6

CHAPTER 6

Advanced financial mathematics

LESSONS

KEY KNOWLEDGE

- use of a first-order linear recurrence relation to model and analyse (numerically and graphically) the amortisation of a reducing balance loan, including the use of a recurrence relation to determine the value of the loan or investment after n payments for an initial sequence from first principles
- use of a table to investigate and analyse the amortisation of a reducing balance loan on a step-by-step basis, the payment made, the amount of interest paid, the reduction in the principal and the balance of the loan
- use of technology with financial modelling functionality to solve problems involving reducing balance loans, such as repaying a personal loan or a mortgage, including the impact of a change in interest rate on repayment amount, time to repay the loan, total interest paid and the total cost of the loan
- use of a first-order linear recurrence relation to model and analyse (numerically and graphically) the amortisation of an annuity, including the use of a recurrence relation to determine the value of the annuity after n payments for an initial sequence from first principles
- use of a table to investigate and analyse the amortisation of an annuity on a step-by-step basis, the payment made, the interest earned, the reduction in the principal and the balance of the annuity
- use of technology to solve problems involving annuities including determining the amount to be invested in an annuity to provide a regular income paid, for example, monthly, quarterly
- simple perpetuity as a special case of an annuity that lasts indefinitely
- use of a first-order linear recurrence relation to model and analyse (numerically and graphically) annuity investments, including the use of a recurrence relation to determine the value of the investment after n payments have been made for an initial sequence from first principles
- use of a table to investigate and analyse the growth of an annuity investment on a step-by-step basis after each payment is made, the payment made, the interest earned and the balance of the investment
- use of technology with financial modelling functionality to solve problems involving annuity investments, including determining the future value of an investment after a number of compounding periods, the number of compounding periods for the investment to exceed a given value and the interest rate or payment amount needed for an investment to exceed a given value in a given time.

6A Introducing financial applications

STUDY DESIGN DOT POINT

- prerequisite lesson

6A 6B 6C 6D 6E 6F

KEY SKILLS

During this lesson, you will be:

- using an amortisation table to model financial problems
- using financial applications of technology.

KEY TERMS

- Amortisation table
- Financial solver
- PV (present value)
- PMT (payment amount)
- FV (future value)

Calculations for financial scenarios can get complex when there is a combination of payments and interest being calculated on the balance of a loan or investment. Amortisation tables and technology can be used to break down these complex scenarios and assist greatly in completing these calculations.

Using an amortisation table to model financial problems

An **amortisation table** is a table that can be used to determine the reducing balance of a loan or investment. These tables can model a range of financial problems by showing step-by-step calculations for the resultant balances in a series of consecutive compounding periods.
A standard table will have five columns:

- 'Payment number' – the number of payments that have been made to, or received from, the loan or investment. This number will increase by one with each iteration or change in balance, even if the payment value is zero.
- 'Payment' – the amount that has been paid to or from the loan or investment.
- 'Interest' – the amount that has been added to the balance of the loan or investment due to interest. This is calculated using the interest rate, per compounding period, and previous balance.
- 'Principal reduction' – the overall change in the balance of the loan or investment after payment and interest have been taken into account for the payment period.
- 'Balance of loan/investment' – the balance of the loan or investment after the payment period, which considers the previous balance and the principal increase/reduction for the payment period.

A new row is added to the amortisation table each time the loan or investment balance is altered. A '0' payment line is also included to show the establishment of a loan or investment.

For example, the following amortisation table shows the first four lines of a $50 000 account with a fixed amount of interest earned each month. Payments of $1000 are taken out of the account each month.

Calculations have been shown to demonstrate how the principal reduction is calculated.

payment number	payment	interest	principal reduction	balance
0	0.00	0.00	0.00	50 000.00
1	1000.00	50.00	= 1000.00 − 50.00 = 950.00	= 50 000.00 − 950.00 = 49 050.00
2	1000.00	50.00	= 1000.00 − 50.00 = 950.00	= 49 050.00 − 950.00 = 48 100.00
3	1000.00	50.00	= 1000.00 − 50.00 = 950.00	= 48 100.00 − 950.00 = 47 150.00

Worked example 1

Ella has borrowed $10 000 from her parents to pay for a trip to Europe.

They have decided not to charge interest on the loan and Ella will make equal weekly repayments once she gets back from her trip.

The first five lines of the amortisation table for this loan is shown.

The loan balance for the third week is missing.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	10 000.00
1	150.00	0.00	150.00	9850.00
2	150.00	0.00	150.00	9700.00
3	150.00	0.00	150.00	
4	150.00	0.00	150.00	9400.00

a. How much is Ella paying each week?

Explanation

Read the 'payment' column.

The repayments each week are equal in this scenario.

payment number	payment	interest	principal reduction	balance of loan
1	150.00	0.00	150.00	9850.00

Answer

$150

b. Calculate the loan balance after the third repayment.

Explanation

Step 1: Determine the relevant cells in the amortisation table.

Since the loan is decreasing in value, the new balance is the difference between the previous loan balance and the principal reduction.

payment number	payment	interest	principal reduction	balance of loan
2	150.00	0.00	150.00	9700.00
3	150.00	0.00	150.00	

Step 2: Calculate the loan balance after the third payment.

balance of loan

= *previous loan balance* − *principal reduction*

= 9700.00 − 150.00

= 9550.00

Answer

$9550

Using financial applications of technology

The **financial solver** is a useful calculator program that can be used to solve complex financial problems.

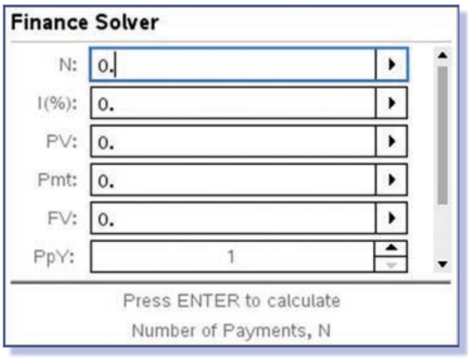
The notation of the financial solver is as follows:

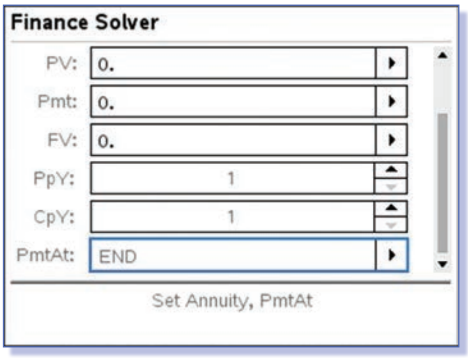
- 'N' – the number of payments made.
- 'I%' – the annual interest rate.
- '**PV**' – the present value of the loan or investment. It is also known as the current value.
- '**PMT**' – the payment amount made per compounding period.
- '**FV**' – the future value of the loan or investment, after *n* compounding periods.
- 'PpY' or 'P/Y' – how often payments are made or received per year.
- 'CpY' or 'C/Y' – how often interest is compounded per year. In General Mathematics, the CpY is usually the same as the PpY.
- 'PmtAT' (TI-Nspire only) – when the payment is made (BEGIN or END). In General Mathematics, all payments occur at the end of each period.

The financial solver can be used to determine unknown figures. For example, the financial solver could be used to calculate how long (N) it would take for an investment to reach a certain value, provided that all of the other variables are known.

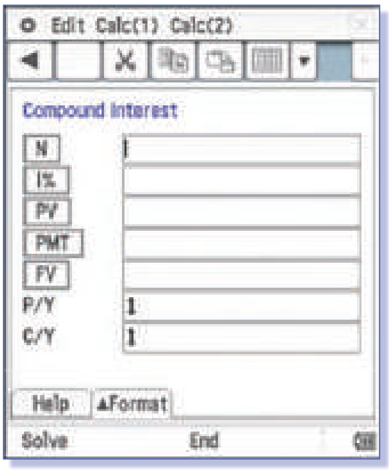
On the TI-Nspire, the Finance Solver can be used to model financial applications. It is accessed by pressing menu and selecting '8: Finance' → '1: Finance Solver'.

When first opening the Finance Solver, most values will be set at zero by default.





On the Casio ClassPad, the financial tool can be used to model financial applications. It is accessed by tapping Financial from the main menu and tapping 'Compound Interest'.



The direction of money exchange must be specified when using a financial calculator. The PV, PMT and FV are either positive or negative to specify whether money is being received or paid.

- The sign is positive if an individual receives money.
- The sign is negative if an individual pays money.

Worked example 2

Isabelle takes out an interest-free loan of $30 000. She makes repayments of $1000 per week to pay off the loan. Isabelle wants to know how much she owes the bank 10 weeks after establishing the loan.

a. Determine the inputs for the financial solver.

Explanation

N: There are 10 weeks and repayments are made weekly, so there are 10 repayments made during this timeframe.

I(%): There is no interest for this loan.

PV: Isabelle borrows $30 000. The PV is positive because she receives the money that she borrowed.

PMT: Isabelle pays back $1000 per week. The PMT is negative because she gives the money to the lender.

FV: This is the value that the financial solver will calculate, so it needs to be left blank.

PpY: The repayments are made weekly, which is equivalent to 52 times in a year.

CpY: Even though there is no interest, this can also be set to 52 to be consistent with the PpY.

Answer

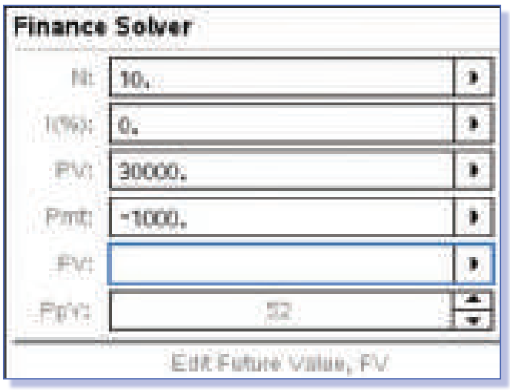
N	10
I(%)	0
PV	30 000
PMT	−1000
FV	
PpY	52
CpY	52

b. Use the financial solver to determine the balance of the loan after 10 weeks. Round to the nearest cent.

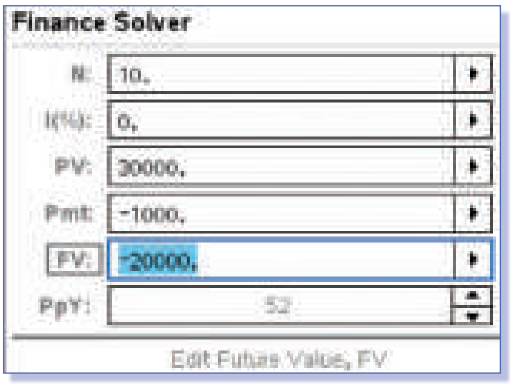
Explanation – Method 1: TI-Nspire

Step 1: Open the Finance Solver application and input the values from the table in part **a**.

Make sure the FV is blank.



Step 2: Move the cursor to the FV input box and press enter.

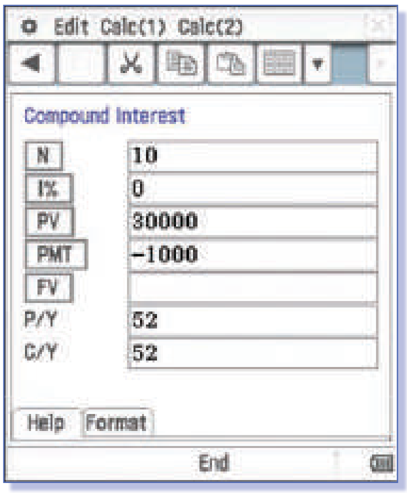


The FV is negative because this represents how much she still needs to pay.

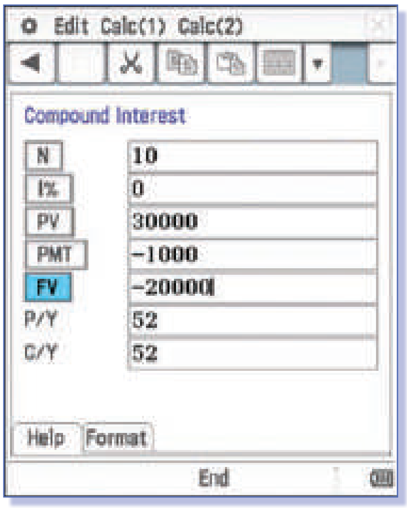
Continues →

Explanation – Method 2: Casio ClassPad

Step 1: From the main menu, tap Financial and input the values from the table in part **a**. Make sure the FV is blank.



Step 2: Tap 'FV'.



The FV is negative because this represents how much she still needs to pay.

Answer – Method 1 and 2

\$20 000

6A Questions

Using an amortisation table to model financial problems

1. Stephanie takes out a loan of \$50 000 with her bank who doesn't charge interest for the first three months. She makes monthly repayments of \$1200 towards the loan.

She has set up an amortisation table to model the amortisation of her loan. There are a few cells missing.

payment number	payment	interest	principal reduction	balance of loan
0	1200.00	0.00	0.00	50 000.00
1	1200.00	0.00	1200.00	48 800.00
2	1200.00	0.00	1200.00	47 600.00
3	1200.00			

a. Which of the following expressions can be used to calculate the principal reduction in the third month?

A. $principal\ reduction = 1200 - 0$

B. $principal\ reduction = 50\ 000 - 1200$

C. $principal\ reduction = 50\ 000 + 48\ 800$

D. $principal\ reduction = 48\ 800 - 1200$

b. What is the balance of Stephanie's loan after 3 months?

A. \$40 000

B. \$46 400

C. \$48 800

D. \$50 000

2. Hannes has put $500 into a safe. Each week, he will take out $50 to pay for petrol.

The amortisation table shows the balance of the money in the safe.

payment number	payment	interest	principal reduction	balance
0	0.00	0.00	0.00	500.00
1	50.00	0.00	50.00	450.00
2	50.00	0.00	50.00	400.00
3	50.00	0.00	50.00	350.00

Explain why the balance is decreasing in value each week. Include a calculation in your explanation.

3. Florence has borrowed $500 from his grandmother, Florina.

Florina will charge interest at a flat rate of $15 per week while Florence will make weekly payments of $50.

Florence's weekly payments will go towards paying off the interest before reducing the principal.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	500.00
1	50.00	15.00		
2	50.00	15.00		
3	50.00	15.00		
4	50.00	15.00		

a. Show that the principal reduction in the first week is $35.

b. Show that the loan balance is $465 after the first week.

c. Fill out the remaining missing cells in the table.

Using financial applications of technology

4. Alo has established a loan of $15 000 to pay for her car.

She will be charged interest at a rate of 15% per annum, compounding twice a month, with repayments of $100 every half-month.

If she wants to know how much she still owes after one year, the value of N will be

A. 1 **B.** 12 **C.** 24 **D.** 26

5. In 2003, Zo opened up a savings account and put $215 into it.

Her bank offered an interest rate of 1.5% per annum, compounding monthly. However, she has since forgotten about the account.

In 2023, she discovers a document that contains details about this account and finally remembers it. She wants to know how much it is worth and uses a financial solver to help her.

Identify the value of the following:

a. I(%) b. PV c. PMT

6. Hubert wants to start investing his earnings. He has put $4000 in a savings account.

The interest rate for Hubert's investment is 2% per annum, compounding daily.

Identify the value of the following:

a. I(%) b. PV c. CpY

Joining it all together

7. The following amortisation table models the balance of a loan worth $500.

The interest rate is 5% per annum, compounding monthly.

The repayments are also made monthly and go towards paying off the interest before reducing the principal.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	500.00
1	15.00	2.08	12.92	487.08
2	15.00	2.03	12.97	474.11
3	15.00	1.98	13.02	461.09
4	15.00	1.92		

a. Fill in the missing cells, giving values to the nearest cent.

b. If this was modelled using the financial solver,

i. explain why the PV value would be positive.

ii. explain why the PMT value would be negative.

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8. The following amortisation table models the balance of an investment worth $500.

The interest rate is 10% per annum, compounding quarterly.

Payments of $50 are also received quarterly.

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	500.00
1	50.00	12.50	37.50	462.50
2	50.00	11.56	38.44	424.06
3	50.00	10.60	39.40	384.66
4	50.00	9.62		

a. Calculate the balance of the investment after payment number 4, to the nearest cent.

b. If this was modelled using the financial solver,

i. would the PV value be positive or negative? Explain why.

ii. would the PMT value be positive or negative? Explain why.

Questions from multiple lessons

Recursion and financial modelling *Year 11 content*

9. The first four terms of a sequence are:

3, 9, 33, 129

Which of the following recurrence relations could generate this sequence?

A. $V_0 = 3, \quad V_{n+1} = 3V_n$

B. $V_0 = 3, \quad V_{n+1} = 2V_n + 3$

C. $V_0 = 3, \quad V_{n+1} = 4V_n - 3$

D. $V_0 = 3, \quad V_{n+1} = V_n + 6$

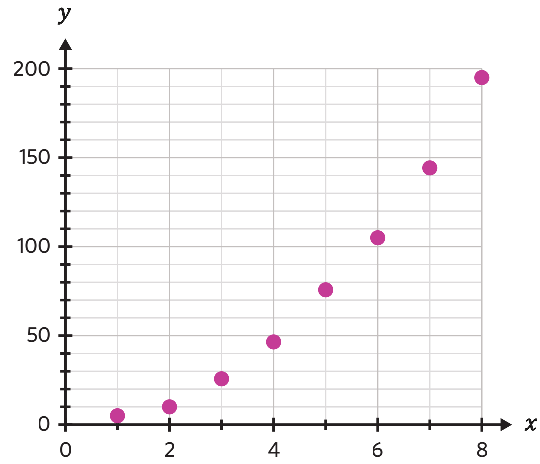
E. $V_0 = 3, \quad V_{n+1} = 5V_n - 6$

Adapted from VCAA 2017 Exam 1 Recursion and financial modelling Q18

Data analysis

10. The data in the table provided generates the following scatterplot.

x	y
1	3
2	11
3	28
4	48
5	77
6	105
7	143
8	195



A squared transformation is applied to the variable x to linearise the data. What is the equation of the least squares regression line for the transformed data? Round all values to two decimal places.

