AOS 2

Matrices

CALCULATOR QUICK LOOK-UP GUIDE

7

CHAPTER 7

Matrices

LESSONS

KEY KNOWLEDGE

- matrix arithmetic: the order of a matrix, types of matrices (row, column, square, diagonal, symmetric, triangular, zero, binary and identity), the transpose of a matrix, and elementary matrix operations (sum, difference, multiplication of a scalar, product and power)
- inverse of a matrix, its determinant, and the condition for a matrix to have an inverse
- use of matrices to represent numerical information presented in tabular form, and the use of a rule for the $a_{ij}{}^{\text{th}}$ element of a matrix to construct the matrix
- binary and permutation matrices, and their properties and applications
- communication and dominance matrices and their use in analysing communication systems and ranking players in round-robin tournaments
- use of the matrix recurrence relation: S_0 = initial state matrix, $S_{n+1} = TS_n$ or $S_{n+1} = LS_n$ where T is a transition matrix, L is a Leslie matrix, and S_n is a column state matrix, to generate a sequence of state matrices (assuming the next state only relies on the current state)
- informal identification of the equilibrium state matrix in the case of regular transition matrices (no noticeable change from one state matrix to the next state matrix)
- use of transition diagrams, their associated transition matrices and state matrices to model the transitions between states in discrete dynamical situations and their application to model and analyse practical situations such as the modelling and analysis of an insect population comprising eggs, juveniles and adults
- use of the matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$ to extend modelling to populations that include culling and restocking.

Image: ChristianChan/Shutterstock.com

7A Introduction to matrices

STUDY DESIGN DOT POINTS

- matrix arithmetic: the order of a matrix, types of matrices (row, column, square, diagonal, symmetric, triangular, zero, binary and identity), the transpose of a matrix, and elementary matrix operations (sum, difference, multiplication of a scalar, product and power)
- use of matrices to represent numerical information presented in tabular form, and the use of a rule for the $a_{ij}{}^{\text{th}}$ element of a matrix to construct the matrix

7A 7B 7C 7D 7E 7F 7G 7H 7I

KEY SKILLS

During this lesson, you will be:

- identifying matrix properties and types
- constructing and interpreting matrices.

KEY TERMS

- Matrix
- Row
- Column
- Order
- Element
- Row matrix
- Column matrix
- Square matrix
- Zero matrix
- Leading diagonal
- Symmetric matrix
- Diagonal matrix
- Upper triangular matrix
- Lower triangular matrix
- Identity matrix

Matrices are useful tools for displaying data. Simple and complex applications can be modelled using matrices. Before this can be conducted, it is important to understand the fundamental properties of matrices.

Identifying matrix properties and types

A **matrix** is a tool for displaying a collection of numerical values that is sorted into rows and columns depending on what it represents.

A **row** is a horizontal list of numbers and is counted from top to bottom.

$$\begin{array}{cc} \begin{bmatrix} 2 & 3 & 7 & 2 \\ 5 & 11 & 3 & 5 \\ 7 & 7 & 8 & 2 \end{bmatrix} & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \end{array}$$

See worked example 1

A **column** is a vertical list of numbers and is counted from left to right.

$$\begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 2 & 3 & 7 & 2 \\ 5 & 11 & 3 & 5 \\ 7 & 7 & 8 & 2 \end{bmatrix} \end{array}$$

The **order** of a matrix describes its dimensions, and is expressed as *number of rows* $\times$ *number of columns*.

For example, if a matrix has three rows and four columns, it is referred to as a 'three-by-four matrix', and is expressed as 3×4.

An **element** is an entry in a matrix. The total number of elements can be found by multiplying the number of rows with the number of columns. Matrices are usually defined with a capital letter, such as A. To refer to a particular element in a matrix, the lowercase letter of the matrix is written, followed by the row and column number written in subscript to the right.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

For matrix A, element a_{mn} refers to the entry in the m^{th} row and n^{th} column.

$$D = \begin{array}{cc} & \begin{array}{cc} 1 & \end{array} \\ & \begin{bmatrix} 3 & 6 \\ 8 & 7 \end{bmatrix} \begin{array}{c} \\ 2 \end{array} \end{array}$$

For matrix D, element d_{21} refers to the entry in the 2nd row and the 1st column, which is 8.

There are various different types of matrices.

See worked example 2

A **row matrix** has only one row and any number of columns.

$[21 \ \ 31]$ and $[8 \ \ 17 \ \ 42 \ \ 52]$

A **column matrix** has only one column and any number of rows.

$$\begin{bmatrix} 42 \\ 56 \\ 74 \end{bmatrix} \text{ and } \begin{bmatrix} 19 \\ 63 \\ 17 \\ 42 \end{bmatrix}$$

A **square matrix** has an equal number of rows and columns.

$$\begin{bmatrix} 46 & 29 \\ 62 & 83 \end{bmatrix} \text{ and } \begin{bmatrix} 11 & 8 & 53 \\ 6 & 98 & 5 \\ 23 & 55 & 72 \end{bmatrix}$$

A **zero matrix** is a matrix where every element is '0'.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The **leading diagonal** is the series of elements in a square matrix that goes from the top left element to the bottom right element.

$$D = \begin{bmatrix} d_{11} & \cdots & 0 & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{mn} \end{bmatrix}$$

A **symmetric matrix** is a square matrix that is symmetric along the leading diagonal.

$$\begin{bmatrix} 3 & 6 & 5 \\ 6 & 0 & 1 \\ 5 & 1 & 2 \end{bmatrix}$$

A **diagonal matrix** is a square, symmetric matrix in which all elements not in the leading diagonal are '0'.

$$\begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix} \text{ and } \begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 32 & 0 & 0 \\ 0 & 0 & 57 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

An **upper triangular matrix** is a square matrix where all elements below the leading diagonal are '0'.

$$\begin{bmatrix} 4 & 5 & 7 \\ 0 & 1 & 6 \\ 0 & 0 & 8 \end{bmatrix}$$

A **lower triangular matrix** is a square matrix where all elements above the leading diagonal are '0'.

