q19

54% of students answered this question correctly.

Questions from multiple lessons

Data analysis

13. The following table displays the long-term average *number of rainy days* each month in Melbourne.

month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
number of rainy days	8.7	6.6	9.3	10.5	12.2	13.5	14.4	15.3	14.0	13.3	11.3	10.0

The data in the table is used to calculate the seasonal indices for each month. The seasonal index for August is closest to

A. 0.68 **B.** 0.76 **C.** 1.10 **D.** 1.24 **E.** 1.32

Adapted from VCAA 2018NH Exam 1 Data analysis Q16

Recursion and financial modelling

14. A company has had an increase in profit of 4% per year for the past two years. This year, the company's profit was $123 400. The company's profit 2 years ago is closest to

A. $114 090 **B.** $116 728 **C.** $118 653 **D.** $128 336 **E.** $133 469

Adapted from VCAA 2013 Exam 1 Business-related mathematics Q6

Recursion and financial modelling

15. Aidan bought an industrial printer for a purchase price of $4000. After six years, the printer was worth $2200.

a. What was the average depreciation in the value of the printer per year? (1 MARK)

b. Let V_n be the value of the printer n years after it was purchased.

One way of calculating the depreciation of the printer's value is the flat-rate method of depreciation.

Determine the recurrence relation, in terms of V_0, V_{n+1} and V_n, that models the value of the printer under flat-rate depreciation. (1 MARK)

c. The printer has printed 2000 pages each year since it was purchased.

The value of the printer can also be depreciated using the unit cost method of depreciation.

Using this method, by how much does the value of the printer reduce for each page printed? (1 MARK)

Adapted from VCAA 2016 Exam 2 Recursion and financial modelling Q6