A. $y = -0.21 + 3.00x$

B. $y = -45.04 + 26.95x$

C. $y = -0.21 + 3.00x^2$

D. $y = -45.04 + 26.95x$

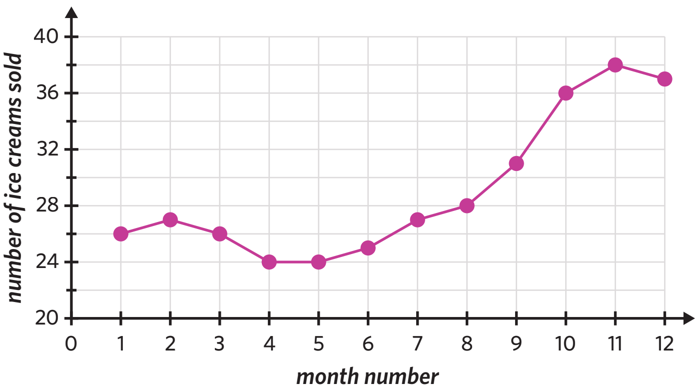
E. $y = 0.10 + 0.33x^2$

Adapted from VCAA 2018 Exam 1 Data analysis Q11

Data analysis

11. The average *number of ice creams sold* per day is recorded over 12 months.

The data is plotted for each month from January (month 1) to December (month 12) in the following time series plot.



a. State the overall trend in the time series plot. (1 MARK)

b. The trend in the data can be modelled by a least squares regression line.
The data used to obtain this line is shown.

month number	1	2	3	4	5	6	7	8	9	10	11	12
number of ice creams sold	26	27	26	24	24	25	27	28	31	36	38	37

Calculate the equation of the regression line, using *month number* as the explanatory variable.
Give values correct to four significant figures. (3 MARKS)

Adapted from VCAA 2017NH Exam 2 Data analysis Q3a, b

6B Reducing balance loans

STUDY DESIGN DOT POINTS

- use of a first-order linear recurrence relation to model and analyse (numerically and graphically) the amortisation of a reducing balance loan, including the use of a recurrence relation to determine the value of the loan or investment after n payments for an initial sequence from first principles
- use of a table to investigate and analyse the amortisation of a reducing balance loan on a step-by-step basis, the payment made, the amount of interest paid, the reduction in the principal and the balance of the loan
- use of technology with financial modelling functionality to solve problems involving reducing balance loans, such as repaying a personal loan or a mortgage, including the impact of a change in interest rate on repayment amount, time to repay the loan, total interest paid and the total cost of the loan

6A 6B 6C 6D 6E 6F

KEY SKILLS

During this lesson, you will be:

- using recurrence relations to model reducing balance loans
- using amortisation tables to solve problems involving reducing balance loans
- using financial applications of technology to solve problems involving reducing balance loans.

KEY TERMS

- Reducing balance loan

Reducing balance loans can be modelled using recurrence relations and amortisation tables. These show the amortisation of a loan from one compounding period to another. In addition to this, a financial solver can be used to solve problems over longer periods of time. There are many examples of reducing balance loans in the real world, such as mortgages and bank loans, where interest is charged and regular repayments are made to reduce the amount that is owed.

Using recurrence relations to model reducing balance loans

A **reducing balance loan** is a compound interest loan with repayments made at regular intervals. The repayments need to be greater than the amount of interest charged at each compounding period. This reduces the loan balance until it reaches zero, and the borrower no longer owes the lender.

The value of these loans, V_n, after n compounding periods, can be modelled by a recurrence relation of the form

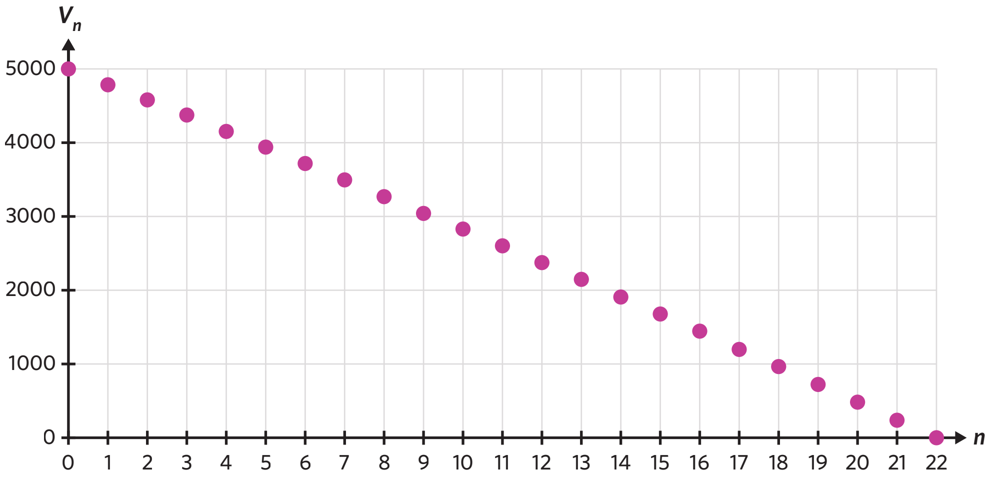
$V_0 = principal, \quad V_{n+1} = R \times V_n - d$, where

- $R = 1 + \frac{r}{100}$, where r is the interest rate (%) per compounding period
- d is the payment made per compounding period

For example, Rupert takes out a \$5000 loan with an interest rate of 10% per annum, compounding monthly and makes repayments of \$250 every month. A recurrence relation that Rupert can use to determine V_n, the value of the loan after n months is

$V_0 = 5000, \quad V_{n+1} = 1.00833 \times V_n - 250.$

The balance of the loan can also be represented on a graph.



There is a slight curve in the graph. This is because, at the beginning of the loan, a smaller portion of the repayment amount goes towards repaying the principal as the amount of interest owed is larger.

Worked example 1

Dawn has borrowed $5000 from Barry, who charges interest at a rate of 5% per annum, compounding weekly. Dawn will make weekly $500 repayments.

a. Construct a recurrence relation that can be used to model V_n, the value of the loan after n weeks. Round R to 5 decimal places.

Explanation

Step 1: Determine the principal, interest rate and payment amount.

$$V_0 = 5000$$

$$r = \frac{5}{52}\% \text{ per compounding period}$$

$$d = 500$$

Step 2: Calculate R.

$$R = 1 + \frac{5}{52 \times 100}$$

$$= 1 + \frac{5}{5200}$$

$$= 1.000961...$$

Step 3: Construct the recurrence relation.

Answer

$V_0 = 5000, \quad V_{n+1} = 1.00096 \times V_n - 500$

b. Calculate the amount that Dawn owes Barry after 3 weeks.

Explanation

Step 1: Calculate V_1, the value of the loan after 1 week.

$$V_1 = 1.00096 \times V_0 - 500$$

$$= 1.00096 \times (5000) - 500$$

$$= 4504.80$$

Step 2: Calculate V_2, the value of the loan after 2 weeks.

$$V_2 = 1.00096 \times V_1 - 500$$

$$= 1.00096 \times (4504.80) - 500$$

$$= 4009.124...$$

Step 3: Calculate V_3, the value of the loan after 3 weeks.

$$V_3 = 1.00096 \times V_2 - 500$$

$$= 1.00096 \times (4009.124...) - 500$$

$$= 3512.973...$$

Answer

$3512.97

Using amortisation tables to solve problems involving reducing balance loans

Amortisation tables are used to keep track of the repayments made and the remaining balance in reducing balance loans.

In reducing balance loans, a portion of each payment repays the full amount of interest owed and the remainder is used to pay off the principal. In turn, this reduces the interest owed for the next compounding period and increases the portion of the payment that reduces the balance of the loan.

For example, the amortisation table shown models a \$3500 reducing balance loan with an interest rate of 9% per annum, compounding monthly, and monthly repayments of \$600.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	3500.00
1	600.00	26.25	573.75	2926.25
2	600.00	21.95	578.05	2348.20
3	600.00	17.61	582.39	1765.81
4	600.00	13.24	586.76	1179.05
5	600.00	8.84	591.16	587.89
6	592.30	4.41	587.89	0.00

Note: Often the final payment is adjusted so that the balance can be fully paid off. The final payment amount does not equal the previous loan balance because there is additional interest that needs to be repaid.

The columns of an amortisation table for a reducing balance loan can be calculated using the following formulas:

$$\textit{interest} = \frac{r}{100} \times \textit{previous loan balance}$$

$$\textit{principal reduction} = \textit{payment} - \textit{interest}$$

$$\textit{balance of loan} = \textit{previous loan balance} - \textit{principal reduction}$$

Worked example 2

The first four lines of an amortisation table for a reducing balance loan of \$500 000 are shown.

Interest is charged at a rate of 6.2% per annum, compounding quarterly.

The loan is to be repaid with quarterly payments.

There are a number of cells missing.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	500 000.00
1	10 000.00	7750.00	2250.00	497 750.00
2	**a.**	7715.13	2284.87	495 465.13
3	10 000.00	**b.**	**c.**	**d.**

a. Calculate the missing payment value for payment number 2.

Explanation

Step 1: Determine the interest and principal reduction for payment number 2.

$\textit{interest} = 7715.13$

$\textit{principal reduction} = 2284.87$

Step 2: Calculate the payment.

$\textit{principal reduction} = \textit{payment} - \textit{interest}$

$$\begin{aligned}\textit{payment} &= \textit{principal reduction} + \textit{interest}\\ &= 7715.13 + 2284.87\\ &= 10\ 000.00\end{aligned}$$

Answer

\$10 000.00

Continues →

b. Calculate the missing interest for payment number 3.

Explanation

Step 1: Identify the interest rate and previous loan balance.

$r = \frac{6.2}{4}\%$ per compounding period

$\textit{previous loan balance} = 495\,465.13$

Step 2: Calculate the interest.

$$\begin{aligned} \textit{interest} &= \frac{r}{100} \times \textit{previous loan balance} \\ &= \frac{6.2}{400} \times 495\,465.13 \\ &\approx 7679.71 \end{aligned}$$

Answer

$7679.71

c. Calculate the missing principal reduction for payment number 3.

Explanation

Step 1: Identify the payment and interest.

$\textit{payment} = 10\,000.00$

$\textit{interest} = 7679.71$ (from part **b**)

Step 2: Calculate the principal reduction.

$$\begin{aligned} \textit{principal reduction} &= \textit{payment} - \textit{interest} \\ &= 10\,000.00 - 7679.71 \\ &= 2320.29 \end{aligned}$$

Answer

$2320.29

d. Calculate the missing loan balance after payment number 3.

Explanation

Step 1: Identify the previous loan balance and principal reduction.

$\textit{previous loan balance} = 490\,465.13$

$\textit{principal reduction} = 2320.29$ (from part **c**)

Step 2: Calculate the loan balance.

$$\begin{aligned} \textit{balance of loan} &= \textit{previous loan balance} \\ &\quad - \textit{principal reduction} \\ &= 495\,465.13 - 2320.29 \\ &= 493\,144.84 \end{aligned}$$

Answer

$493 144.84

Using financial applications of technology to solve problems involving reducing balance loans

The financial solver is a useful tool that can be used to solve problems involving reducing balance loans.

The present value, PV, is always positive since the borrower receives this amount from the lender.

The payment, PMT, is always negative since the borrower makes a payment to the lender.

The future value, FV, is always negative as it represents money that is owed, or yet to be paid.

A future value of 0 indicates that the loan has been fully repaid.

Worked example 3

Cilian takes out a reducing balance loan of \$35 000 with a bank that charges an interest rate of 5.9% per annum, compounding monthly. He makes a payment of \$1000 each month to pay off the loan.

a. How much does Cilian still owe after 21 months?

Explanation

Step 1: Determine the financial solver inputs.

N	21	(there are 21 months)
I(%)	5.9	(annual interest rate)
PV	35 000	(this is positive because Cilian receives it from the lender)
PMT	−1000	(this is negative because Cilian pays the lender)
FV		
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

Step 2: Use the financial solver to solve for FV.

FV	−16 731.708...

The negative FV indicates that Cilian still owes the bank.

Answer

\$16 731.71

b. The bank changes the interest rate for Cilian's loan after 21 months to 6.4% per annum compounding monthly. Cilian's monthly payments remain unchanged.

To ensure the loan is fully repaid, the final repayment will be lower.

If Cilian manages to pay off the rest of the loan, how much interest will he pay in total?

Use the rounded value found in part **a**.

Explanation

Step 1: Calculate the number of compounding periods it will take Cilian to pay off the loan.

Cilian still owes \$16 731.71 when the interest rate changes.

This is now the present value of the loan.

N		
I(%)	6.4	(annual interest rate)
PV	16 731.71	(this is positive since it's the amount owing)
PMT	−1000	(this is negative since Cilian pays the lender)
FV	0	(the loan is to be fully repaid)
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

N	17.572...

This means that after the first 21 months, there will be 17 full payments of \$1000 and 1 final payment less than \$1000.

Continues →

Step 2: Calculate the value of the final payment.

N	17	
I(%)	6.4	(annual interest rate)
PV	16 731.71	(this is positive since it's the amount owing)
PMT	−1000	(this is negative since Cilian pays the lender)
FV		
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

FV	−570.161...

This means that after 17 payments since the interest rate change, Cilian still owes $570.16.

This is not equal to the final payment amount because additional interest still needs to be paid.

N	1	(one final payment)
I(%)	6.4	(annual interest rate)
PV	570.161...	(this is positive since it's the amount owing)
PMT		
FV	0	(the loan is to be fully repaid)
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

PMT	−573.201...

The final payment is $573.20.

Step 3: Calculate the total amount of money that Cilian pays the lender.

The 21 payments before the interest rate was changed need to be included as well.

$$\begin{aligned} \textit{total amount paid} &= 21 \times 1000 + 17 \times 1000 + 573.20 \\ &= 38\,573.20 \end{aligned}$$

Step 4: Calculate the total interest that Cilian pays the lender.

The interest is the difference between the total amount paid and the amount that is borrowed.

$$\begin{aligned} \textit{interest} &= \textit{total amount paid} - \textit{total amount borrowed} \\ &= 38\,573.20 - 35\,000 \\ &= 3573.20 \end{aligned}$$

Answer

$3573.20

Exam question breakdown

VCAA 2019 Exam 2 Recursion and financial modelling Q9a

Phil would like to purchase a block of land.

He will borrow \$350 000 to make this purchase.

Interest on this loan will be charged at the rate of 4.9% per annum, compounding fortnightly.

After three years of equal fortnightly repayments, the balance of Phil's loan will be \$262 332.33.

What is the value of each fortnightly repayment Phil will make?

Round to the nearest cent. (1 MARK)

Explanation

Step 1: Determine the financial solver inputs.

N	78	(there are 78 fortnights in 3 years)
I(%)	4.9	(annual interest rate)
PV	350 000	(this is positive because Phil receives it from the lender)
PMT		
FV	−262 332.33	(this is negative because Phil still owes the lender)
PpY	26	(payments made fortnightly)
CpY	26	(interest compounds fortnightly)

Step 2: Use the financial solver to solve for PMT.

PMT	−1704.0300...

The PMT is negative because Phil pays the lender.

27% of students answered this question correctly.

A significant number of students incorrectly entered a positive FV value into the financial solver. The future value of a loan needs to be negative as it represents the money that is owed, or yet to be paid. A few students incorrectly rounded to \$1704.05 or \$1704.

Answer

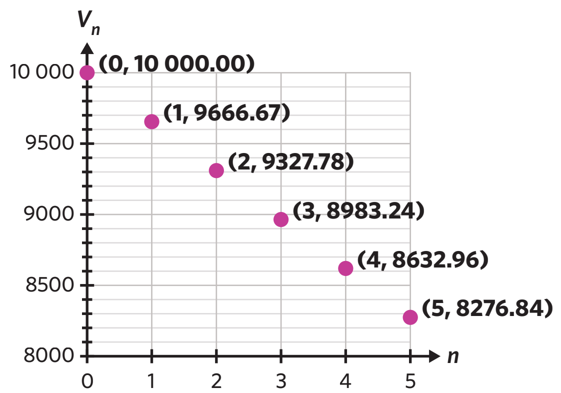
\$1704.03

6B Questions

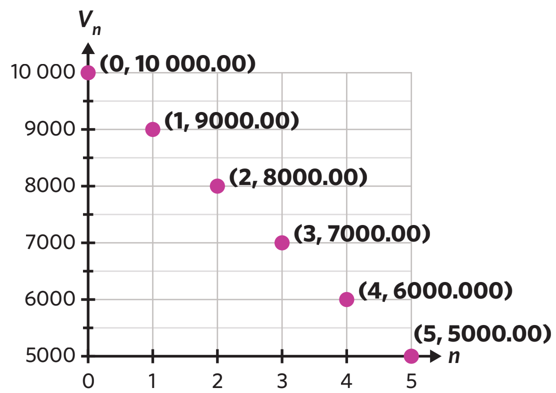
Using recurrence relations to model reducing balance loans

1. Which of the following graphs is most likely to represent the value of a reducing balance loan over 5 years?

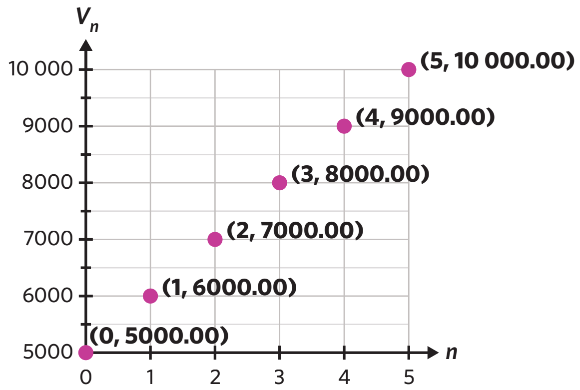
A.