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 4 & 2 \end{bmatrix}$$

An **identity matrix** is a square, diagonal matrix in which the leading diagonal consists only of '1', and '0' elsewhere. This matrix is denoted with I.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Worked example 1

Matrix A contains a list of multiple elements.

$$A = \begin{bmatrix} 101 & 2 & 85 \\ 41 & 56 & 73 \end{bmatrix}$$

a. What is the order of matrix A?

Explanation

Step 1: Count the number of rows.

The matrix has 2 rows.

Step 2: Count the number of columns.

The matrix has 3 columns.

Answer

2×3

b. What is element a_{21}?

Explanation

Step 1: Locate the row.

The 2 in a_{21} refers to the second row.

$$\begin{bmatrix} 101 & 2 & 85 \\ 41 & 56 & 73 \end{bmatrix}$$

Step 2: Locate the column.

The 1 in a_{21} refers to the first column.

$$\begin{bmatrix} 101 & 2 & 85 \\ 41 & 56 & 73 \end{bmatrix}$$

Answer

41

Worked example 2

Matrices P, Q, R, S and T are all 3×3 matrices. Each of the following matrices fall into one or more classifications, namely diagonal, identity, symmetric, upper triangular, and lower triangular.

a. How can matrix P be classified?

$$P = \begin{bmatrix} 6 & 5 & 2 \\ 0 & 5 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Explanation

Consider the properties of the matrix.

All elements below the main diagonal are '0'.

Answer

Upper triangular matrix

b. How can matrix Q be classified?

$$Q = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

Continues →

Explanation

Consider the properties of the matrix.

Each element is symmetric across the leading diagonal.

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

Answer

Symmetric matrix

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c. How can matrix R be classified?

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Explanation

Consider the properties of the matrix.

Each element is symmetric across the leading diagonal.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All elements not in the leading diagonal are ‘0’.

It is only ‘1’ on the leading diagonal and ‘0’ elsewhere.

All elements below the main diagonal are ‘0’.

All elements above the main diagonal are ‘0’.

Answer

Symmetric, diagonal, identity, upper triangular, lower triangular matrix

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d. How can matrix S be classified?

$$S = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Explanation

Consider the properties of the matrix.

Each element is symmetric across the leading diagonal.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

All elements not in the leading diagonal are ‘0’.

All elements below the main diagonal are ‘0’.

All elements above the main diagonal are ‘0’.

Answer

Symmetric, diagonal, upper triangular, lower triangular matrix

Continues →

e. How can matrix T be classified?

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 11 & 2 & 0 \\ 2 & 5 & 8 \end{bmatrix}$$

Explanation

Consider the properties of the matrix.

All elements above the main diagonal are '0'.

Answer

Lower triangular matrix

Constructing and interpreting matrices

See worked example 3

Matrices can be used to represent numerical information. In most cases, storing data in a matrix makes it easier to visualise and manipulate. For example, the following table is used to represent the number of days with snow (S) and without snow (N) in July (J) and August (A).

	July (J)	August (A)
number of days with snow (S)	24	17
number of days without snow (N)	7	13

A matrix can be constructed that represents the information in the table.

$$\begin{array}{cc} \begin{array}{cc} J & A \end{array} & \\ \begin{bmatrix} 24 & 17 \\ 7 & 13 \end{bmatrix} & \begin{array}{c} S \\ N \end{array} \end{array}$$

See worked example 4

Matrices can also be constructed using element rules. These element rules often use the row and column number of a particular element, as part of the rule.

For example, matrix A is a 2×2 matrix with the element rule $a_{ij} = i + j$, where a_{ij} is the element in the i^{th} row and j^{th} column. The element rule can be applied to each element in A to give the following.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad A = \begin{bmatrix} 1+1 & 1+2 \\ 2+1 & 2+2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Worked example 3

A group of 100 people were surveyed about their favourite fruit. Their results are shown in the table.

	favourite fruit		
	banana	**dragon fruit**	**kiwi fruit**
adult	14	19	16
child	23	11	17

a. Construct matrix F, to represent the information in the table.

Explanation

Step 1: Set up an empty matrix.

There are 2 rows and 3 columns of data presented in the table. As such, the order of the matrix should be 2×3. Rows can be labelled A and C for 'adult' and 'child' respectively. Columns can be labelled B, D and K for banana, dragon fruit and kiwifruit respectively.

$$F = \begin{array}{cc} \begin{array}{ccc} B & D & K \end{array} & \\ \begin{bmatrix} & & \\ & & \end{bmatrix} & \begin{array}{c} A \\ C \end{array} \end{array}$$

Continues →

Step 2: Fill in the first row.

Element f_{11} corresponds to the number of adults who prefer bananas, which is 14.

Element f_{12} corresponds to the number of adults who prefer dragon fruit, which is 19.

Element f_{13} corresponds to the number of adults who prefer kiwifruit which is 16

$$F = \begin{array}{c} \begin{array}{ccc} B & D & K \end{array} \\ \begin{bmatrix} 14 & 19 & 16 \\ & & \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ C \end{array}$$

Step 3: Repeat for the second row.

This row corresponds to the favourite fruit of children.

Answer

$$F = \begin{array}{c} \begin{array}{ccc} B & D & K \end{array} \\ \begin{bmatrix} 14 & 19 & 16 \\ 23 & 11 & 17 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ C \end{array}$$

b. Construct a row matrix to represent the favourite fruit of adults surveyed.

Explanation

Step 1: Identify the row on the table which corresponds to the favourite fruits of adults.

The first row on the table represents the favourite fruit of adults.

Step 2: Construct a row matrix representing the data.

A row matrix will only have one row. As there are three fruits listed on the table, the row matrix will have three columns.

Answer

$\begin{bmatrix} 14 & 19 & 16 \end{bmatrix}$

c. Interpret the sum of the matrix constructed in **b**.

Explanation

Step 1: Sum the elements in the matrix.

$14 + 19 + 16 = 49$

Step 2: Interpret the sum.

The matrix constructed in part **b** corresponds to the adults surveyed and their favourite fruits.

Answer

The total number of adults surveyed, which was 49.

Worked example 4

Matrix C is a matrix with an order of 3×2.

Construct matrix C, with the element rule $c_{ij} = i \times j$, where c_{ij} refers to the entry in the i^{th} row and j^{th} column.