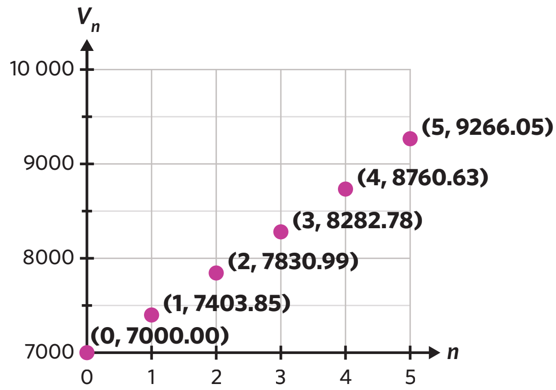
B.



C.



D.



2. Skyla took out a loan that can be modelled by the following recurrence relation, where V_n represents the value of the loan after n months.

$V_0 = 10\,000, \quad V_{n+1} = 1.004 \times V_n - 600$

a. How much did Skyla borrow?

b. How much does Skyla pay the lender each month?

c. What is the annual interest rate of Skyla's loan?

3. Construct a recurrence relation for each scenario, where V_n represents the value of the reducing balance loan after n compounding periods.

a. Brycen takes out a loan of \$8500 with an annual interest rate of 4.68% per annum, compounding fortnightly. A payment of \$250 is made every fortnight to pay off the loan.

b. Roxie takes out a mortgage of \$985 000 with an annual interest rate of 3.12% per annum, compounding weekly, and repayments of \$1200 each week.

4. Grant takes out a loan to start his own food truck business.

The loan can be modelled by the recurrence relation

$V_0 = 28\,000, \quad V_{n+1} = 1.0056 \times V_n - 2100$

where n represents the number of months after the loan is established.

a. What is the annual interest rate of Grant's loan? Round to two decimal places.

b. How much will Grant owe after three months? Round to the nearest cent.

c. After how many months will Grant owe less than \$20 000?

5. Candice took out a loan of \$15 000 to buy her car. Interest is charged at 39% per annum, compounding fortnightly.

Candice has been making repayments of \$1800 each fortnight. The current balance of the loan is \$10 203.77. How many fortnights has it been since Candice established the loan?

Using amortisation tables to solve problems involving reducing balance loans

6. The first four lines of an amortisation table for a reducing balance loan are shown. Interest is compounded on a fortnightly basis.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	94 000.00
1	5600.00	455.54	5144.46	88 855.54
2	5600.00	430.61	5169.39	83 686.15
3	5600.00	405.56	5194.44	78 491.71

a. The amount that was borrowed is

A. $0.00 B. $455.54 C. $5600.00 D. $94 000.00

b. The principal reduction after payment number 2 is

A. $0.00 B. $5144.46 C. $5169.39 D. $5194.44

c. The interest rate, per annum, is closest to

A. 0.48% B. 1.26% C. 4.80% D. 12.60%

7. A loan of $18 000 has an interest rate of 6% per annum, compounding monthly.

Repayments of $1627 are made each month.

Use this information to complete the amortisation table.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	18 000.00
1				

8. The following amortisation table models a reducing balance loan. Fill in the empty cells.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	1200.00
1	242.42	4.02	238.40	961.60
2	242.42		239.20	722.40
3	242.42	2.42	240.00	
4	242.42	1.62		241.60
5		0.81	241.60	0.00

9. Consider the following amortisation table for a reducing balance loan. Interest is compounded quarterly.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	50 000.00
1	1450.00	300.00	1150.00	48 850.00
2	1450.00	293.10	1156.90	47 693.10
3	1450.00	286.16	1163.84	46 529.26
4	1450.00	279.18	1170.82	45 358.44

Calculate the interest rate, per annum, of the loan.

10. Joan has agreed to a loan of $3000 with her bank. The loan has an interest rate of 4% per annum, compounding monthly. Joan makes monthly payments of $500 towards the loan.

Fill out the amortisation table for the first three payments.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	3000.00
1				
2				
3				

Using financial applications of technology to solve problems involving reducing balance loans

11. Jasmine takes out a loan of $560 000 to pay for her mortgage. She makes monthly repayments of $3006.20 and will pay off the loan after 30 years.

The annual interest rate is closest to

A. 4% p.a. **B.** 5% p.a. **C.** 10% p.a. **D.** 15% p.a.

12. Use a financial solver for each of the following.

a. Clay takes out a $20 000 loan from the bank to pay for a holiday trip to Abu Dhabi. The interest rate is 10.2% per annum, compounding quarterly. Clay makes quarterly repayments of $1000.

How long will it take for Clay to owe $16 866.08? Round to the nearest quarter.

b. Blaine borrows $42 000 from a loan shark to pay for a car. The loan shark charges interest and forces Blaine to make weekly payments of $2000 a week. 12 weeks after establishing the loan, Blaine still owes $21 003.35.

What is the annual interest rate that is being charged? Round to the nearest percentage.

c. Sabrina needs to borrow some money to pay for a house. Her bank offers an interest rate of 6.4% per annum, compounding biannually. The loan is to be paid off after 5 years of biannual payments of $4974.07.

How much does Sabrina borrow? Round to the nearest thousand dollars.

d. Rinto took out a loan of $25 000 to pay for a car. Interest is calculated at a rate of 12.4% per annum, compounding fortnightly. He makes repayments on a fortnightly basis and owes $21 979.13 after four years.

How much is Rinto paying each fortnight? Round to the nearest cent.

13. Drake wants to purchase the latest phone for $1299 but doesn't have enough money on hand. He places a 10% deposit and borrows the remainder of the money from a tech store that charges an interest rate of 1% p.a. compounding weekly.

Drake is required to pay the tech store $50 on a weekly basis.

If Drake now owes the tech store $920.13, how long has it been since he purchased the phone?

14. Miette borrows $19 000 with an interest rate of 7.8% per annum, compounding monthly. She makes monthly repayments of $240.

a. How much does she owe after two years? Round to the nearest cent.

The bank decides to increase the interest rate for Miette's loan after two years to 9.5% per annum. Miette's monthly repayments remain unchanged.

To ensure the loan is fully repaid, the final repayment will be a larger amount.

b. After the two years, how many months will it take for Miette to fully repay the loan? Use the rounded value found in part **a**.

c. How much does Miette pay in total?

d. How much interest does Miette pay in total?

Joining it all together

15. An amortisation table for a reducing balance loan is shown. Interest compounds monthly.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	5000.00
1	1200.00	30.00	1170.00	3830.00
2	1200.00	22.98	1177.02	2652.98
3	1200.00	15.92	1184.08	1468.90

a. Calculate the annual interest rate.

b. Construct a recurrence relation that can be used to model the value of this loan.

c. If the loan is to be fully repaid in 5 months, calculate the value of the final repayment.

---

16. The value of a reducing balance loan, V_n, after n quarters, can be modelled by the recurrence relation, as shown.

$V_0 = 24\,745, \quad V_{n+1} = R \times V_n - 3816$

The loan is to be fully repaid in 7 quarters. Calculate the value of R, rounded to four decimal places.

---

17. Frank wishes to buy a high-end gaming chair. He borrows \$8000 at 10.98% per annum, compounding weekly. Frank's weekly repayments are chosen so he pays off the chair in one year.

a. How much is Frank paying each week, correct to the nearest cent?

After 10 weeks, Frank decides to increase his repayments by \$50 per week.

b. To ensure the loan is fully repaid, the final repayment will be lower.

In total, how long will it take Frank to pay off the loan?

c. What is the total amount of interest that Frank will pay in the first 16 weeks?

Use the rounded value found in part a.

Exam practice

18. The value of a reducing balance loan, in dollars, after n months, V_n, can be modelled by the recurrence relation shown.

$V_0 = 26\,000, \quad V_{n+1} = 1.003 \times V_n - 400$

What is the value of this loan after five months?

A. \$24 380.31 **B.** \$24 706.19 **C.** \$25 031.10

D. \$25 355.03 **E.** \$25 678.00

VCAA 2017 Exam 1 Recursion and financial modelling Q17

82% of students answered this question correctly.

---

19. The first three lines of an amortisation table for a reducing balance home loan are shown.

The interest rate for this home loan is 4.8% per annum compounding monthly.

The loan is to be repaid with monthly payments of \$1500.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	250 000.00
1	1500.00	1000.00	500.00	249 500.00
2	1500.00			

The amount of payment number 2 that goes towards reducing the principal of the loan is

A. \$486 **B.** \$502 **C.** \$504

D. \$996 **E.** \$998

VCAA 2016 Exam 1 Recursion and financial modelling Q22

54% of students answered this question correctly.

---

20. Bimal has a reducing balance loan.

The balance, in dollars, of the loan from month to month, B_n, is modelled by the recurrence relation shown.

$B_0 = 450\,000, \quad B_{n+1} = R \times B_n - 2633$

Given that the loan will be fully repaid in 20 years, the value of R is closest to

A. 1.003 **B.** 1.0036 **C.** 1.03

D. 1.036 **E.** 1.36

VCAA 2021 Exam 1 Recursion and financial modelling Q23

31% of students answered this question correctly.

---

21. For renovations to her coffee shop, Sienne took out a reducing balance loan of \$570 000, with interest calculated fortnightly.

The balance of the loan, in dollars, after n fortnights, S_n, can be modelled by the recurrence relation

$S_0 = 570\,000, \quad S_{n+1} = 1.001 \times S_n - 1193$

Show that the compound interest rate for this loan is 2.6% per annum. (1 MARK)

VCAA 2021 Exam 2 Recursion and financial modelling Q8b

27% of students answered this question correctly.

---

22. Ken has borrowed \$70 000 to buy a new caravan.

He will be charged interest at a rate of 6.9% per annum, compounding monthly.

Ken will make monthly repayments of \$800.

After three years, Ken will make a lump sum payment of \$$L$ in order to reduce the balance of his loan.

This lump sum payment will ensure that Ken's loan is fully repaid in a further three years.

Ken's repayment amount and interest remains the same.

What is the value of Ken's lump sum payment, \$$L$? Round to the nearest dollar. (2 MARKS)

VCAA 2016 Exam 2 Recursion and financial modelling Q7b

The average mark on this question was **0.2**.

Questions from multiple lessons

Data analysis

23. Dayne works at a restaurant each day of the week.

The table shows the daily seasonal indices for the amount of money Dayne earns each day.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
seasonal index	0.68	0.72	0.73	0.93	1.06	1.38	

The deseasonalised amount of money Dayne earned last Sunday was \$185.

How much money did Dayne actually make last Sunday?

A. \$123.33 **B.** \$185.00 **C.** \$255.30 **D.** \$277.50 **E.** \$134.06

Adapted from VCAA 2018NH Exam 1 Data analysis Q14

Recursion and financial modelling

24. A plasma screen TV was purchased for \$3500.

The TV depreciates 12% each year using the reducing balance method.

A recurrence relation that can be used to determine the value of the TV after n years, T_n, is

A. $T_n = 3500, \quad T_{n+1} = 1.12T_n$

B. $T_n = 3500, \quad T_{n+1} = 0.88T_n$

C. $T_n = 3500, \quad T_{n+1} = 0.88T_n - 12$

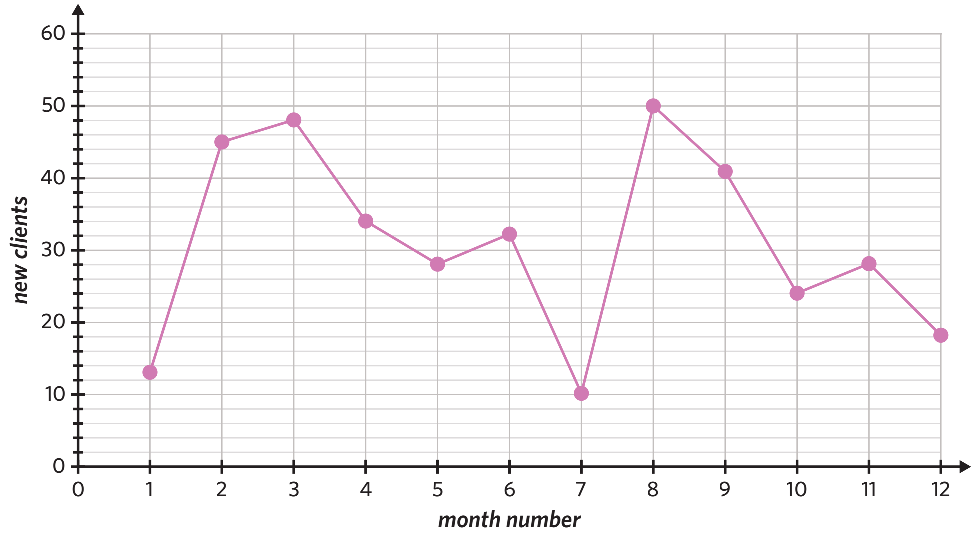
D. $T_n = 3500, \quad T_{n+1} = T_n + 12$

E. $T_n = 3500, \quad T_{n+1} = T_n - 12$

Adapted from VCAA 2016 Exam 1 Recursion and financial modelling Q19

Data analysis

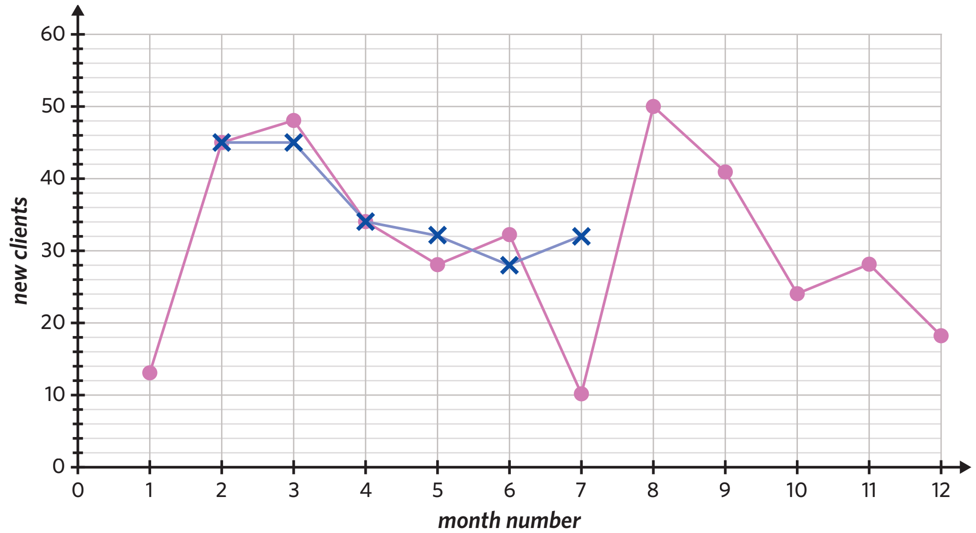
25. The following time series plot shows the total number of *new clients* a consultancy agency acquires each month for 2018.



a. Find the range of *new clients*? (1 MARK)

b. Determine the five-median smoothed number of *new clients* for month 9. (1 MARK)

Three-median smoothing has been used to smooth the time series plot up to month 7, as shown.



c. Complete the three-median smoothing by marking each remaining smoothed point with a cross (×). (2 MARKS)

Adapted from VCAA 2017NH Exam 2 Data analysis Q4

6C Interest-only loans

STUDY DESIGN DOT POINTS

- use of a first-order linear recurrence relation to model and analyse (numerically and graphically) the amortisation of a reducing balance loan, including the use of a recurrence relation to determine the value of the loan or investment after *n* payments for an initial sequence from first principles
- use of a table to investigate and analyse the amortisation of a reducing balance loan on a step-by-step basis, the payment made, the amount of interest paid, the reduction in the principal and the balance of the loan
- use of technology with financial modelling functionality to solve problems involving reducing balance loans, such as repaying a personal loan or a mortgage, including the impact of a change in interest rate on repayment amount, time to repay the loan, total interest paid and the total cost of the loan

6A 6B 6C 6D 6E 6F

KEY SKILLS

During this lesson, you will be:

- using recurrence relations to model interest-only loans
- using financial applications of technology to solve problems involving interest-only loans.

KEY TERMS

- Interest-only loan

When making big purchases, such as a home, it generally requires a large sum of money to be borrowed from the bank. In order to make repayments more affordable, the borrower can opt to make interest-only payments at the start of the loan. This makes the initial regular repayments cheaper as the borrower will not repay any of the principal, however it may increase the total amount paid in the long run.

Using recurrence relations to model interest-only loans

In an **interest-only loan**, the borrower only repays the interest that is charged. As a result of this, the principal that must be paid back remains the same after each compounding period.

See worked example 1

The value of an interest-only loan, V_n, after n compounding periods, can be modelled by a recurrence relation of the same form as a reducing balance loan.

$V_0 = principal, \quad V_{n+1} = R \times V_n - d$, where

- $R = 1 + \frac{r}{100}$, where r is the interest rate (%) per compounding period
- d is the payment made per compounding period

The payment is equal to the interest earned each compounding period and can therefore be calculated as

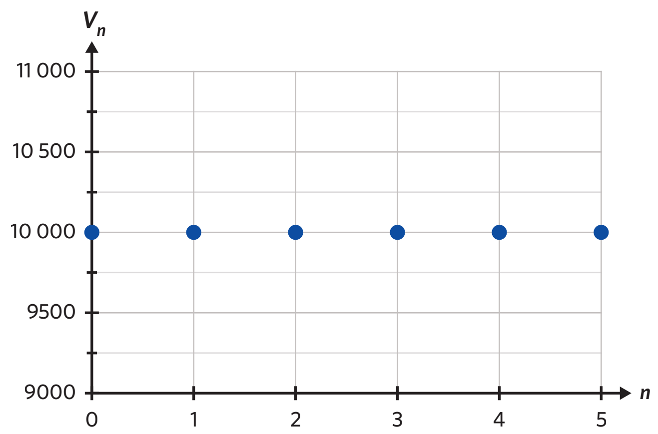
$d = \frac{r}{100} \times V_0$

For example, \$10 000 is borrowed at an interest rate of 4% per annum, compounding annually. The interest charged, 4% of \$10 000, is \$400. If the borrower pays back \$400 at the end of each year, this keeps the principal balance at \$10 000 indefinitely. A recurrence relation that can be used to model the value of the loan, V_n, after n years is

$V_0 = 10\,000, \quad V_{n+1} = 1.04 \times V_n - 400.$

The balance of the loan over a period of 5 years can also be represented on a graph.