Explanation

Step 1: Set up the matrix.

$$C = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

Step 2: Calculate the matrix elements.

$$C = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 \times 1 & 1 \times 2 \\ 2 \times 1 & 2 \times 2 \\ 3 \times 1 & 3 \times 2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

Continues →

Answer

$C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$

Exam question breakdown

VCAA 2019 Exam 1 Matrices Q3

Consider the matrix P, where $P = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$.

The element in row i and column j of matrix P is p_{ij}.

The elements in matrix P are determined by the rule

A. $p_{ij} = 4 - j$ **B.** $p_{ij} = 2i + 1$ **C.** $p_{ij} = i + j + 1$ **D.** $p_{ij} = i + 2j$ **E.** $p_{ij} = 2i - j + 2$

Explanation

Step 1: Set up the matrix.

$$P = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Step 2: Calculate the matrix elements for each option and compare it P.

A:

$$P_A = \begin{bmatrix} 4-1 & 4-2 & 4-3 \\ 4-1 & 4-2 & 4-3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$P_A \neq P$ ✗

B:

$$P_B = \begin{bmatrix} 2 \times 1 + 1 & 2 \times 1 + 1 & 2 \times 1 + 1 \\ 2 \times 2 + 1 & 2 \times 2 + 1 & 2 \times 2 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 3 \\ 5 & 5 & 5 \end{bmatrix}$$

$P_B \neq P$ ✗

C:

$$P_C = \begin{bmatrix} 1+1+1 & 1+2+1 & 1+3+1 \\ 2+1+1 & 2+2+1 & 2+3+1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

$P_C \neq P$ ✗

D:

$$P_D = \begin{bmatrix} 1 + 2 \times 1 & 1 + 2 \times 2 & 1 + 2 \times 3 \\ 2 + 2 \times 1 & 2 + 2 \times 2 & 2 + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$$

$P_D \neq P$ ✗

E:

$$P_E = \begin{bmatrix} 2 \times 1 - 1 + 2 & 2 \times 1 - 2 + 2 & 2 \times 1 - 3 + 2 \\ 2 \times 2 - 1 + 2 & 2 \times 2 - 2 + 2 & 2 \times 2 - 3 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$$

$P_E = P$ ✓

59% of students answered this question correctly.

A total of **34%** of students incorrectly selected B, C and D. This is likely because they had only worked through a select number of elements in matrix P. In such questions, it is important to spend the time to calculate each matrix element individually.

Answer

E

7A Questions

Identifying matrix properties and types

1. What matrix has an order of 2×3?

A. $\begin{bmatrix} 3 & 7 & 5 \\ 1 & 8 & 9 \end{bmatrix}$ **B.** $\begin{bmatrix} 2 & 5 & 2 & 9 \\ 0 & 8 & 6 & 4 \\ 6 & 8 & 6 & 3 \\ 6 & 6 & 2 & 5 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 & 4 \\ 7 & 2 \\ 3 & 5 \end{bmatrix}$ **D.** $\begin{bmatrix} 5 & 2 & 1 \\ 1 & 3 & 9 \\ 8 & 1 & 3 \end{bmatrix}$

2. Consider matrix B.

$$B = \begin{bmatrix} 5 & 2 \\ 1 & 3 \\ 8 & 1 \\ 1 & 7 \end{bmatrix}$$

a. How many rows and columns can be found in B?

b. What is the order of B?

c. What entry corresponds with b_{31}?

3. A, B, C and D are all matrices of different orders.

$$A = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 & 7 \\ 1 & 3 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 7 & 4 \\ 4 & 8 & 0 \\ 12 & 5 & 7 \end{bmatrix} \quad D = \begin{bmatrix} 11 & 77 & 72 & 24 \\ 46 & 26 & 89 & 23 \end{bmatrix}$$

a. What is the order of A?

b. What is the order of B?

c. What is the order of C?

d. What is the order of D?

4. The following matrix is a 3×5 matrix.

$$D = \begin{bmatrix} 1 & 89 & 67 & 4 & 111 \\ 32 & 4 & 46 & 53 & 72 \\ 74 & 3 & 67 & 12 & 47 \end{bmatrix}$$

a. What entry corresponds to d_{23}?

b. What entry corresponds to d_{35}?

c. What entry corresponds to d_{15}?

d. Which entry corresponds to d_{24}?

5. M, N, O, P, and Q are all 3×3 matrices.

$$M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 9 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad Q = \begin{bmatrix} 7 & 0 & 0 \\ 1 & 5 & 0 \\ 3 & 5 & 8 \end{bmatrix}$$

Identify the matrices that can be classified as

a. an upper triangular matrix.

b. a diagonal matrix.

c. the identity matrix.

d. a symmetric matrix.

Constructing and interpreting matrices

6. A group of 100 people were surveyed about their favourite pets. The results are shown in the following table.

	favourite pets		
	cats	**dogs**	**fish**
class A	18	12	16
class B	19	20	15

Choose the matrix that correctly represents the information in the table.

A. $\begin{array}{cc} \text{B} & \\ \begin{bmatrix} 19 \\ 15 \\ 20 \end{bmatrix} & \begin{array}{c} \text{C} \\ \text{D} \\ \text{F} \end{array} \end{array}$

B. $\begin{array}{cc} \begin{array}{ccc} \text{C} & \text{D} & \text{F} \end{array} & \\ \begin{bmatrix} 18 & 16 & 12 \end{bmatrix} & \text{A} \end{array}$

C. $\begin{array}{cc} \begin{array}{ccc} \text{C} & \text{D} & \text{F} \end{array} & \\ \begin{bmatrix} 18 & 12 & 16 \\ 19 & 20 & 15 \end{bmatrix} & \begin{array}{c} \text{A} \\ \text{B} \end{array} \end{array}$

D. $\begin{array}{cc} \begin{array}{cc} \text{A} & \text{B} \end{array} & \\ \begin{bmatrix} 19 & 18 \\ 12 & 20 \\ 16 & 15 \end{bmatrix} & \begin{array}{c} \text{C} \\ \text{D} \\ \text{F} \end{array} \end{array}$

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7. The following table and matrix both represent the number of residents and cats in three different households.

	number of residents	*number of cats*
household 1	5	2
household 2	3	1
household 3	1	9

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 1 & 9 \end{bmatrix}$$

a. What does element a_{21} represent?

b. What does element a_{32} represent?

c. What does column 1 of matrix A represent?

d. What does column 2 of matrix A represent?