Notice that it is constant.



Interest-only loans can be modelled using a table as well, similar to an amortisation table. However, there will be no change from one payment to the next because the principal balance will not reduce.

See worked example 2

Worked example 1

Brock borrows \$8000 in an interest-only loan at 3% per annum, compounding monthly.

a. What is the monthly amount that he will be required to pay?

Explanation

Step 1: Determine the principal and interest rate for the interest-only loan.

$V_0 = 8000$

$r = \frac{3}{12}\%$ per compounding period

Step 2: Calculate d.

$$d = \frac{3}{12 \times 100} \times 8000$$
$$= \frac{3}{1200} \times 8000$$
$$= 20$$

Answer

\$20

b. Construct a recurrence relation that can be used to model the value of Brock's loan, V_n, after n months.

Explanation

Step 1: Determine the principal, interest rate and payment amount.

$V_0 = 8000$

$r = \frac{3}{12}\%$ per compounding period

$d = 20$

Step 2: Calculate R.

$$R = 1 + \frac{3}{12 \times 100}$$
$$= 1 + \frac{3}{1200}$$
$$= 1.0025$$

Step 3: Construct the recurrence relation.

Answer

$V_0 = 8000, \quad V_{n+1} = 1.0025 \times V_n - 20$

Worked example 2

Felicity has an interest-only loan of $45 000, interest is charged at a rate of 3% per annum, compounding weekly. This is modelled in the following table similar to an amortisation table.

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	45 000.00
1	25.96	25.96	0.00	45 000.00
2	25.96	25.96	0.00	45 000.00
3	25.96			

Fill in the missing cells for payment number 3.

Explanation

Step 1: Calculate the interest for payment number 3.

The interest amount is equal to the payment received, which is $25.96.

Step 2: Calculate the principal reduction for payment number 3.

The value of an interest-only loan remains constant, so there is no reduction of the principal.

Step 3: Calculate the loan balance after payment number 3.

The balance does not decrease in an interest-only loan, so it will remain at $45 000.00.

Answer

payment number	payment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	45 000.00
1	25.96	25.96	0.00	45 000.00
2	25.96	25.96	0.00	45 000.00
3	25.96	25.96	0.00	45 000.00

Using financial applications of technology to solve problems involving interest-only loans

The financial solver is a handy program that can be used to solve problems involving interest-only loans.

The number of payments, N, can be set to any value since the value of the interest-only loan remains constant.

The present value, PV, is always positive since the borrower receives this amount from the lender.

The payment, PMT, is always negative since the borrower makes a payment to the lender.

The future value, FV, is always negative as it represents money that is owed, or yet to be paid. It is also the negative value of the PV.

The financial solver can be used to calculate an unknown interest rate I(%) or payment amount PMT, but not an unknown principal value, since both the PV and FV would be unknown.

Worked example 3

Mark borrows $40 000 in an interest-only loan to help fund his business. He is required to make quarterly payments to offset the interest, compounding quarterly at 4.7% p.a.

a. How much is Mark required to pay each quarter?

Explanation

Step 1: Determine the financial solver inputs.

N	1	(can be any value since the balance is unchanged)
I(%)	4.7	(annual interest rate)
PV	40 000	(this is positive because Mark receives it from the lender)
PMT		
FV	−40 000	(this is negative because Mark still owes the lender)
PpY	4	(payments made quarterly)
CpY	4	(interest compounds quarterly)

Step 2: Use the financial solver to solve for PMT.

PMT	−470

The PMT is negative because Mark pays it to the lender.

Answer

$470

b. Mark decides that after he has paid more than $10 000 in interest, he will close the account and end the interest-only loan. How many quarters is it before he closes the account?

Explanation

Step 1: Determine the payment amount made each quarter.

payment = 470

Step 2: Calculate the number of quarters it takes to reach $10 000.

$$\frac{10\,000}{470} = 21.2765...$$

This must be rounded up. At 21 quarters, Mark will not have exceeded the $10 000 threshold.

Answer

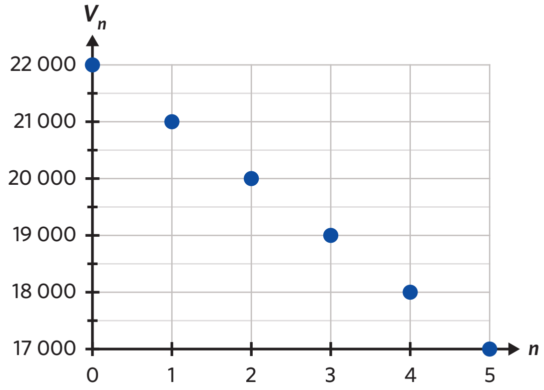
22 quarters

6C Questions

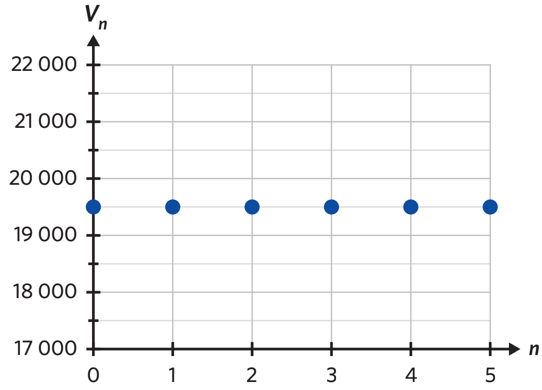
Using recurrence relations to model interest-only loans

1. Which of the following graphs is most likely to represent the value of an interest-only loan across 5 years?

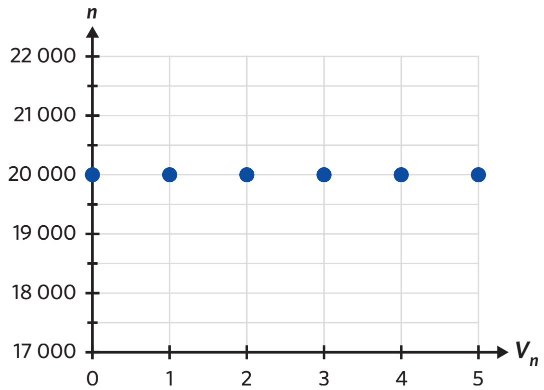
A.



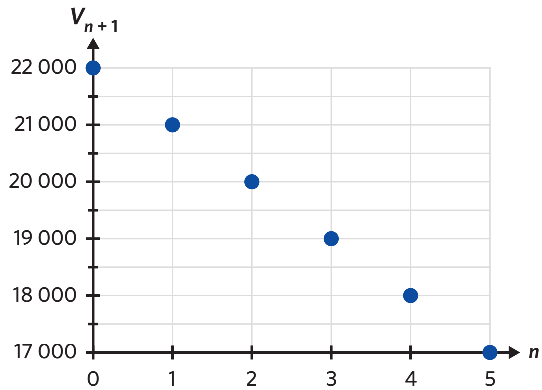
B.



C.



D.



2. The monthly repayments of an interest-only loan are described by the following lines of working.

$$\begin{aligned} d &= \frac{r}{100} \times V_0 \\ &= \frac{36}{1200} \times 8000 \\ &= 240 \end{aligned}$$

a. What is the principal of this interest-only loan?

b. What is the monthly repayment?

c. What is the annual interest rate?

d. Construct a recurrence relation that can be used to model the value of the interest-only loan, V_n, after n compounding periods.

3. Calculate the regular repayment amount required for each of the following interest-only loans. Round to the nearest cent.

a. A \$12 000 loan with 2.8% interest per month, compounding monthly. Payments are made monthly.

b. A \$22 000 loan with 14% interest per annum, compounding annually. Payments are made annually.

c. A \$1600 loan with 18.9% interest per annum, compounding quarterly. Payments are made quarterly.

d. A \$4850 loan with 2.6% interest per annum, compounding fortnightly. Payments are made fortnightly.

4. Terry has taken out an interest-only loan. Every month, he makes a payment of $465. The interest rate is 24% per annum and interest is compounded monthly.

a. What is the principal of the loan?

b. Construct a recurrence relation that can be used to model V_n, the value of the loan after n months.

5. Find the annual interest rates for the following interest-only loans, rounded to one decimal place.

a. An interest-only loan of $21 000, compounding monthly, with monthly payments of $315.

b. An interest-only loan of $16 000, compounding quarterly, with quarterly payments of $86.

c. An interest-only loan of $82 000, compounding fortnightly, with fortnightly payments of $1353.

6. Jennifer borrows $750 000 from the bank to purchase an investment property. She will make monthly interest-only payments for the first 5 years. Her bank charges interest at a rate of 5% per annum, compounding monthly.

The following table is a snippet of this loan.

payment number	payment	interest	principal reduction	balance of loan
11				
12				
13				
14				

Fill in the missing cells. Round values to the nearest cent.

Using financial applications of technology to solve problems involving interest-only loans

7. What is the annual interest rate required to establish an interest-only loan of $4500, with monthly repayments of $9? Interest compounds on a monthly basis.

A. 1.2% p.a. **B.** 2.0% p.a. **C.** 2.3% p.a. **D.** 2.4% p.a.

8. Determine the annual interest rate required for each of the following interest-only loans. Round to one decimal place.

a. An interest-only loan of $48 000 with interest compounding monthly. Payments of $415.10 are made monthly.

b. An interest-only loan of $8122.50 with interest compounding fortnightly. Payments of $162.45 are made fortnightly.

c. An interest-only loan of $12 700 with interest compounding quarterly. Payments of $506.13 are made quarterly.

9. Determine the payment required for each of the following interest-only loans. Give answers correct to the nearest cent.

a. An interest-only loan of $16 000 at 5.7% interest per annum compounding yearly. Payments are made annually.

b. An interest-only loan of $250 000 at 7.6% interest per annum compounding fortnightly. Payments are made fortnightly.

c. An interest-only loan of $550 000 at 5.3% interest per annum compounding monthly. Payments are made monthly.

10. Lee wants to purchase a sports car worth \$547 841. He places a 30% deposit and finances the rest of the car. The car dealer offers an interest-only loan for the first 3 years of the loan and Lee wants to take full advantage of this offer. Interest compounds weekly and Lee is required to make weekly payments.

a. How much did Lee borrow?

b. If Lee is required to pay \$221.24 each week, then what is the annual interest rate for this loan? Give the answer correct to two significant figures.

c. How much interest has Lee paid by the end of the interest-only period?

Joining it all together

11. Harvey is looking to borrow \$500 000 and invest it wisely, in the hope of making a decent profit over the next four years. He likes the idea of an interest-only loan and has been presented with various offers.

Melbourne Bank offers Harvey an interest rate of 6.9% per annum, compounding monthly. Harvey must make monthly repayments.

a. How much interest will Harvey pay across the four years?

Sydney Bank offers the \$500 000 loan with the total interest amount being \$150 000 across the four years. Payments are to be made monthly and interest compounds monthly.

b. What is the annual interest rate of this option?

c. Write a recurrence relation that can be used to model the value of the loan, V_n, after n payment periods using Sydney Bank's offer.

d. Which option is better for Harvey? Explain why.

---

12. Cassandra wants to borrow some money to pay for a beach house in Byron Bay. She places a deposit of 10% and decides to take out an interest-only loan to cover the remaining cost of the house.

Her bank offers an interest rate of 7.3% per annum, compounding daily. She will be paying \$180 every day.

a. How much does the beach house cost?

b. Write a recurrence relation that can be used to model the value of Cassandra's loan, V_n after n days.

In three years, her bank changes the interest rate and she is required to increase her daily repayments by \$90 to maintain the interest-only loan. Interest continues to compound daily.

c. How much interest has Cassandra paid in the first three years? Assume that there were no gap years during this time.

d. How much does Cassandra still owe?

e. How much has the annual interest rate increased by?

Exam practice

13. Bob borrowed \$400 000 to buy an apartment.

The interest rate for this loan was 3.14% per annum, compounding monthly.

A scheduled monthly repayment that allowed Bob to fully repay the loan in 20 years was determined.

Bob decided, however, to make interest-only repayments for the first two years.

After these two years, the interest rate changed. Bob was still able to pay off the loan in the 20 years by repaying the scheduled amount each month.

The interest rate, per annum, for the final 18 years of the loan was closest to

A. 1.85% **B.** 2.21% **C.** 2.79%
D. 3.14% **E.** 4.07%

VCAA 2021 Exam 1 Recursion and financial modelling Q24

33% of students answered this question correctly.

Questions from multiple lessons

Recursion and financial modelling

14. Percy has a compound interest investment that earns interest compounding quarterly.

The balance of Percy's compound interest investment was $27 060.80 after one year.

After three years, the balance of the investment was $31 706.04.

The value of Percy's initial investment is closest to

A. $20 000 **B.** $21 337 **C.** $24 669 **D.** $25 000 **E.** $25 669

Adapted from VCAA 2017NH Exam 1 Recursion and financial modelling Q24

Data analysis

15. Jeff is conducting some research on data from 2016 on how the distance a second-hand car was driven (in 000's of km) impacted the car's price. A least squares line has been fitted to the data and its equation is

$price = 15\,129 - 68 \times distance$

In 2016, the price of a second-hand car that had driven 123 000 km was $6299. When the least squares line is used to predict the price of this car, the residual is closest to

A. −$8 348 871 **B.** −$6765 **C.** −$466 **D.** $466 **E.** $6765

Adapted from VCAA 2016 Exam 1 Data analysis Q10

Recursion and financial modelling

16. A 3D printer was purchased for $4600.

The value of the 3D printer can be depreciated using the reducing balance method of depreciation. Its value after one year was $4324 and its value after two years was $4064.56.

a. Show that the depreciation rate of the 3D printer is 6% per annum. (1 MARK)

b. Let V_n be the value of the 3D printer after n years.

Determine a recurrence relation in terms of V_{n+1}, V_n and V_0 that displays the value of the 3D printer after n years. (1 MARK)

Adapted from VCAA 2017NH Exam 2 Recursion and financial modelling Q5d

6D Amortising annuities

STUDY DESIGN DOT POINTS

- use of a first-order linear recurrence relation to model and analyse (numerically and graphically) the amortisation of an annuity, including the use of a recurrence relation to determine the value of the annuity after n payments for an initial sequence from first principles
- use of a table to investigate and analyse the amortisation of an annuity on a step-by-step basis, the payment made, the interest earned, the reduction in the principal and the balance of the annuity
- use of technology to solve problems involving annuities including determining the amount to be invested in an annuity to provide a regular income paid, for example, monthly, quarterly

6A 6B 6C **6D** 6E 6F

KEY SKILLS

During this lesson, you will be:

- using recurrence relations to model amortising annuities
- using amortisation tables to solve problems involving amortising annuities
- using financial applications of technology to solve problems involving amortising annuities.

KEY TERMS

- Annuity
- Amortising annuity

An amortising annuity can take on the form of a retirement investment fund. This occurs when an individual's superannuation, a large lump sum, is invested with a bank or super fund where interest is earned but regular payments are taken out for the retiree to use. It is also referred to as a retirement income stream.

Using recurrence relations to model amortising annuities

An **annuity** is an investment that involves a fixed sum of money paid to an investor at regular intervals, and typically earns compound interest. In the case of an **amortising annuity**, the balance of the investment will decrease over time until it drops to zero or the account is closed.

The value of an amortising annuity, V_n, after n compounding periods, can be modelled by a recurrence relation of the form

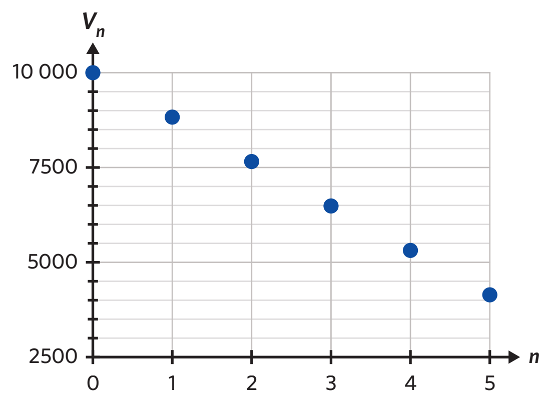
$V_0 = principal, \quad V_{n+1} = R \times V_n - d$, where

- $R = 1 + \frac{r}{100}$, where r is the interest rate (%) per compounding period
- d is the payment received per compounding period

For example, Lara has an annuity of \$10 000 that is opened at an interest rate of 12% per annum, compounding monthly. The annuity provides an income of \$1250 per month. A recurrence relation that can be used to determine V_n, the value of the investment after n months is

$V_0 = 10\,000, \quad V_{n+1} = 1.01 \times V_n - 1250.$

The balance of Lara's investment over a period of 5 months can also be represented on a graph.



There is a slight curve in the graph. This is because, at the beginning of the annuity, a larger amount of interest is earned, so a smaller portion of the payment reduces the principal.

Worked example 1

Tom invests \$25 000 in an annuity with an interest rate of 7% per annum, compounding monthly. This annuity will provide him with a monthly income of \$2500.

a. Construct a recurrence relation that can be used to model V_n, the value of this annuity after n compounding periods. Round R to 5 decimal places.

Explanation

Step 1: Determine the principal, interest rate and payment amount.

$V_0 = 25\,000$

$r = \frac{7}{12}\%$ per compounding period

$d = 2500$

Step 2: Calculate R.

$R = 1 + \frac{7}{12 \times 100}$

$R = 1 + \frac{7}{1200}$

$R = 1.00583...$

Step 3: Construct the recurrence relation.

Answer

$V_0 = 25\,000, \quad V_{n+1} = 1.00583 \times V_n - 2500$

b. Calculate the value of the investment after 4 months.

Explanation

Step 1: Calculate V_1, the value of the investment after 1 month.

$V_1 = 1.00583 \times V_0 - 2500$

$= 1.00583 \times (25\,000) - 2500$

$= 22\,645.75$

Step 2: Calculate V_2, the value of the investment after 2 months.

$V_2 = 1.00583 \times V_1 - 2500$

$= 1.00583 \times (22\,645.75) - 2500$

$= 20\,277.774...$

Step 3: Calculate V_3, the value of the investment after 3 months.

$V_3 = 1.00583 \times V_2 - 2500$

$= 1.00583 \times (20\,277.774...) - 2500$

$= 17\,895.994...$

Step 4: Calculate V_4, the value of the investment after 4 months.

$V_4 = 1.00583 \times V_3 - 2500$

$= 1.00583 \times (17\,895.994...) - 2500$

$= 15\,500.327...$

Answer

\$15 500.33

Using amortisation tables to solve problems involving amortising annuities

Amortisation tables are used to keep track of the payments received by an investor and the remaining balance in an amortising annuity.

In an amortising annuity, a portion of each payment received by the investor consists of the interest calculated each compounding period, and the remainder is taken out of the investment balance.

For example, the amortisation table shown models the duration of an annuity.

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	10 000.00
1	2000.00	100.00	1900.00	8100.00
2	2000.00	81.00	1919.00	6181.00
3	2000.00	61.81	1938.19	4242.81
4	2000.00	42.43	1957.57	2285.24
5	2000.00	22.85	1977.15	308.09
6	311.17	3.08	308.09	0.00

Note: Often the final payment is adjusted so that the entire balance of the investment is paid out. The final payment amount does not equal the previous balance because there is interest that needs to be calculated.

The columns of an amortisation table for an amortising annuity can be calculated using the following formulas:

$$\textit{interest} = \frac{r}{100} \times \textit{previous investment balance}$$

$$\textit{principal reduction} = \textit{payment} - \textit{interest}$$

$$\textit{balance of investment} = \textit{previous investment balance} - \textit{principal reduction}$$

Worked example 2

Mike invests \$5000 in an annuity account paying interest at 10% per annum. Mike wants to receive a payment of \$2010 per year. This is modelled using the following amortisation table.

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	5000.00
1	2010.00	500.00	1510.00	3490.00
2	2010.00	349.00	1661.00	1829.00
3				0.00

Mike wants the final payment amount adjusted so that the annuity will be fully paid out after payment number 3.

a. Calculate the missing interest for payment number 3.

Explanation

Step 1: Identify the interest rate and previous loan balance.

$r = 10\%$ per compounding period

$\textit{previous loan balance} = 1829.00$

Step 2: Calculate the interest amount.

$\textit{interest} = \frac{10}{100} \times 1829.00$

$\textit{interest} = 182.90$

Answer

\$182.90

b. Calculate the final payment that Mike will receive for this annuity.

Explanation

Calculate the payment amount for payment number 3.

The final payment will be the sum of the remaining investment balance and the interest calculated.

$$\begin{aligned}\textit{final payment} &= 1829.00 + 182.90 \\ &= 2011.90\end{aligned}$$

Continues →

Answer

$2011.90

c. How much interest has the account earned over three years?

Explanation

Step 1: Calculate the total amount received in payments for three years.

$$\begin{aligned} \textit{total amount} &= 2 \times 2010.00 + 1 \times 2011.90 \\ &= 6031.90 \end{aligned}$$

Step 2: Calculate the total reduction in principal over three years.

The investment was depleted after three payments, so the total principal reduction is the full $5000 investment.

Step 3: Calculate the total interest earned over three years.

This is the difference between the total amount received in payments and the total reduction in principal.

$$\begin{aligned} \textit{interest earned} &= 6031.90 - 5000.00 \\ &= 1031.90 \end{aligned}$$

Note: Adding the values in the 'interest' column can also be used to calculate the total interest earned for this period.

Answer

$1031.90

Using financial applications of technology to solve problems involving amortising annuities

The financial solver is a useful tool for solving problems involving amortising annuities.

The present value, PV, is always negative since the investor gives this amount to the financial institution.

The payment, PMT, is always positive since the investor receives payments, taken out of the investment.

The future value, FV, is always positive as it represents future payments that can be made to the investor.

A future value of 0 indicates that the annuity has been fully paid out.

Worked example 3

Chuck invests $400 000 in an annuity with a bank who offers an interest rate of 4% per annum, compounding monthly. The annuity pays out $5000 per month.

a. What will the annuity be worth after 6 months?

Explanation

Step 1: Determine the financial solver inputs.

N	6	(there are 6 months)
I(%)	4	(annual interest rate)
PV	−400 000	(this is negative because Chuck gives it to the bank)
PMT	5000	(this is positive because Chuck receives it from the bank)
FV		
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

Continues →

Step 2: Use the financial solver to solve for FV.

FV	377 815.849...

This is positive because it represents future payments that can be made to Chuck.

Answer

$377 815.85

b. After 6 months, the bank decreases the interest rate to 3.6% per annum, compounding monthly. Chuck has opted to keep receiving the same payment amount each month.

To ensure the annuity is fully paid out, the final payment will be lower.

In total, how many payments will it take for the annuity to reach a zero balance?

Use the rounded value found in part **a**.

Explanation

Step 1: Determine the financial solver inputs.

Chuck still has $377 815.85 invested with the bank when the interest rate changes. This is now represented as the PV of the annuity.

N		
I(%)	3.6	(annual interest rate)
PV	−377 815.85	(this is negative because the amount is still with the bank)
PMT	5000	(this is positive because Chuck receives this payment)
FV	0	(annuity to be fully paid out)
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

Step 2: Use the financial solver to solve for N.

N	85.820...

This means that after the first six months, there will be 85 full payments of $5000 and 1 final payment less than $5000 before the annuity reaches a balance of $0. Hence, there will be 86 payments after the interest rate is changed.

Step 3: Calculate the total number of payments.

$86 + 6 = 92$

Answer

92 payments

Exam question breakdown

VCAA 2016 Exam 1 Recursion and financial modelling Q24

Mai invests in an annuity that earns interest at the rate of 5.2% per annum, compounding monthly.

Monthly payments are received from the annuity.

The balance of the annuity will be $130 784.93 after five years.

The balance of the annuity will be $66 992.27 after ten years.

The monthly payment that Mai receives from the annuity is closest to

A. $1270 **B.** $1400 **C.** $1500 **D.** $2480 **E.** $3460

Continues →

Explanation

Step 1: Determine the financial solver inputs.

The value of Mai's annuity after 5 years is the present value because this is the earliest time point in the scenario.

The value of Mai's annuity after 10 years can subsequently be treated as the future value, 5 years later.

N	60	(there are 60 months in 5 years)
I(%)	5.2	(annual interest rate)
PV	−130 784.93	(this is negative because Mai invests it with the bank)
PMT		
FV	66 992.27	(this is positive because it represents future payments that can be made to Mai)
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

Step 2: Use the financial solver to solve for PMT.

PMT	1500.000...

This is positive because Mai receives it from the bank.

30% of students answered this question correctly.

23% of students incorrectly answered option E. This was caused by the incorrect input of figures into the financial solver, with the wrong positive and negative signs for the PV and FV. Students may have struggled with this question because of the unconventional method of modelling the value of a loan in the middle of its lifespan, rather than the start or end of its duration.

Answer

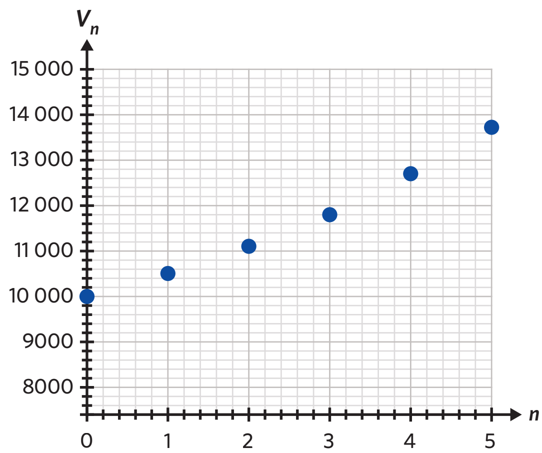
C

6D Questions

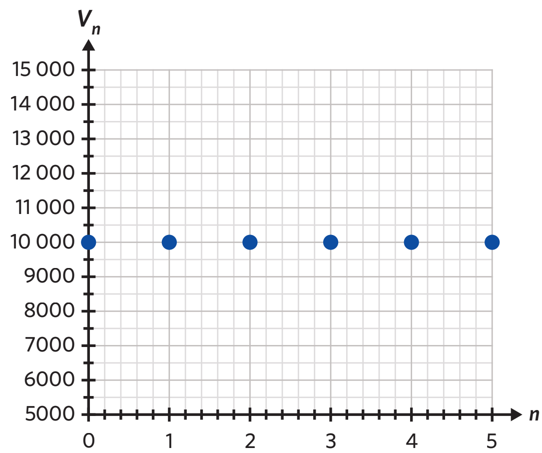
Using recurrence relations to model amortising annuities

1. Which of the following graphs is most likely to represent the value of an amortising annuity?

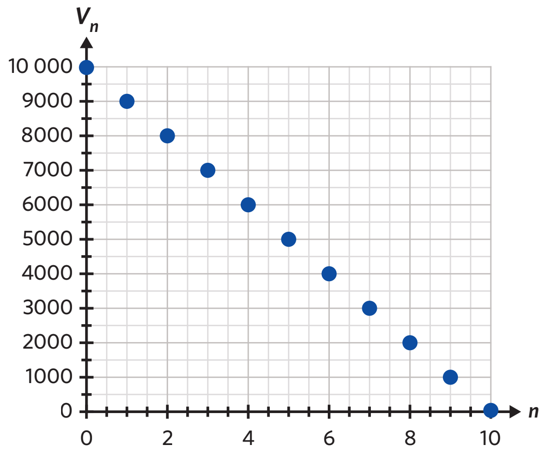
A.



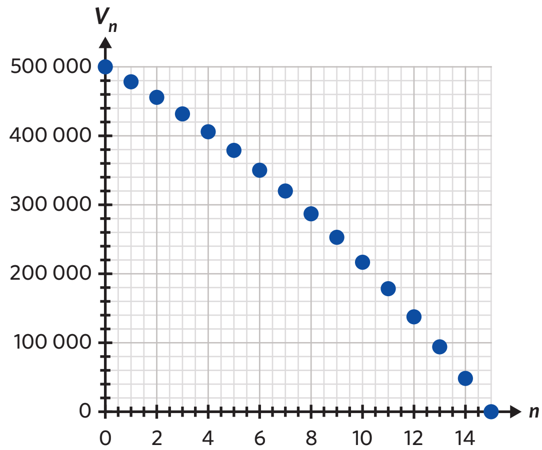
B.



C.



D.



2. The value of an annuity V_n, after n months can be modelled using the recurrence relation

$V_0 = 25\,000, \quad V_{n+1} = 1.012 \times V_n - 1500$

a. How much was originally invested in this annuity?

b. How much does the investor receive as a payment from the investment every month?

c. What is the annual interest rate?

3. Carlos has invested in an annuity modelled by the recurrence relation

$V_0 = 50\,000, \quad V_{n+1} = 1.150 \times V_n - 10\,000$

a. Using recursive calculations, calculate the value of the annuity after Carlos has received four payments.

b. Carlos' friend, Gabby, takes out the same annuity, but receives payments of \$9000 instead. What is the balance of her annuity after three payments?

4. Generate a recurrence relation to model the value of the annuity V_n, after n compounding periods for each of the following scenarios. If necessary, round values to 5 significant figures.

a. Amy invests \$2500 in an annuity with an interest rate of 6.9% per annum, compounding monthly. The annuity provides her with a monthly income of \$100.

b. Bella invests \$6200 in an annuity with an interest rate of 4.1% per annum, compounding quarterly. The annuity provides her with a quarterly income of \$250.

c. Cameron has an annuity with an interest rate of 5.0% per annum, compounding fortnightly. He initially invests \$1500 and receives a fortnightly income of \$60.

5. The balance of an annuity after n fortnights, B_n can be modelled by the recurrence relation

$B_0 = 168\,714, \quad B_{n+1} = 1.002 \times B_n - 14\,562$

Calculate the interest, in dollars, the annuity has earned after five payments.

Using amortisation tables to solve problems involving amortising annuities

6. The first four lines of the amortisation table for an annuity are shown.

The payment that the investor received for payment number 3 is

A. \$954.00

B. \$1000.00

C. \$1150.44

D. \$12 091.93

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	14 521.00
1	954.00	145.21	808.79	13 712.21
2	954.00	137.12	816.88	12 895.33
3		128.95	825.05	12 070.28
4	954.00	120.70	833.30	11 236.98

7. John invests his superannuation in an annuity. Interest compounds monthly and he receives a retirement income stream of $2000 each month.

An amortisation table that models the annuity is shown.

payment number	payment	interest	principal reduction	balance of investment
20	2000.00	482.26	1517.74	136 270.19
21	2000.00	476.95	1523.05	134 747.14
22	2000.00	471.61	1528.39	133 218.75
23	2000.00	466.27	1533.73	131 685.02
24	2000.00	460.90	1539.10	130 145.92
25	2000.00			

a. Show that the interest rate is 4.2% per annum.

For payment number 25, calculate

b. the amount of interest earned.

c. the reduction in principal.

d. the resultant balance of the annuity.

8. Find x, y and z in each of the following amortisation tables.

a.

payment number	payment	interest	principal reduction	balance of investment
0	x	0.00	0.00	400.00
1	50.00	y	10.00	390.00
2	50.00	39.00	11.00	z

b.

payment number	payment	interest	principal reduction	balance of investment
0	0	0	0	1600.00
1	150.00	64.00	x	y
2	150.00	z	89.44	1424.56

c.

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	5000.00
1	2000.00	120.00	1880.00	3120.00
2	2000.00	x	1925.12	1194.88
3	y	28.68	z	0.00

Using financial applications of technology to solve problems involving amortising annuities

9. Ashton's parents invested $25 000 in an annuity that pays out $5000 each year to cover his school expenses. After two years of high school, the annuity is worth $17 455.23. Interest compounds annually.

What is the annual interest rate for this annuity?

A. 5.0%

B. 5.3%

C. 6.0%

D. 6.1%

10. Find the current value of the following annuities.

a. A \$20 000 annuity began three years ago and has been paying out \$5000 per year. The annuity pays interest at a rate of 8% per annum.

b. A \$1200 annuity opened six months ago, with an interest rate of 13.2% per annum, compounding monthly. It pays out \$150 per month.

c. A \$6500 annuity began one and a half years ago, with an interest rate of 8.8% per annum, compounding every three months, paying \$400 each quarter.

d. Five years ago, \$81 000 was invested in an annuity. It has an interest rate of 8.24% per annum, compounding weekly, and pays \$250 weekly.

---

11. When Jack retires, he wants to have enough superannuation invested in an annuity to pay him \$1000 a month for exactly 20 years. His superfund offers an annual interest rate of 5.2% compounding monthly. How much does Jack need to have in his super account before he can retire?

---

12. Dawn has invested \$500 000 in an annuity. Her bank offers an interest rate of 4.7% per annum, compounding fortnightly. She will receive fortnightly payments of \$1500.

a. How much interest does the annuity earn in the 1st year?

b. What will the annuity be worth in 4 years?

After 4 years, Dawn opts to have the payments increase to \$2000 each fortnight for the remainder of the annuity. The interest rate and compounding frequency remains constant.

c. To ensure the annuity is fully paid out, the final payment will be lower.

In total, how many fortnights will the annuity last for?

Joining it all together

13. Dale invests his life savings in an annuity, where interest compounds fortnightly. A recurrence relation that models the value of the annuity, V_n, after n fortnights is shown.

$V_0 = 1\ 250\ 000, \quad V_{n+1} = 1.004 \times V_n - 8500$

a. How much did Dale invest?

b. Calculate the annual interest rate.

Dale's accountant, Randall, has created an amortisation table to model the annuity after n fortnights. However, due to an error with his fax machine, two lines didn't print.

The balance of the investment after payment 9 is \$1 217 991.27.

payment number	payment	interest	principal reduction	balance of investment
10				
11				

c. Fill in the missing rows.

d. After 11 fortnights, how much interest has the account earned?

Use the rounded value found in part **c**.

14. Bobby has recently retired after 40 years of hard work. She has $759 231 invested in an annuity with her superannuation fund who offers an interest rate of 8.9% per annum, compounding weekly. She will receive a weekly retirement income stream. Her grandchild has set up an amortisation table to model the annuity. The first five lines are shown.

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	759 231.00
1	2000.00	1299.45	700.55	758 530.45
2	2000.00	1298.25	701.75	757 828.70
3	2000.00	1297.05	702.95	757 125.75
4	2000.00	1295.85	704.15	756 421.60
5		1294.64	705.36	755 716.24

a. Determine the payment amount for payment number 5.

b. Construct the recurrence relation that can be used to model the value of the annuity, V_n after n weeks. Round to 4 decimal places where necessary.

After 2 years, the interest rate decreases to 7.4% p.a, compounding weekly.

c. To ensure the annuity is fully paid out, the final payment will be lower.

If everything else remains the same, how many more weeks will the annuity last?

15. The value of an annuity, V_n after n months can be modelled by the recurrence relation

$V_0 = 645\,628.00, \quad V_{n+1} = R \times V_n - 5478.47$

If the annuity is to be fully paid out in 10 years, calculate R, rounded to 4 decimal places.

16. Ella invests $100 000 in an annuity. Her bank offers a 4.68% per annum interest rate, compounding fortnightly. She will receive payments of $1250 every fortnight for 3 years.

a. Construct a recurrence relation that can be used to model the value of the annuity V_n after n fortnights. Round to 4 decimal places where necessary.

b. What is the value of the annuity after three years?

After three years, Ella decreases the fortnightly payments to $900.

To ensure the annuity is fully paid out, the final payment will be lower.

c. How long, in years and fortnights, does the annuity last in total?
Use the rounded value found in part **b**.

Exam practice

17. The value of an annuity, V_n, after n monthly payments of $555 have been made, can be determined using the recurrence relation

$V_0 = 100\,000, \quad V_{n+1} = 1.0025 \times V_n - 555$

The value of the annuity after five payments have been made is closest to

A. $97 225

B. $98 158

C. $98 467

D. $98 775

E. $100 224

VCAA 2016 Exam 1 Recursion and financial modelling Q18

77% of students answered this question correctly.