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8. A group of children were surveyed on their favourite animal between flamingoes (F), porcupines (P) and reindeer (R). The results are shown in the given table.

	favourite animal		
	flamingoes	**porcupines**	**reindeer**
boys	15	8	9
girls	13	14	5

a. Convert the table into a matrix with the following form.

$$\begin{array}{cc} \begin{array}{ccc} \text{F} & \text{P} & \text{R} \end{array} & \\ \begin{bmatrix} & & \\ & & \end{bmatrix} & \begin{array}{c} \text{boys} \\ \text{girls} \end{array} \end{array}$$

b. What does the entry in the second row and second column of the matrix represent?

c. What does the sum of all elements in the first row of the matrix represent?

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9. Construct a matrix using the given order and element rule, where a_{ij} is the element in the i^{th} row and j^{th} column.

a. 2×2 and $a_{ij} = 2i + 2j$

b. 3×2 and $a_{ij} = 2 \times i \times j$

c. 2×4 and $a_{ij} = (i + j)^2$

Joining it all together

10. Kyle noted how many scoops of each ice cream flavour he sold over three days. The results are in the table shown.

	day 1	day 2	day 3
cookies and cream	23	19	28
salted caramel	13	20	9
strawberry	14	11	26

a. Construct matrix F, a square matrix that represents the number of scoops of each flavour sold each day.

b. What does the sum of all elements in matrix F represent?

c. Construct matrix G, a column matrix that represents the number of scoops of each flavour sold on day 2.

d. What does the sum of all elements in matrix G represent?

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11. Matrix W shows the total amount of water stored, in litres, in a water tank from Monday to Thursday.

$$\begin{array}{cc} & \begin{array}{cccc} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thu} \end{array} \\ W = & \begin{bmatrix} 300 & 600 & 900 & 1200 \end{bmatrix} \end{array}$$

a. What is the order of this matrix?

b. What type of matrix is this?

c. What does w_{13} represent?

Exam practice

12. A toll road is divided into three sections E, F and G. The *cost*, in dollars to drive one journey on each section is shown in matrix C.

$$C = \begin{array}{c} \begin{bmatrix} 3.58 \\ 2.22 \\ 2.87 \end{bmatrix} \end{array} \begin{array}{c} \text{E} \\ \text{F} \\ \text{G} \end{array}$$

a. What is the cost of one journey on section G? (1 MARK)

b. Write down the order of matrix C. (1 MARK)

c. One day, Kim travels once on section E and twice in section G. Construct a row matrix that shows this. (1 MARK)

VCAA 2018 Exam 2 Matrices Q1

Part **a**: **99%** of students answered this question correctly.
Part **b**: **94%** of students answered this question correctly.
Part **c**: **66%** of students answered this question correctly.

Questions from multiple lessons

Data analysis

13. Data was collected on how employees commuted to work, to investigate the association between the two following variables.

- *rides a bicycle to work* (yes, no)
- *distance from office* (under 2 km, 2–10 km, over 10 km)

Which one of the following is appropriate to use in the statistical analysis of this association?

A. Back to back stem plot
B. Segmented bar chart
C. The coefficient of determination
D. Residual plot
E. Parallel boxplots

Adapted from VCAA 2018 Exam 1 Data analysis Q6

Recursion and financial modelling

14. The value of an investment, in dollars, after n years, V_n , can be modelled by the recurrence relation shown.

$V_0 = 32\ 000, \quad V_{n+1} = 1.0038V_n + 350$

What is the value of the regular payment added to the principal of this investment?

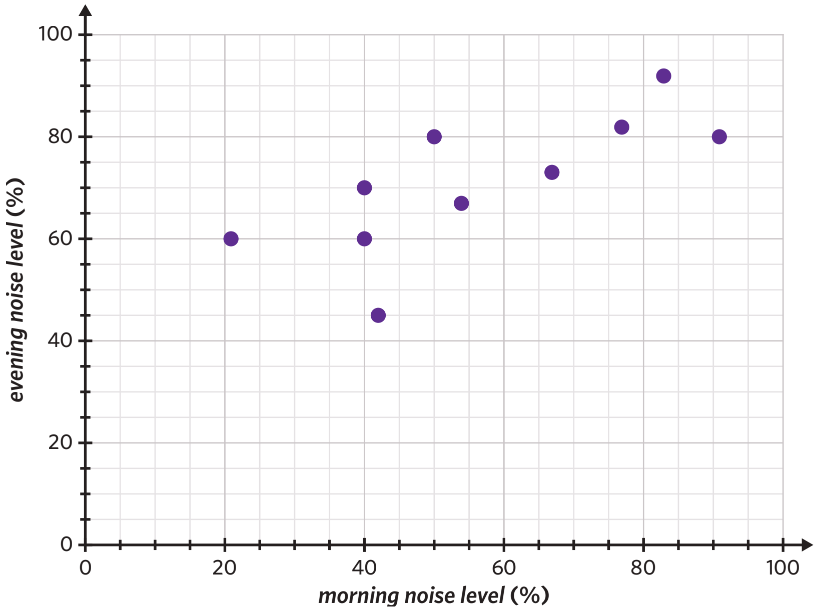
A. \$38.00 **B.** \$350.00 **C.** \$32 000.00 **D.** \$325.00 **E.** \$700.00

Adapted from VCAA 2018 Exam 1 Recursion and financial modelling Q17

Data analysis

15. The amount of noise in a suburb can be recorded as a percentage of the maximum amount of noise allowed by the council. This is called the percentage noise level.

The percentage noise levels for the morning and evening peak periods for 10 large suburbs are plotted on the following scatterplot.



A least squares line is to be fitted to the data with the aim of predicting evening noise level from the morning noise level.

The equation of this line is

evening noise level $= 45.3 + 0.45 \times$ *morning noise level*

a. Use the equation of the least squares line to predict the evening noise level when the morning noise level is 75%. Round to one decimal place. (1 MARK)

b. Determine the residual value when the equation of the least squares line is used to predict the evening noise level when the morning noise level is 50%. Round to one decimal place. (2 MARKS)

c. The value of the correlation coefficient r is 0.74. What percentage of the variation in the evening noise level can be explained by the variation in the morning noise level? Round to the nearest percent. (1 MARK)

Adapted from VCAA 2018 Exam 2 Data analysis Q2

7B Operations with matrices

STUDY DESIGN DOT POINT

- matrix arithmetic: the order of a matrix, types of matrices (row, column, square, diagonal, symmetric, triangular, zero, binary and identity), the transpose of a matrix, and elementary matrix operations (sum, difference, multiplication of a scalar, product and power)

7A 7B 7C 7D 7E 7F 7G 7H 7I

KEY SKILLS

During this lesson, you will be:

- adding and subtracting matrices
- multiplying matrices by a scalar
- determining the transpose of a matrix.

KEY TERMS

- Scalar multiplication
- Transpose

Once matrices have been created, it can be helpful to perform operations on them in order to make calculations with the data. It is important to know how each of the basic operations differ when performing them with matrices rather than single values.

Adding and subtracting matrices

Matrix addition or subtraction is only defined if the matrices are of the same order. If they are not, then the solution is undefined.

When adding or subtracting matrices, elements in the same position are added or subtracted from each other.

Matrix addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} a+e & b+g \\ c+f & d+h \end{bmatrix}$$

Matrix subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} a-e & b-g \\ c-f & d-h \end{bmatrix}$$

Worked example 1

Consider the matrices $A = \begin{bmatrix} 2 & 2 \\ 1 & 4 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 0 & 2 \\ 4 & 5 \end{bmatrix}$.

a. Is $A + B$ defined?

Explanation

Step 1: Identify the order of each matrix.

Matrix A has three rows and two columns. Its order is 3×2.

Matrix B has three rows and two columns. Its order is 3×2.

Step 2: Determine whether the matrix sum is defined.

The matrices are of the same order.

Hence, the matrix sum is defined.

Answer

Yes

Continues →

b. If possible, calculate $A - B$.

Explanation – Method 1: By hand

Step 1: Write down the calculation.

$$A - B = \begin{bmatrix} 2 & 2 \\ 1 & 4 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 0 & 2 \\ 4 & 5 \end{bmatrix}$$

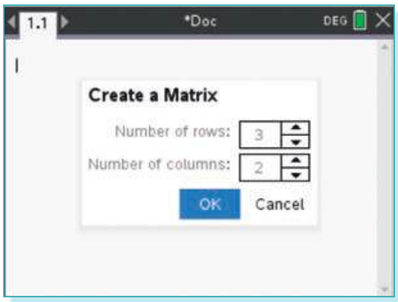
Step 2: Subtract each element in matrix B from its corresponding element in matrix A.

$$= \begin{bmatrix} 2-4 & 2-1 \\ 1-0 & 4-2 \\ 5-4 & 3-5 \end{bmatrix}$$

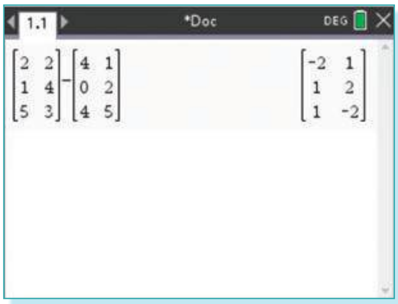
Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

Step 2: Press and select . On the settings window, set 'Number of rows' as 3 and 'Number of columns' as 2. Select 'OK'. Enter the values for matrix A.



Step 3: Type '-' and then repeat step 2 for matrix B. Press enter to calculate.

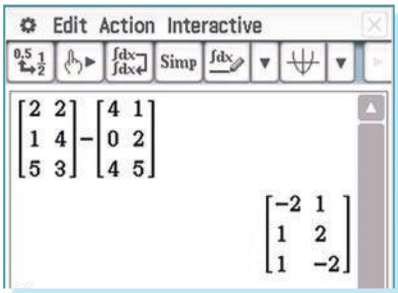


Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap Main.

Step 2: Press keyboard and tap Math2. Tap to create a matrix, and to add an extra row. Enter the values for matrix A.

Step 3: Type '-' and then repeat step 2 for matrix B. Press EXE to calculate.



Answer – Method 1, 2 and 3

$$\begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}$$

Multiplying matrices by a scalar

Scalar multiplication refers to multiplying a matrix by a number (the scalar). Each element in the matrix is multiplied by the scalar.

$$k \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} k \times a & k \times b \\ k \times c & k \times d \end{bmatrix}$$

Worked example 2

Given that $C = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 2 \\ 1 & 0 & 4 \end{bmatrix}$, calculate $3C$.

Explanation – Method 1: By hand

Multiply each individual element by 3.

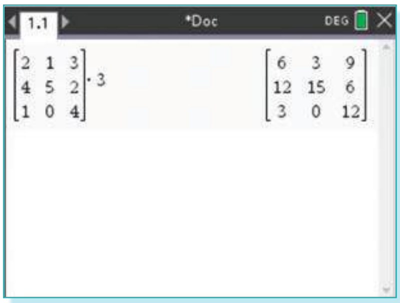
$$3 \times \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 2 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 \times 2 & 3 \times 1 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 2 \\ 3 \times 1 & 3 \times 0 & 3 \times 4 \end{bmatrix}$$

Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

Step 2: Press and select . Select 'OK'. Enter the values for matrix C.

Step 3: Type '× 3' and press enter to calculate.

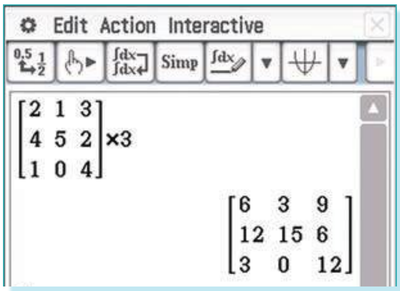


Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Press keyboard and tap Math2. Tap to create a matrix, to add an extra row, and to add an extra column. Enter the values for matrix C.

Step 3: Type '× 3' and press EXE to calculate.