18. Julie has retired from work and has received a superannuation payment of $492 800.

Julie could invest the $492 800 in an annuity.

The annuity earns interest at the rate of 4.32% per annum, compounding monthly.

The balance of Julie's annuity at the end of the first year would be $480 242.25.

What monthly payment, in dollars, would Julie receive? (1 MARK)

VCAA 2018 Exam 2 Recursion and financial modelling Q6bi

48% of students answered this question correctly.

---

19. Deepa invests $500 000 in an annuity that provides an annual payment of $44 970.55. Interest is calculated annually.

The first five lines of the amortisation table are shown.

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	500 000.00
1	44 970.55	20 000.00	24 970.55	475 029.45
2	44 970.55	19 001.18	25 969.37	449 060.08
3	44 970.55	17 962.40	27 008.15	422 051.93
4	44 970.55	16 882.08	28 088.47	393 963.46

The number of years, in total, for which Deepa will receive the regular payment of $44 970.55 is closest to

A. 12 **B.** 15 **C.** 16
D. 18 **E.** 20

VCAA 2021 Exam 1 Recursion and financial modelling Q19

44% of students answered this question correctly.

---

20. Sienna invests $152 431 into an annuity from which she will receive a regular monthly payment of $900 for 25 years. The interest rate for this annuity is 5.1% per annum, compounding monthly.

a. Let V_n be the balance of the annuity after n monthly payments. A recurrence relation written in terms of V_0, V_{n+1} and V_n can model the value of this annuity from month to month.

Showing recursive calculations, determine the value of the annuity after two months. Round the answer to the nearest cent. (2 MARKS)

b. After two years, the interest rate for this annuity will fall to 4.6%.

To ensure that she will still receive the same number of $900 monthly payments, Sienna will add an extra one-off amount into the annuity at this time.

Determine the value of this extra amount that Sienna will add.

Round the answer to the nearest cent. (1 MARK)

VCAA 2021 Exam 2 Recursion and financial modelling Q9

Part **a**: The average mark on this question was **0.7**.

Part **b**: **8%** of students answered this question correctly.

Questions from multiple lessons

Recursion and financial modelling

21. Jessie took out a $15 000 loan to renovate her apartment. Interest for the loan was charged at a rate of 13.8% per annum, compounding quarterly.

For the first two years of the loan, Jessie made quarterly repayments of $800.

For the next two years of the loan, Jessie made quarterly repayments of $1000.

The amount Jessie paid in interest only during this four-year period is closest to

A. $6300 **B.** $6700 **C.** $7100 **D.** $7300 **E.** $7700

Adapted from VCAA 2018NH Exam 1 Recursion and financial modelling Q24

Data analysis

22. The population of a bee farm was monitored over a period of time.

The time series plot shows the *population* of the farm, in thousands, over a thirty-month period.

The five-median smoothed population in thousands, for the 15$^{\text{th}}$ month is closest to

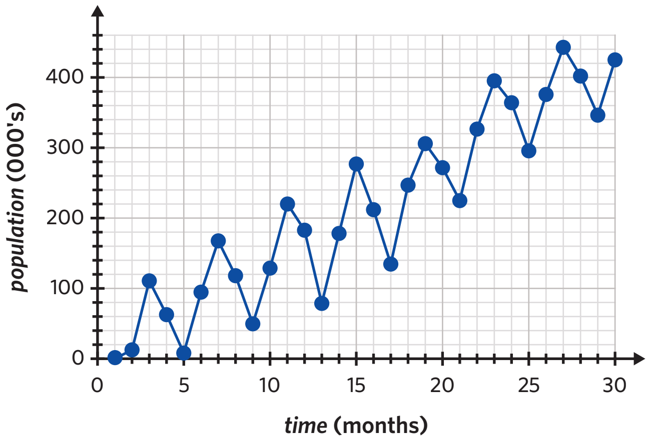
A. 79

B. 135

C. 178

D. 212

E. 277



Adapted from VCAA 2017 Exam 1 Data analysis Q14

Recursion and financial modelling

23. Igor has just purchased a new jet ski for \$22 000.

The value of his jet ski depreciates using the reducing balance method.

The value of Igor's jet ski, in dollars, after n years, J_n, can be modelled by the recurrence relation shown.

$J_0 = 22\,000, \quad J_{n+1} = R \times J_n$

a. For each of the first two years of the reducing balance depreciation, the value of R is 0.75. What is the annual rate of depreciation during the first two years? (1 MARK)

b. For the next three years of reducing balance depreciation, the annual rate of depreciation is changed to 15.9%. What is the value of the jet ski five years after it was purchased? Round the answer to the nearest cent. (2 MARKS)

Adapted from VCAA 2018 Exam 2 Recursion and financial modelling Q5

6E Perpetuities

STUDY DESIGN DOT POINT

- simple perpetuity as a special case of an annuity that lasts indefinitely

6A 6B 6C 6D **6E** 6F

KEY SKILLS

During this lesson, you will be:

- using recurrence relations to model perpetuities
- using financial applications of technology to solve problems involving perpetuities.

KEY TERM

- Perpetuity

An investment can last indefinitely if the investor receives a payment each period that is equal to the interest earned. This type of investment could be useful for an investor who wants a regular stream of income over a long period of time, such as a retiree who has received their superannuation.

Using recurrence relations to model perpetuities

A **perpetuity** is a special type of annuity where the payment received is equal to the interest earned, effectively allowing the annuity to last indefinitely. The value of the perpetuity remains the same for each compounding period.

The value of a perpetuity, V_n, after n compounding periods, can be modelled by a recurrence relation of the same form as an annuity.

See worked example 1

$V_0 = principal, \quad V_{n+1} = R \times V_n - d$, where

- $R = 1 + \frac{r}{100}$, where r is the interest rate (%) per compounding period
- d is the payment received per compounding period

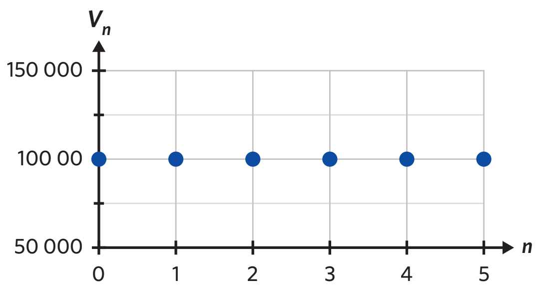
However, the payment is equal to the interest earned each compounding period, and can therefore be calculated as

$d = \frac{r}{100} \times V_0$

For example, \$100 000 is invested in a perpetuity with an interest rate of 4.2% per annum, compounding annually. The interest amount is 4.2% of \$100 000, which is \$4200. If the investor receives a payment of \$4200 at the end of each year, the principal balance will remain constant at \$100 000. A recurrence relation that can be used to model the value of the investment, V_n, after n years is

$V_0 = 100\,000, \quad V_{n+1} = 1.042 \times V_n - 4200.$

The balance of the perpetuity over a period of 5 years can also be represented on a graph. Notice that it is constant.



Perpetuities can also be modelled using a table, similar to an amortisation table. However, there will be no change with each payment because the principal balance remains constant.

See worked example 2

Worked example 1

Elise invests \$259 920 in a perpetuity. Her bank offers an interest rate of 4.8% per annum, compounding monthly.

a. What is the monthly amount she will receive from the bank?

Explanation

Step 1: Determine the principal and interest rate for the perpetuity.

$V_0 = 259\,920$

$r = \frac{4.8}{12}\%$ per compounding period

Step 2: Calculate d.

$$d = \frac{4.8}{12 \times 100} \times 259\,920$$
$$= \frac{4.8}{1200} \times 259\,920$$
$$= 1039.68$$

Answer

\$1039.68

b. Construct a recurrence relation that can be used to model the value of Elise's investment, V_n, after n months.

Explanation

Step 1: Determine the principal, interest rate and payment amount.

$V_0 = 259\,920$

$r = \frac{4.8}{12}\%$ per compounding period

$d = 1039.68$

Step 2: Calculate R.

$$R = 1 + \frac{4.8}{12 \times 100}$$
$$= 1 + \frac{4.8}{1200}$$
$$= 1.004$$

Step 3: Construct the recurrence relation.

Answer

$V_0 = 259\,920, \quad V_{n+1} = 1.004 \times V_n - 1039.68$

Worked example 2

Sona has \$490 000 invested in a perpetuity account. Interest is earned at a rate of 1.04% per annum, compounding weekly. This is modelled in the following table.

payment number	payment	interest	principal reduction	balance of investment
10	98.00	98.00	0.00	490 000.00
11	98.00	98.00	0.00	490 000.00
12	98.00	98.00	0.00	490 000.00
13	98.00	98.00	0.00	490 000.00
14	98.00			

Fill in the missing cells for payment number 14.

Explanation

Step 1: Determine the interest for payment number 14.

The interest amount is equal to the payment received, which is \$98.00

Continues →

Step 2: Determine the principal reduction for payment number 14.

The value of a perpetuity remains constant, so there is no reduction of the principal.

Step 3: Determine the loan balance after payment number 14.

The balance doesn't decrease in a perpetuity, so it will remain at $490 000.00.

Answer

payment number	payment	interest	principal reduction	balance of investment
10	98.00	98.00	0.00	490 000.00
11	98.00	98.00	0.00	490 000.00
12	98.00	98.00	0.00	490 000.00
13	98.00	98.00	0.00	490 000.00
14	98.00	98.00	0.00	490 000.00

Using financial applications of technology to solve problems involving perpetuities

The financial solver can be used to solve problems involving perpetuities.

The number of payments, N, can be set to any value since the value of the perpetuity remains constant.

The present value, PV, is always negative since the investor deposits the initial investment.

The payment, PMT, is always positive since the investor receives payments, taken out of the investment.

The future value, FV, is always positive since it represents a future payment that will be made to the investor. It is also the positive value of the PV.

The financial solver can be used to calculate an unknown interest rate I(%) or payment amount PMT, but not an unknown principal value, since both the PV and FV would be unknown.

Worked example 3

May invests $123 201 in a perpetuity. Her bank offers an interest rate of 1.3% per annum, compounding fortnightly.

a. How much will May receive each fortnight?

Explanation

Step 1: Determine the financial solver inputs.

N	1	(can be any value since the balance is unchanged)
I(%)	1.3	(annual interest rate)
PV	−123 201	(this is negative because May invests it with the bank)
PMT		
FV	123 201	(this is positive because it represents a future payment that will be made to May if she closes the perpetuity)
PpY	26	(payments made fortnightly)
CpY	26	(interest compounds fortnightly)

Continues →

Step 2: Use the financial solver to solve for PMT.

PMT	61.60

This is positive because May receives it from the bank.

Answer

$61.60

b. May decides that once she receives a cumulative total of $6400, she will close the perpetuity account. For how many fortnights will the account be open?

Explanation

Step 1: Recall the payment amount made each fortnight.

$payment = 61.60$

Step 2: Calculate the number of fortnights to receive a cumulative total of $6400.

$\dfrac{6400}{61.40} = 103.896...$

This must be rounded up. At 103 fortnights, May will not have exceeded the $6400 threshold.

Answer

104 fortnights

Exam question breakdown

VCAA 2018 Exam 2 Recursion and financial modelling Q6a

Julie has retired from work and has received a superannuation payment of $492 800.

Julie could invest the $492 800 in a perpetuity. She would then receive $887.04 each fortnight for the rest of her life.

At what annual percentage rate is interest earned by this perpetuity? (1 MARK)

Explanation

Step 1: Determine the financial solver inputs.

N	1	(can be any value since the balance is unchanged)
I(%)		
PV	−492 800	(this is negative because Julie invests it with the bank)
PMT	887.04	(this is positive because Julie receives it from the bank)
FV	492 800	(this is positive because it represents a future payment that will be made to Julie once she closes the perpetuity)
PpY	26	(payments made fortnightly)
CpY	26	(interest compounds fortnightly)

Step 2: Use the financial solver to solve for I(%).

I(%)	4.68

This is the annual interest rate.

Answer

4.68% p.a.

41% of students answered this question correctly.

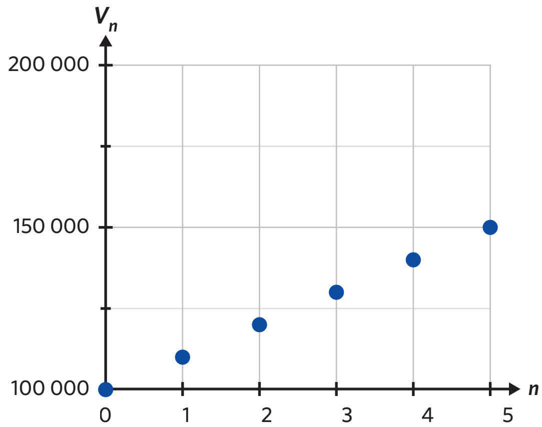
A number of students incorrectly answered **0.18%**, the fortnightly interest rate. This does not account for the fortnightly payment frequency.

6E Questions

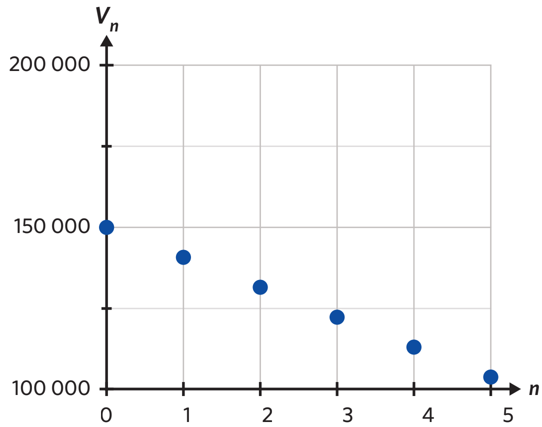
Using recurrence relations to model perpetuities

1. Which of the following graphs is most likely to represent the value of a perpetuity after 5 years?

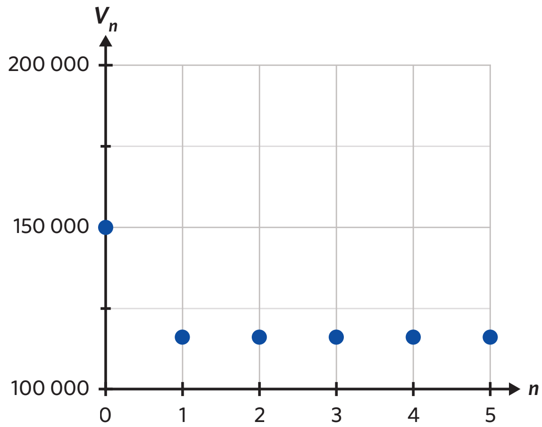
A.



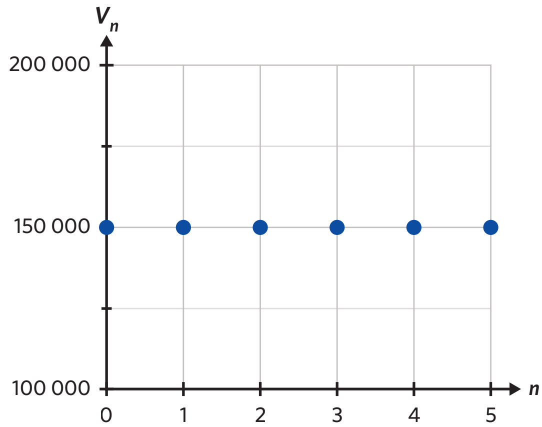
B.



C.



D.



2. The following working calculates the monthly payment from a perpetuity investment.

$$d = \frac{r}{100} \times V_0$$
$$= \frac{0.3}{100} \times 90\,000$$
$$= 270$$

a. What is the value of the initial investment?

b. What is the monthly payment?

c. What is the annual interest rate?

d. Construct a recurrence relation that can be used to model the value of the perpetuity, V_n, after n compounding periods.

3. The following recurrence relation models the value of a perpetuity, V_n, after n months.

$V_0 = 168\,429, \quad V_{n+1} = 1.005 \times V_n - 842.15$

a. How much does the investor receive each month?

b. Calculate the annual interest rate.

c. What is the value of the perpetuity after 100 months?

4. Answer the following questions without using a financial solver. Round to two decimal places where necessary.

   a. A perpetuity with an initial investment of $80 000 has an interest rate of 3.5% per annum, compounding monthly. How much are the monthly payments?

   b. A perpetuity with an initial investment of $250 000 has an interest rate of 2.8% per annum, compounding quarterly. How much are the quarterly payments?

   c. A perpetuity has a monthly payment of $225 to the investor. If the interest rate is 3.4% per annum, how much was the initial investment?

   d. An investor puts $145 000 into a perpetuity. They receive weekly payments of $200. If interest compounds weekly, what is the annual interest rate?

5. Solt has a perpetuity that pays $2153.85 on a fortnightly basis. Interest is calculated at a rate of 4% per annum, compounding fortnightly.

   a. Calculate the amount that Solt invested.

   b. Construct a recurrence relation that can be used to model the value of this perpetuity, V_n, after n fortnights. Round R to 4 decimal places.

6. Peppa has invested her superannuation in a perpetuity. Her super fund offers an interest rate of 7.2% per annum, compounding monthly. She receives a retirement income stream of $4525.62 each month.

   The following table can be used to model this investment over a number of periods.

payment number	payment	interest	principal reduction	balance of investment
20	4525.62			
21				

   a. Calculate the amount that Peppa invested with her super fund.

   b. Fill in the table.

Using financial applications of technology to solve problems involving perpetuities

7. The annual interest rate required to establish a perpetuity with a balance of $64 482 and fortnightly payments of $128.96 is closest to

   A. 0.2% p.a. **B.** 2.0% p.a. **C.** 5.2% p.a. **D.** 6.7% p.a.

8. A perpetuity with an initial investment of $28 500 has an interest rate of 3.16% per annum, compounding monthly. What is the value of the monthly payments?

9. The initial investment in a perpetuity was $1 287 995. The perpetuity pays $1451.27 per fortnight. The interest is compounded fortnightly. What is the perpetuity's annual interest rate, correct to three significant figures?