Answer – Method 1, 2 and 3

$$\begin{bmatrix} 6 & 3 & 9 \\ 12 & 15 & 6 \\ 3 & 0 & 12 \end{bmatrix}$$

Determining the transpose of a matrix

The **transpose** of a matrix can be determined by swapping its rows and columns.

The transpose of matrix A is denoted A^T.

If the order of matrix A is 2×4, the order of its transpose A^T will be 4×2.

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}^T = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}$$

Worked example 3

If $D = \begin{bmatrix} 1 & 0 & 9 \\ 5 & 2 & 3 \end{bmatrix}$, determine D^T.

Explanation – Method 1: By hand

Step 1: Determine the order of D^T.

The order of matrix D is 2×3.

Hence, the order of matrix D^T will be 3×2.

Step 2: Copy the values from the first row of D into the first column of D^T.

$$D^T = \begin{bmatrix} 1 & \\ 0 & \\ 9 & \end{bmatrix}$$

Step 3: Copy the values from the second row of D into the second column of D^T.

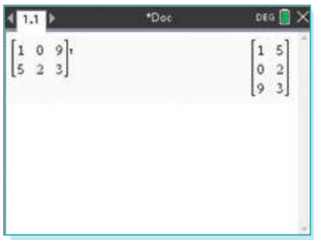
$$D^T = \begin{bmatrix} 1 & 5 \\ 0 & 2 \\ 9 & 3 \end{bmatrix}$$

Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

Step 2: Press and select . On the settings window, set 'Number of rows' as 2 and 'Number of columns' as 3. Select 'OK'. Enter the values for matrix D.

Step 3: Press menu and select '7: Matrices' → '2: Transpose'. Press enter.

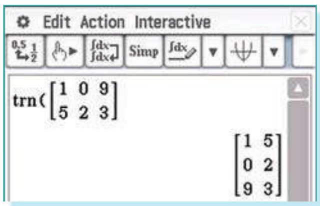


Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap Main.

Step 2: Tap 'Action' → 'Matrix' → 'Create' → 'trn'.

Step 3: Press keyboard and tap Math2. Tap to create a matrix and to add an extra column. Enter the values for matrix D. Press EXE.



Answer – Method 1, 2 and 3

$$\begin{bmatrix} 1 & 5 \\ 0 & 2 \\ 9 & 3 \end{bmatrix}$$

Exam question breakdown

VCAA 2021 Exam 1 Matrices Q1

If matrix $M = \begin{bmatrix} 3 & 2 \\ 8 & 9 \\ 13 & 7 \end{bmatrix}$, then its transpose, M^T, is

A. $\begin{bmatrix} 2 & 3 \\ 9 & 8 \\ 7 & 13 \end{bmatrix}$ **B.** $\begin{bmatrix} 2 & 9 & 7 \\ 3 & 8 & 13 \end{bmatrix}$ **C.** $\begin{bmatrix} 7 & 9 & 2 \\ 13 & 8 & 3 \end{bmatrix}$ **D.** $\begin{bmatrix} 3 & 8 & 13 \\ 2 & 9 & 7 \end{bmatrix}$ **E.** $\begin{bmatrix} 13 & 8 & 3 \\ 7 & 9 & 2 \end{bmatrix}$

Explanation – Method 1: By hand

Step 1: Determine the order of M^T.

The order of matrix M is 3×2.

Hence, the order of matrix M^T will be 2×3.

Step 2: Transpose matrix M by swapping its rows and columns.

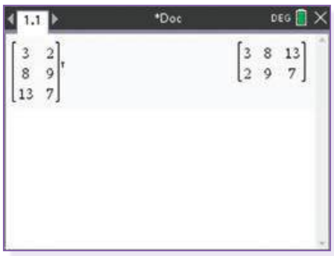
$$M^T = \begin{bmatrix} 3 & 8 & 13 \\ 2 & 9 & 7 \end{bmatrix}$$

Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

Step 2: Press [template key] and select [matrix template]. On the settings window, set 'Number of rows' as 3 and 'Number of columns' as 2. Select 'OK'. Enter the values for matrix M.

Step 3: Press menu and select '7: Matrices' → '2: Transpose'. Press enter.

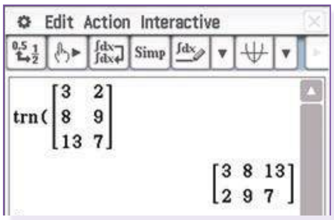


Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Tap 'Action' → 'Matrix' → 'Create' → 'trn'.

Step 3: Press keyboard and tap Math2. Tap [matrix icon] to create a matrix and [row icon] to add an extra row. Enter the values for matrix M. Press EXE.



Answer – Method 1, 2 and 3

D

82% of students answered this question correctly.

7B Questions

Adding and subtracting matrices

1. Which of the following matrix operations is defined?

A. $\begin{bmatrix} 4 & 1 & 9 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 8 & 9 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 11 & 6 & 15 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 4 \\ 12 & 19 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 20 & -1 \\ 6 & 7 \\ 8 & 15 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 9 & 5 \\ 3 & 6 & 1 \\ 5 & 3 & 7 \end{bmatrix}$

2. For each of the following matrix operations, determine if the matrix operation is defined. If the operation is defined, evaluate.

a. $\begin{bmatrix} 5 & -9 & 7 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 6 & -3 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 1 & 7 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ 5 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 3 & 8 \\ 8 & 5 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 6 & 4 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 9 & 5 \\ -13 & 8 \\ 4 & -3 \\ 11 & 7 \end{bmatrix} - \begin{bmatrix} 10 & 11 \\ 4 & -2 \\ 1 & 8 \\ 9 & 7 \end{bmatrix}$

3. Simon and Darren are both part-time Uber drivers who drive on Mondays, Wednesdays, and Fridays. The average number of passengers driven by each of them on each day are shown in the following table.

day	Simon	Darren
Mon	10	9
Wed	8	8
Fri	15	12

a. Construct two 3×1 matrices, S and D, that represent the average number of passengers driven each day by Simon and Darren respectively.

b. Calculate $S - D$. What does this matrix represent?