10. A perpetuity had an initial investment of $323 185. If the interest rate is 2.98% per annum, and interest is compounded quarterly, how much are the quarterly payments?

11. Farah won $236 064 in a lottery and decided to invest it in a perpetuity. She invests 64% of her total lottery winnings. Interest is compounded monthly and she will receive a monthly income stream.

    a. How much was her initial investment?

    b. The annual interest rate of the perpetuity is 3.41% p.a. What are the monthly payments?

    Farah closes the perpetuity after two and a half years.

    c. What is the total interest she received from the perpetuity?

Joining it all together

12. Rebecca has \$245 105 invested in a perpetuity. Her bank offers an interest rate of 4.8% per annum, compounding quarterly.

a. How much does Rebecca receive each quarter from the perpetuity?

b. Construct a recurrence relation that can be used to model the value of Rebecca's annuity, V_n, after n quarters.

c. Calculate the value of the perpertuity after three years.

---

13. The following table shows the value of a perpetuity over a period of time.

payment number	payment	interest	principal reduction	balance of investment
0	0.00	0.00	0.00	135 560.00
1	271.12	271.12	0.00	135 560.00
2	271.12	271.12	0.00	135 560.00

a. Calculate the interest rate per compounding period for this investment.

b. Construct a recurrence relation that can be used to model the value of the annuity, V_n, over n periods.

c. Construct a graph that displays the value of this perpetuity over the first 5 compounding periods.

---

14. Hart wants to have enough superannuation so that, once invested in a perpetuity, he will receive a retirement income stream of \$736.44 per fortnight. His super fund offers an interest rate of 4.2% per annum, compounding fortnightly.

a. How much superannuation does Hart need to have before he can retire?

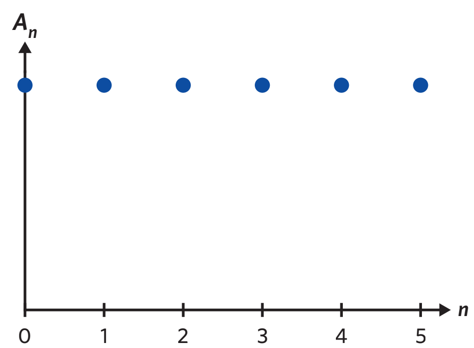
Seven years after Hart retired, his super fund decreased its interest rate to 2.7% per annum, compounding fortnightly.

b. By how much will the fortnightly payments decrease as a result? Use the rounded value found in part **a**.

c. How much interest will Hart have received 20 years after establishing the perpetuity?

Exam practice

15. The following graph represents the value of an annuity investment, A_n, in dollars, after n time periods.



A recurrence relation that could match this graphical representation is

A. $A_0 = 200\ 000, \quad A_{n+1} = 1.015 \times A_n - 2500$

B. $A_0 = 200\ 000, \quad A_{n+1} = 1.025 \times A_n - 5000$

C. $A_0 = 200\ 000, \quad A_{n+1} = 1.03 \times A_n - 5500$

D. $A_0 = 200\ 000, \quad A_{n+1} = 1.04 \times A_n - 6000$

E. $A_0 = 200\ 000, \quad A_{n+1} = 1.05 \times A_n - 8000$

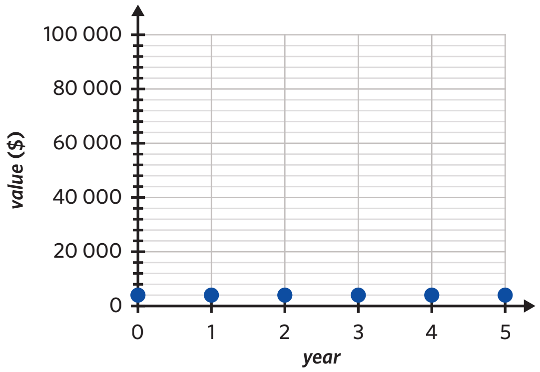
VCAA 2020 Exam 1 Recursion and financial modelling Q25

79% of students answered this question correctly.

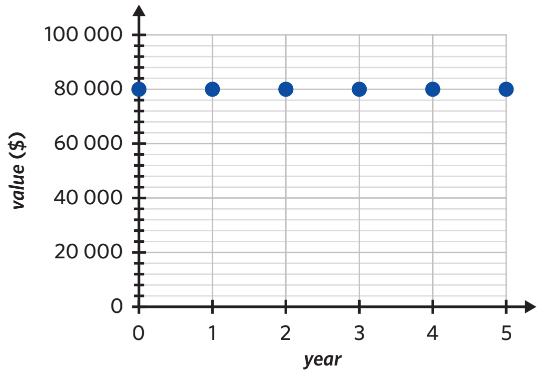
16. Juanita invests \$80 000 in a perpetuity that will provide \$4000 per year to fund a scholarship at a university.

The graph that shows the value of this perpetuity over a period of five years is

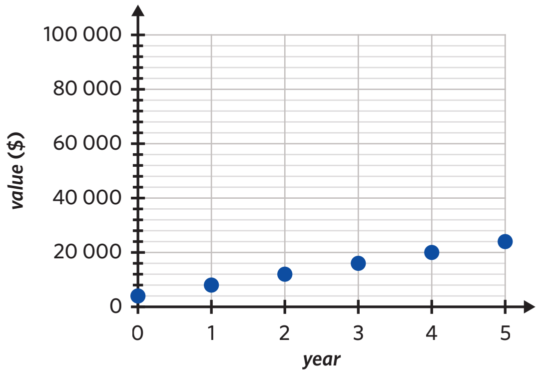
A.



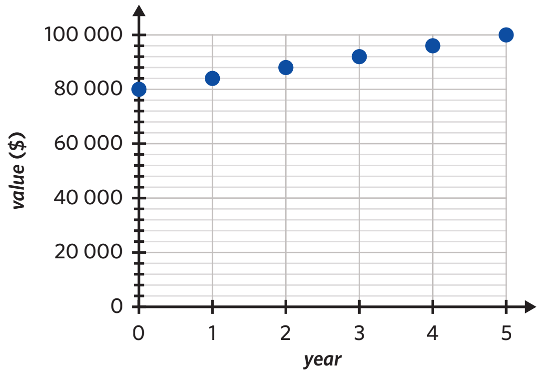
B.



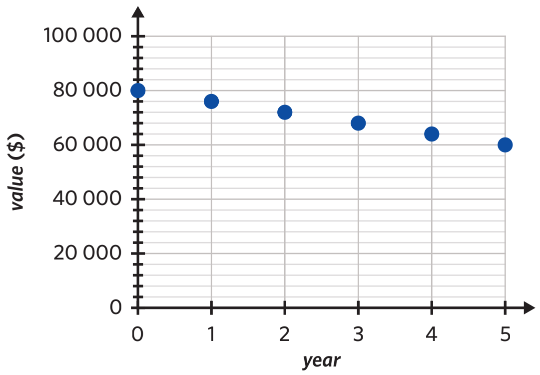
C.



D.



E.



VCAA 2016 Exam 1 Recursion and financial modelling Q21

50% of students answered this question correctly.

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17. Sienna invests \$420 000 in a perpetuity from which she will receive a regular monthly payment of \$1890.

The perpetuity earns interest at the rate of 5.4% per annum.

a. Determine the total amount, in dollars, that Sienna will receive after one year of monthly payments. (1 MARK)

b. Write down the value of the perpetuity after Sienna has received one year of monthly payments. (1 MARK)

c. Let S_n be the value of Sienna's perpetuity after n months.

Complete the recurrence relation, in terms of S_0, S_{n+1} and S_n, that would model the value of the perpetuity over time. (1 MARK)

$S_0 = \boxed{}, \quad S_{n+1} = \boxed{} \times S_n - 1890$

VCAA 2021 Exam 2 Recursion and financial modelling Q6

Part **a**: **59%** of students answered this question correctly.

Part **b**: **47%** of students answered this question correctly.

Part **c**: **38%** of students answered this question correctly.

Questions from multiple lessons

Recursion and financial modelling

18. Josh has a personal loan with a present value of \$12 356.89 that he took out to fund his round-the-world gap year trip.

The interest rate for Josh's loan is 15.7% per annum, compounding quarterly.

His quarterly repayment is \$800.

The loan is to be fully repaid after six years.

Josh knows that the loan cannot be exactly repaid with 24 repayments of \$800.

To solve this problem, Josh will make 23 payments of \$800. He will then adjust the value of the final repayment so that the loan is fully repaid on the 24^{th} payment.

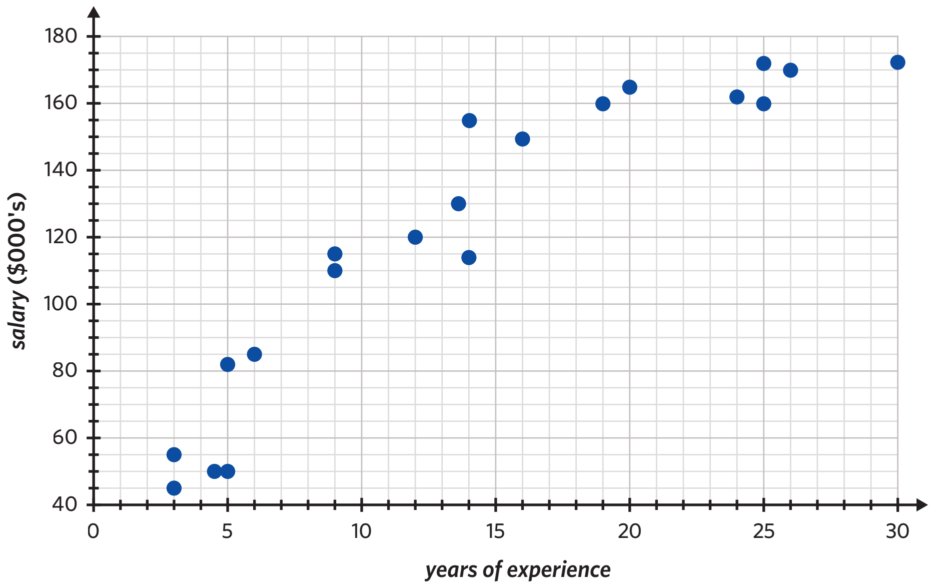
The value of the 24^{th} payment is closest to

A. \$127.64 **B.** \$164.05 **C.** \$635.95 **D.** \$927.64 **E.** \$964.05

Adapted from VCAA 2018 Exam 1 Recursion and financial modelling Q22

Data analysis

19. The following table provides information on *years of experience* and *salary* (\$000's) of 20 workers at a large company. A scatterplot of the data is also shown.



years of experience	*salary* (\$000's)
3	45
4.5	50
5	50
3	55
9	110
5	81
6	85
9	115
14	115
12	120
13.5	130
16	150
14	155
19	160
25	160
24	162
20	165
26	170
30	171
25	171

A log transformation is applied to the variable *years of experience* to linearise the scatterplot.

With *salary* as the response variable, the equation of the least squares line fitted to the linearised data is closest to

A. $salary = -22.98 + 136.82 \times years\ of\ experience$

B. $\log_{10}(salary) = -22.98 + 136.82 \times years\ of\ experience$

C. $salary = -22.98 + 136.82 \times \log_{10}(years\ of\ experience)$

D. $\log_{10}(years\ of\ experience) = -61.93 + 37.22 \times salary$

E. $salary = -61.93 + 37.22 \times \log_{10}(years\ of\ experience)$

Adapted from VCAA 2016 Exam 1 Data analysis Q11

Recursion and financial modelling

20. Denzel borrows $35 000 to buy his girlfriend a car.

He will be charged interest at 5.2% per annum, compounding monthly.

Denzel makes monthly repayments of $400 to repay his loan.

a. How much money does Denzel owe after the first 24 payments? (1 MARK)

b. What is the total interest that Denzel will have paid after 12 repayments? (1 MARK)

After four years, Denzel makes a lump sum payment of $\$L$ to reduce the balance of the loan.
This payment will ensure that his loan is fully repaid in a further two years.

The repayment amount remains at $400 per month, and the interest rate remains at 5.2% per annum, compounding monthly.

c. What is the value of Denzel's lump-sum payment, $\$L$? Round to the nearest dollar. (2 MARKS)

Adapted from VCAA 2016 Exam 2 Recursion and financial modelling Q7

6F Annuity investments

STUDY DESIGN DOT POINTS

- use of a first-order linear recurrence relation to model and analyse (numerically and graphically) annuity investments, including the use of a recurrence relation to determine the value of the investment after n payments have been made for an initial sequence from first principles
- use of a table to investigate and analyse the growth of an annuity investment on a step-by-step basis after each payment is made, the payment made, the interest earned and the balance of the investment
- use of technology with financial modelling functionality to solve problems involving annuity investments, including determining the future value of an investment after a number of compounding periods, the number of compounding periods for the investment to exceed a given value and the interest rate or payment amount needed for an investment to exceed a given value in a given time

6A 6B 6C 6D 6E **6F**

KEY SKILLS

During this lesson, you will be:

- using recurrence relations to model annuity investments
- using tables to solve problems involving annuity investments
- using financial applications of technology to solve problems involving annuity investments.

KEY TERMS

- Annuity investment

An annuity investment occurs when an investor puts away a sum of money into an account that earns compound interest and makes regular additional payments to increase the principal balance. Due to these additional payments, annuity investments tend to grow faster than accounts that only earn compound interest. As such, annuity investments are particularly useful for an investor who wishes to have a sum of money stored away for use later in life.

Using recurrence relations to model annuity investments

An **annuity investment** is a special type of annuity where the investor deposits a fixed sum of money every compounding period in addition to the interest earned on the account. This allows the investment to increase in value. At this point in time, money will not be paid out to the investor until they close the account.

The value of an annuity investment, V_n, after n compounding periods, can be modelled by a recurrence relation of the form

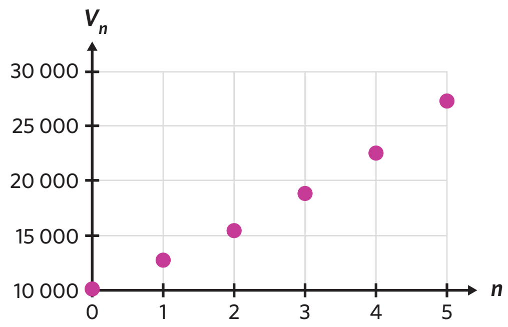
$V_0 = principal, \quad V_{n+1} = R \times V_n + d$, where

- $R = 1 + \frac{r}{100}$, where r is the interest rate (%) per compounding period
- d is the payment made per compounding period

For example, John initially deposited \$10 000 in an annuity that earns 15% interest per annum, compounding annually. He deposits an additional \$1000 to the account at the end of each year. The recurrence relation that can be used to determine V_n, the value of the investment after n years is

$V_0 = 10\ 000, \quad V_{n+1} = 1.15 \times V_n + 1000.$

The balance of the investment over a period of 5 years can also be represented by the following graph.



There is a slight curve in the graph. This is because, as the investment grows, the amount of interest earned increases.

Worked example 1

An annuity investment currently has a balance of \$3600 and earns interest at a rate of 3.6% per annum, compounding monthly. The investor adds an additional \$500 into the account at the end of each month.

a. Construct a recurrence relation that can be used to model the value of the investment, V_n, after n months.

Explanation

Step 1: Determine the principal, interest rate and payment amount.

$V_0 = 3600$

$r = \frac{3.6}{12}\%$ per compounding period

$d = 500$

Step 2: Calculate R.

$$R = 1 + \frac{3.6}{12 \times 100}$$
$$= 1 + \frac{3.6}{1200}$$
$$= 1.003$$

Step 3: Construct the recurrence relation.

Answer

$V_0 = 3600, \quad V_{n+1} = 1.003 \times V_n + 500$

b. How much will the account be worth after three months?

Explanation

Step 1: Calculate V_1, the value of the investment after 1 month.

$V_1 = 1.003 \times V_0 + 500$

$V_1 = 1.003 \times 3600 + 500$

$V_1 = 4110.80$

Step 2: Calculate V_2, the value of the investment after 2 months.

$V_2 = 1.003 \times V_1 + 500$

$V_2 = 1.003 \times 4110.80 + 500$

$V_2 = 4623.132...$

Step 3: Calculate V_3, the value of the investment after 3 months.

$V_3 = 1.003 \times V_2 + 500$

$V_3 = 1.003 \times 4623.132... + 500$

$V_3 = 5137.001...$

Answer

\$5137.00

Using tables to solve problems involving annuity investments

Tables similar to amortisation tables can be used to keep track of the payments added by the investor, and the resultant balance in an annuity investment.

In an annuity investment, the deposited payments and interest earned both contribute to an increase in the principal balance.

The following table models an annuity investment over a period of five years.

payment number	payment	interest	principal addition	balance of investment
0	0.00	0.00	0.00	10 000.00
1	1000.00	1500.00	2500.00	12 500.00
2	1000.00	1875.00	2875.00	15 375.00
3	1000.00	2306.25	3306.25	18 681.25
4	1000.00	2802.19	3802.19	22 483.44
5	1000.00	3372.52	4372.52	26 855.95

The columns of a table for an annuity investment can be calculated using the following formulas:

$$interest = \frac{r}{100} \times previous\ investment\ balance$$

$$principal\ addition = payment + interest$$

$$balance\ of\ investment = previous\ investment\ balance + principal\ addition$$

Worked example 2

The table shown represents an annuity investment with an initial deposit of \$5000 and an interest rate of 5.2% per annum, compounding quarterly. Regular deposits of \$200 are made at the end of each quarter.

payment number	payment	interest	principal addition	balance of investment
0	0.00	0.00	0.00	5000.00
1	**a.**	65.00	265.00	5265.00
2	200.00	68.45	**b.**	5533.45
3	200.00	71.93	271.93	**c.**
4	200.00	**d.**	275.47	6080.85

There are a few cells missing.

a. Determine the missing payment for payment number 1.

Explanation

Determine the payment made to the annuity investment.

Regular deposits of \$200 are made at the end of each quarter.

$payment = 200$

Answer

\$200

Continues →

b. Calculate the missing principal addition for payment number 2.

Explanation

Step 1: Determine the payment and interest.