Their colleague Theodora works on Mondays, Wednesdays, Fridays, and Saturdays. Her average number of passengers each day is shown.

day	Theodora
Mon	14
Wed	11
Fri	16
Sat	20

c. Construct a 4×1 matrix, T, to represent this information.

d. Is it possible to evaluate $S + D + T$ to find the total number of passengers they drove each day? Explain why.

Multiplying matrices by a scalar

4. If $A = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$, $3A$ is

A. $\begin{bmatrix} 54 \end{bmatrix}$

B. $\begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}$

C. $\begin{bmatrix} 12 \\ 14 \\ 16 \end{bmatrix}$

D. $\begin{bmatrix} 12 \\ 18 \\ 24 \end{bmatrix}$

5. Calculate each of the following scalar multiplications.

a. $2\begin{bmatrix} 3 & 6 & 1 & 9 & 5 \end{bmatrix}$

b. $4\begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 4 \end{bmatrix}$

c. $-3\begin{bmatrix} 12 \\ -7 \\ -4 \\ 9 \\ 8 \\ 11 \end{bmatrix}$

d. $1.5\begin{bmatrix} 6 & 10 & 5 \\ 4 & -3 & -20 \\ -1 & 8 & 2 \end{bmatrix}$

6. The prices of men's and women's garments at a clothing store are displayed in matrix P.

$$P = \begin{array}{c} \begin{array}{cc} \text{men's} & \text{women's} \end{array} \\ \begin{bmatrix} 40 & 45 \\ 80 & 90 \\ 100 & 120 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{t-shirt} \\ \text{hoodie} \\ \text{jeans} \end{array}$$

This week, the store is offering a 20% discount on all their stock.

The discounted prices can be calculated by multiplying matrix P by a scalar, k.

a. What is the value of the scalar, k?

b. Calculate $k \times P$.

c. What is the discounted price of a women's hoodie?

Determining the transpose of a matrix

7. Which of the following is the transpose of the matrix $[a \quad b \quad c \quad d]$?

A. $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

B. $\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$

C. $[d \quad c \quad b \quad a]$

D. $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$

8. Determine the transpose of each of the following matrices.

a. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 5 \\ 9 & 12 \end{bmatrix}$

c. $\begin{bmatrix} -7 & 6 \\ 13 & 5 \\ -1 & 15 \end{bmatrix}$

d. $\begin{bmatrix} 8 & 9 & 12 \\ -11 & 1 & 5 \\ 4 & 0 & -2 \end{bmatrix}$

9. Gabrielle owns a patisserie franchise with two locations in Melbourne.

Her Footscray store's sales data over 3 days for 3 of their most popular pastries are shown in matrix F.

$$F = \begin{array}{c} \begin{array}{ccc} \text{Mon} & \text{Tue} & \text{Wed} \end{array} \\ \begin{bmatrix} 34 & 28 & 40 \\ 20 & 15 & 29 \\ 41 & 35 & 37 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{almond croissant} \\ \text{brownie} \\ \text{croissant} \end{array}$$

Her Carlton store also recorded their sales data, but displayed it in the opposite way, in matrix C.

$$C = \begin{array}{c} \begin{array}{ccc} \text{almond croissant} & \text{brownie} & \text{croissant} \end{array} \\ \begin{bmatrix} 51 & 33 & 60 \\ 38 & 21 & 30 \\ 50 & 26 & 62 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Mon} \\ \text{Tue} \\ \text{Wed} \end{array}$$

Display the Carlton store's sales information in matrix C^T, in the same format as the Footscray store's.

Joining it all together

10. For each of the following matrix expressions, determine if the matrix operation is defined. If the operation is defined, evaluate.

a. $2\begin{bmatrix} 11 & 1 & 20 \\ 6 & 8 & 10 \\ -9 & 13 & 4 \\ 4 & 2 & -6 \end{bmatrix} - 3\begin{bmatrix} 8 & -5 & 1 & -6 \\ 7 & -4 & 3 & -7 \\ 6 & -3 & 5 & -8 \end{bmatrix}$

b. $\begin{bmatrix} 12 & 11 & 5 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -9 & 10 & -1 & 2 \end{bmatrix}^T$

c. $\left(\begin{bmatrix} 12 & 11 & 5 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -9 & 10 & -1 & 2 \end{bmatrix}\right)^T$

d. $\begin{bmatrix} 5 & 11 \\ 9 & 7 \\ -10 & 2 \end{bmatrix} - 2\begin{bmatrix} 3 & -5 & 8 \\ 6 & 4 & 1 \end{bmatrix}^T$

---

11. Determine the values of x and y in the following matrix equations.

a. $\begin{bmatrix} 5 & 10 & 11 \\ 9 & x & -8 \\ -3 & 7 & 5 \end{bmatrix} - \begin{bmatrix} 15 & 9 & 13 \\ 4 & -4 & -2 \\ 8 & 2 & y \end{bmatrix} = \begin{bmatrix} -10 & 1 & -3 \\ 5 & 2 & -6 \\ -11 & 5 & -1 \end{bmatrix}$

b. $4\begin{bmatrix} 1 \\ 7 \\ -9 \\ -3 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 6 \\ x \\ 7 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \\ -22 \\ y \\ 18 \end{bmatrix}$

c. $x\begin{bmatrix} 4 & 10 \\ 7 & 8 \end{bmatrix}^T = \begin{bmatrix} y & -10.5 \\ -15 & -12 \end{bmatrix}$

d. $\left(3\begin{bmatrix} 1 & 6 \\ 5 & -3 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 7 & -9 \\ 8 & -10 \\ x & 11 \end{bmatrix}\right)^T = \begin{bmatrix} -4 & 7 & 6 \\ 27 & y & -5 \end{bmatrix}$

---

12. Oxbridge University retains data on the number of commencing students each year. The number of new local and international students in 2020 for 4 select courses are shown in matrix O_{2020}.

$$O_{2020} = \begin{array}{c} \begin{array}{cccc} \text{medicine} & \text{literature} & \text{law} & \text{philosophy} \end{array} \\ \begin{bmatrix} 230 & 240 & 280 & 150 \\ 220 & 150 & 260 & 120 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{local} \\ \text{international} \end{array}$$

a. In 2021, Oxbridge University's enrolments were 20% higher than in 2020. Using scalar multiplication, calculate the matrix O_{2021}, displaying Oxbridge's 2021 local and international student enrolments in these 4 courses.