$payment = 200.00$

$interest = 68.45$

Step 2: Calculate the principal addition.

$$\begin{aligned} principal\ addition &= payment + interest \\ &= 200.00 + 68.45 \\ &= 268.45 \end{aligned}$$

Answer

$268.45

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c. Calculate the missing investment balance after payment number 3.

Explanation

Step 1: Determine the principal addition and previous investment balance.

$previous\ investment\ balance = 5533.45$

$principal\ addition = 271.93$

Step 2: Calculate the investment balance.

$$\begin{aligned} &balance\ of\ investment \\ &= principal\ addition + previous\ investment\ balance \\ &= 5533.45 + 271.93 \\ &= 5805.38 \end{aligned}$$

Answer

$5805.38

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d. Calculate the missing interest for payment number 4.

Explanation

Step 1: Determine the interest rate and previous investment balance.

$r = \frac{5.2}{4}\%$ per compounding period

$previous\ investment\ balance = 5805.38$

Step 2: Calculate the interest.

$$\begin{aligned} interest &= \frac{r}{100} \times previous\ investment\ balance \\ &= \frac{5.2}{4 \times 100} \times 5805.38 \\ &= \frac{5.2}{400} \times 5805.38 \\ &= 75.469... \end{aligned}$$

Answer

$75.47

Using financial applications of technology to solve problems involving annuity investments

The financial solver is a useful tool that can be used to solve problems involving annuity investments.

The present value, PV, is always negative since the investor gives this amount to the bank for investing.

The payment, PMT, is always negative since the investor deposits this amount with the bank every compounding period.

The future value, FV, is always positive as it represents a future payment that will be made to the investor once they close their investment.

Worked example 3

Steve is saving up for a cruise holiday by investing \$10 000 in an account that earns interest at a rate of 3.90% per annum, compounding monthly. He deposits \$500 at the end of each month.

a. After 5 years, how much is in Steve's account?

Explanation

Step 1: Determine the financial solver inputs.

N	60	(there are 60 months in 5 years)
I(%)	3.90	(annual interest rate)
PV	−10 000	(this is negative because Steve invests it with the bank)
PMT	−500	(this is negative because Steve deposits it into the account)
FV		
PpY	12	(payments made monthly)
CpY	12	(interest compounds monthly)

Step 2: Use the financial solver to solve for FV.

FV	45 214.925...

This is positive because it represents a future payment that will be made to Steve once he closes the annuity.

Answer

\$45 214.93

b. How much interest has the account earned after 5 years?

Explanation

Step 1: Calculate the growth of the investment after 5 years.

This growth is the difference between the present value and initial value.

From part **a**, Steve's account is worth \$45 214.93.

$$\begin{aligned} growth &= 45\ 214.93 - 10\ 000 \\ &= 35\ 214.93 \end{aligned}$$

Step 2: Calculate the total amount made in payments during the 5 year period.

$$\begin{aligned} total\ amount &= 500 \times 60 \\ &= 30\ 000 \end{aligned}$$

Step 3: Calculate the interest.

The interest is the difference between the principal growth and the total amount made in payments.

$$\begin{aligned} interest &= 35\ 214.93 - 30\ 000 \\ &= 5214.93 \end{aligned}$$

Answer

\$5214.93

Exam question breakdown

VCAA 2016 Exam 1 Recursion and financial modelling Q23

Sarah invests \$5000 in a savings account that pays interest at the rate of 3.9% per annum, compounding quarterly. At the end of each quarter, immediately after the interest has been paid, she adds \$200 to her investment.

After two years, the value of her investment will be closest to

A. \$5805 **B.** \$6600 **C.** \$7004 **D.** \$7059 **E.** \$9285

Explanation

Step 1: Determine the financial solver inputs.

N	8	(there are 8 quarters in 2 years)
I(%)	3.9	(annual interest rate)
PV	−5000	(this is negative because Sarah invests it with the bank)
PMT	−200	(this is negative because Sarah deposits it into the account)
FV		
PpY	4	(payments made quarterly)
CpY	4	(interest compounds quarterly)

Step 2: Use the financial solver to solve for FV.

FV	7059.249...

This is positive because it represents a future payment that will be made to Sarah once she closes the annuity.

Answer

D

58% of students answered this question correctly.

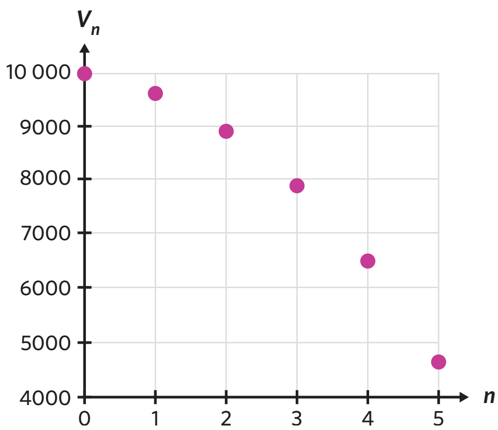
13% of students who chose A did not consider the quarterly compounding periods in their calculations. **12%** of students who chose B calculated the value of the investment as if it was not earning any interest. **11%** of students who chose C would have incorrectly constructed an n^{th} term rule to perform the calculations.

6F Questions

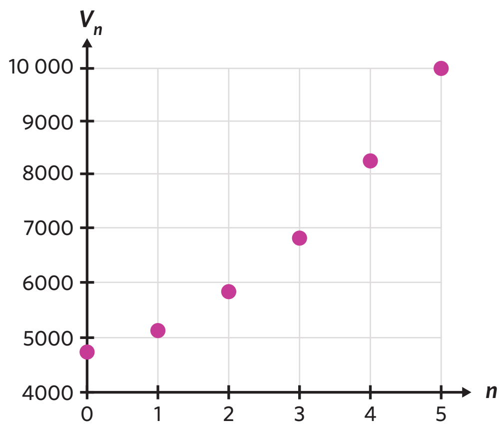
Using recurrence relations to model annuity investments

1. Which of the following graphs is most likely to represent the value of an annuity investment after 5 compounding periods?

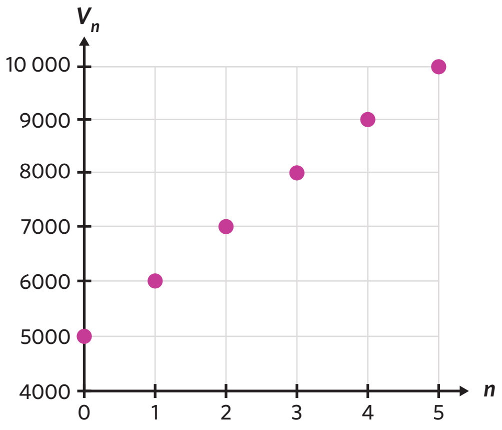
A.



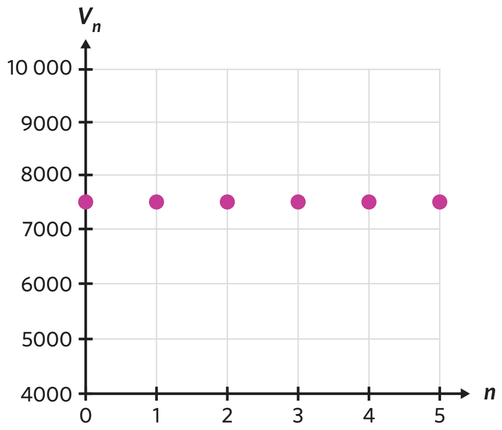
B.



C.



D.



2. Paula's annuity investment can be modelled by the following recurrence relation, where V_n is the value of the account after n months.

$V_0 = 6500, \quad V_{n+1} = 1.0035 \times V_n + 180$

a. How much was the initial investment?

b. How much is Paula depositing into the account every month?

c. What is the annual interest rate?

3. Write a recurrence relation that can be used to model the value of the annuity investments V_n, after n compounding periods, for each of the following scenarios.

a. A principal of \$2000 with a weekly deposit of \$10 made, and an interest rate of 5.2% per annum, compounding weekly.

b. An annuity investment with an annual interest rate of 4.2% per annum, compounding monthly, with an initial deposit of \$500 and additional monthly payments of \$40.

4. Phil wants to predict how much money he will have in his savings account each month. He uses the following recurrence relation to model his account balance from month to month.

$V_0 = 8000, \quad V_{n+1} = 1.0025 \times V_n + 230$

a. Calculate the annual interest rate for this annuity investment.

b. How much would Phil have in his savings account after two months?

c. How many months will it take for Phil's account balance to first reach \$9000?

Using tables to solve problems involving annuity investments

5. The following table can be used to model the value of an annuity investment over a period of five months.

payment number	payment	interest	principal addition	balance of investment
0	0.00	0.00	0.00	15 000.00
1	1000.00	65.00	1065.00	16 065.00
2	1000.00	69.62		17 134.62
3	1000.00	74.25	1074.25	18 208.87
4	1000.00	78.91	1078.91	19 287.78
5	1000.00	83.58	1083.58	20 371.36

The principal addition after the second payment is

A. \$0.00 **B.** \$1065.00 **C.** \$1069.62 **D.** \$1074.25

6. Sheareen has constructed a table to predict the value of her savings for the next six months. Interest compounds monthly and payments are scheduled monthly.

payment number	payment	interest	principal addition	balance of investment
0	0.00	0.00	0.00	100 000.000
1	2500.00	340.00	2840.00	102 840.00
2		349.66	2849.66	105 689.66
3	2500.00	359.34		108 549.00
4	2500.00		2869.07	111 418.07
5	2500.00	378.82	2878.82	
6	2500.00	388.61	2888.61	117 185.50

There are a number of cells missing.

a. Calculate the missing payment value for payment number 2.

b. Calculate the missing principal addition after payment number 3.

c. Calculate the missing interest for payment number 4.

d. Calculate the missing investment balance after payment number 5.

e. Calculate the annual interest rate for this account. Round to three significant figures.

7. An annuity investment has an initial principal of $5600 and an interest rate of 4.1% per annum, compounding monthly. A regular deposit of $200 is made every month.

The following table can be used to model the value of the investment for the first two months.

payment number	payment	interest	principal addition	balance of investment
0	0.00	0.00	0.00	5600.00
1				
2				

Fill in the missing cells.

8. Johnathan puts $104 123 in an annuity investment earning interest at a rate of 1.04% per annum, compounding monthly. He will be adding $1000 to the account each month.

He wants to set up a table that can model the growth of his investment from month to month.

payment number	payment	interest	principal addition	balance of investment
0				
1				
2				
3				

a. Fill in the missing cells.

b. How much interest was earned in the first three months?

c. Explain why the calculated interest is increasing each month.

d. Explain why the principal addition is increasing each month.

Using financial applications of technology to solve problems involving annuity investments

9. Mariana invests $55 000 in an annuity investment which earns interest at a rate of 3.9% per annum, compounding monthly. She deposits a further $450 per month.

After 6 months, the value of the annuity is closest to

A. $53 359.21 **B.** $53 359.22 **C.** $58 803.28 **D.** $58 803.29

10. Jonah needs $17 780 to go on a whale watching trip exactly one year from now. He currently has $8500 that he will put in a savings account earning interest at a rate of 4.1% per annum, compounding monthly. Jonah knows that this alone will not be enough and plans to make monthly deposits into the account.

How much will Jonah have to deposit each month to be able to afford the trip?

11. Jacob wants to have $51 988 saved up for an extended holiday in Dubai. He plans to go in 1.5 years. He deposits $12 000 with his bank and makes regular payments of $1000 to the account each fortnight to save up for the trip. His account earns interest, compounding fortnightly.

Assuming he saves up the exact amount in 1.5 years, what interest rate does he require from the bank? Round your answer to one decimal place.

12. Bel invests $50 000 in an annuity investment and wishes to have enough for a down payment of a house in 5 years. Her dream house is $650 000 and she is required to make a 20% down payment. Her bank currently offers an interest rate of 4.2% per annum, compounding quarterly.

a. How much does the annuity investment need to be worth after the 5 years?

b. How much does Bel need to deposit every quarter in order to have enough for a down payment after 5 years?

Bel is able to add $5000 each quarter to the account instead.

c. How many quarters will it take for her goal to be reached?

13. Rachel wants to purchase a car in four years. She currently has $20 000 which she will put in an annuity investment that earns interest at a rate of 3.4% per annum, compounding fortnightly. She will make regular deposits of $161.02 on a fortnightly basis that will allow her investment to reach the exact value of the car she wishes to purchase.

a. How much is the car worth?

b. How much is the annuity investment worth after two years?

After two years, her bank changes the interest rate and she is required to increase her fortnightly payments to $169.15 in order to reach her goal in the remaining two years. Interest continues to compound fortnightly.

c. What is the new annual interest rate, rounded to two significant figures? Use the rounded values found in parts **a** and **b**.

Joining it all together

14. The following table models an annuity investment from week to week.

payment number	payment	interest	principal addition	balance of investment
0	0.00	0.00	0.00	7000.00
1	30.00	5.81	35.81	7035.81
2	30.00	5.84	35.84	7071.65

a. Calculate the annual interest rate for this annuity investment, rounded to four significant figures.

b. Construct a recurrence relation that can be used to calculate the value of the annuity investment V_n, after n weeks. Round to five decimal places where necessary.

c. How many weeks will it take for the annuity investment to surpass $7200?

15. Ash wants to retire in 30 years with at least $430 000 in savings. He currently has $15 000 which will be used to establish an annuity investment. In addition to this, Ash will need to make equal weekly deposits to ensure that he can retire comfortably. His bank currently offers an interest rate of 2.1% per annum, compounding weekly.

a. What is the minimum amount that Ash needs to deposit each week in order to meet his goals?

b. After 10 years, Ash is offered a pay rise, and therefore is able to increase his weekly payments. His bank continues to offer the same interest rate, compounding weekly. This allows him to retire in the next 15 years.
What is this new payment amount? Use the rounded value found in part **a**.

Exam practice

16. The value of an annuity investment, in dollars, after n years, V_n, can be modelled by the following recurrence relation.

$V_0 = 46\ 000, \quad V_{n+1} = 1.0034 \times V_n + 500$

a. What is the value of the regular payment added to the principal of this annuity investment?

A. $34.00 **B.** $156.40 **C.** $466.00
D. $500.00 **E.** $656.40

b. Between the second and third years, the increase in the value of this investment is closest to

A. $656 **B.** $658 **C.** $661
D. $1315 **E.** $1975

VCAA 2018 Exam 1 Recursion and financial modelling Q17, 18

Part **a**: **79%** of students answered this question correctly.
Part **b**: **61%** of students answered this question correctly.

17. Joanna deposited $12 000 in an investment account earning interest at the rate of 2.8% per annum, compounding monthly.

She would like this account to reach a balance of $25 000 after five years.

To achieve this balance, she will make an extra payment into the account each month, immediately after the interest is calculated.

The minimum value of this payment is closest to

A. $113.85 **B.** $174.11 **C.** $580.16
D. $603.22 **E.** $615.47

VCAA 2021 Exam 1 Recursion and financial modelling Q22

56% of students answered this question correctly.

18. Four lines of a table, similar to an amortisation table, for an annuity investment are shown.

The interest rate for this investment remains constant, but the payment value may vary.

payment number	payment	interest	principal addition	balance of investment
17	100.00	27.40	127.40	6977.50
18	100.00	27.91	127.91	7105.41
19	100.00	28.42	128.42	7233.83
20				7500.00

The balance of the investment after payment number 20 is $7500.

The value of payment number 20 is closest to

A. \$29 **B.** \$100 **C.** \$135
D. \$237 **E.** \$295

VCAA 2017 Exam 1 Recursion and financial modelling Q23

38% of students answered this question correctly.

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19. Alex sold his mechanics' business for \$360 000 and invested this amount in an annuity investment.
This annuity investment earns interest at the rate of 3.8% per annum, compounding monthly.
For the first four years, Alex makes a further payment each month of \$500 to his investment.
This monthly payment is made immediately after the interest is added.
After four years of these regular monthly payments, Alex increases the monthly payment.
This new monthly payment gives Alex a balance of \$500 000 in his annuity after a further two years.
What is the value of Alex's new monthly payment?
Round the answer to the nearest cent. (2 MARKS)

VCAA 2017 Exam 2 Recursion and financial modelling Q7b

The average mark for this question was **0.6**.

Questions from multiple lessons

Recursion and financial modelling

20. The value, R_n, after n years of a reducing balance loan can be modelled by the following recurrence relation.

$R_0 = 100\ 000, \quad R_{n+1} = 1.04 \times R_n - 6500$

What is the balance of the loan after six years?

A. \$81 692.99 **B.** \$83 417.56 **C.** \$86 459.19 **D.** \$87 531.90 **E.** \$89 907.56

Adapted from VCAA 2017 Exam 1 Recursion and financial modelling Q17

Data analysis

21. The number of Instagram followers for a group of Year 11 students is approximately normally distributed.
One of the students, Alejandro, has a standardised number of followers of $z = 1$.
The percentage of students with less Instagram followers than Alejandro is closest to

A. 16% **B.** 34% **C.** 68% **D.** 84% **E.** 98%

Adapted from VCAA 2018 Exam 1 Data analysis Q4

Recursion and financial modelling

22. Archibald has recently moved out of home and opened a savings account to save money for a big couch so that his friends can come over.
The amount of money in the savings account after n years, V_n, can be modelled by the recurrence relation

$V_0 = 1000, \quad V_{n+1} = 1.05 \times V_n$

a. How much money did Archibald initially deposit into the savings account? (1 MARK)

b. Use recursion to show the amount of money that will be in Archibald's savings account after 3 years, V_3, to the nearest cent. (1 MARK)

c. What is the annual percentage compound interest rate for this savings account? (1 MARK)

d. The amount of money in the account after n years, V_n, can also be determined using a rule.

i. Complete the rule by writing the appropriate numbers in the boxes provided. (1 MARK)

$V_n = \boxed{}^{\,n} \times \boxed{}$

ii. How much money, to the nearest cent, will be in Archibald's account after 7 years? (1 MARK)

Adapted from VCAA 2016 Exam 2 Recursion and financial modelling Q5