b. In total, how many international students started studying law at Oxbridge over 2020 and 2021?

c. Comparable data was also collected by Camford University for their 2021 enrolments. The data is shown in matrix C_{2021}.

$$C_{2021} = \begin{array}{c} \begin{array}{cc} \text{local} & \text{international} \end{array} \\ \begin{bmatrix} 305 & 350 \\ 372 & 231 \\ 320 & 318 \\ 266 & 175 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{medicine} \\ \text{literature} \\ \text{law} \\ \text{philosophy} \end{array}$$

Determine the transpose of C_{2021}, to display this data in the same form as Oxbridge's data.

d. Calculate ${C_{2021}}^T - O_{2021}$ to find the difference between the 2021 enrolment numbers of the two universities.

e. How many more local students in 2021 started studying philosophy at Camford than at Oxbridge?

Exam practice

13. Consider the following four matrix expressions.

$\begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 8 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 8 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 8 & 0 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

How many of these four matrix expressions are defined?

A. 0 **B.** 1 **C.** 2

D. 3 **E.** 4

VCAA 2019 Exam 1 Matrices Q1

87% of students answered this question correctly.

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14. The transpose of $\begin{bmatrix} 2 & 7 & 10 \\ 13 & 19 & 8 \end{bmatrix}$ is

A. $\begin{bmatrix} 13 & 19 & 8 \\ 2 & 7 & 10 \end{bmatrix}$ **B.** $\begin{bmatrix} 10 & 7 & 2 \\ 8 & 19 & 13 \end{bmatrix}$ **C.** $\begin{bmatrix} 2 & 13 \\ 7 & 19 \\ 10 & 8 \end{bmatrix}$

D. $\begin{bmatrix} 13 & 2 \\ 19 & 7 \\ 8 & 10 \end{bmatrix}$ **E.** $\begin{bmatrix} 8 & 10 \\ 19 & 7 \\ 13 & 2 \end{bmatrix}$

VCAA 2016 Exam 1 Matrices Q1

81% of students answered this question correctly.

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15. The following table shows information about two matrices, A and B.

matrix	order	rule
A	3×3	$a_{ij} = 2i + j$
B	3×3	$b_{ij} = i - j$

The element in row i and column j of matrix A is a_{ij}.

The element in row i and column j of matrix B is b_{ij}.

The sum $A + B$ is

A. $\begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{bmatrix}$ **B.** $\begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{bmatrix}$ **C.** $\begin{bmatrix} 3 & 6 & 9 \\ 3 & 6 & 9 \\ 3 & 6 & 9 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \end{bmatrix}$ **E.** $\begin{bmatrix} 3 & 6 & 3 \\ 6 & 3 & 9 \\ 3 & 9 & 3 \end{bmatrix}$

VCAA 2017 Exam 1 Matrices Q6

58% of students answered this question correctly.

Questions from multiple lessons

Data analysis

16. The statistical analysis of a set of bivariate data involving variables x and y resulted in the information displayed in the following table.

mean	$\bar{x} = 17.05$	$\bar{y} = 19.92$
standard deviation	$s_x = 1.25$	$s_y = 1.83$
equation of the least squares line	$y = 15.146 + 0.28x$	

Using this information, the value of the correlation coefficient r for this set of bivariate data is closest to

A. 0.15 **B.** 0.19 **C.** 0.33 **D.** 0.41 **E.** 0.44

Adapted from VCAA 2018 Exam 1 Data analysis Q13

Recursion and financial modelling

17. Lindsay took out a loan to buy a new house. Information about her first loan repayment is shown in the following amortisation table.

repayment number	repayment	interest	principal reduction	balance of loan
0	0.00	0.00	0.00	230 000.00
1	1000.00	690.00	310.00	229 310.00
2	1000.00			

How much interest does Lindsay pay in repayment number 2?

A. \$687.00 **B.** \$687.93 **C.** \$688.12 **D.** \$690.00 **E.** \$692.08

Adapted from VCAA 2017NH Exam 1 Recursion and financial modelling Q21

Matrices *Year 11 content*

18. A small ticketing company sells tickets for a variety of events; music concerts (M), theatre (T), sporting events (S), and comedy shows (C).

Matrix N contains the number of each type of booking received last month.

$$N = \begin{bmatrix} 112 \\ 46 \\ 75 \\ 53 \end{bmatrix} \begin{array}{l} \text{M} \\ \text{T} \\ \text{S} \\ \text{C} \end{array}$$

a. What is the order of matrix N? (1 MARK)

b. A booking fee per ticket is collected by the company with each ticket sale.

Matrix F contains the booking fee per ticket for each type of event.

$$F = \begin{array}{c} \begin{array}{cccc} \text{M} & \text{T} & \text{S} & \text{C} \end{array} \\ \begin{bmatrix} 8 & 10 & 7 & 5 \end{bmatrix} \end{array}$$

i. Calculate the matrix product $R = F \times N$. (1 MARK)

ii. What information is represented by matrix R? (1 MARK)

Adapted from VCAA 2016 Exam 2 Matrices Q1

7C Advanced operations with matrices

STUDY DESIGN DOT POINT

- matrix arithmetic: the order of a matrix, types of matrices (row, column, square, diagonal, symmetric, triangular, zero, binary and identity), the transpose of a matrix, and elementary matrix operations (sum, difference, multiplication of a scalar, product and power)

KEY SKILLS

During this lesson, you will be:

- defining matrix products
- calculating a matrix product
- using a summing matrix
- calculating a matrix power.

KEY TERMS

- Matrix product
- Post-multiplication
- Pre-multiplication
- Summing matrix
- Matrix power

A key component of matrix arithmetic involves multiplying matrices. In order to multiply two matrices, the matrix product must be defined. Matrix multiplication has several applications, including summing the rows and columns in matrices and raising matrices to a power.

Defining matrix products

A **matrix product** is the resulting matrix when two or more matrices are multiplied. **Post-multiplication** is the multiplication of one matrix after another matrix. For example, the product AB can be defined as matrix A post-multiplied by matrix B.

Pre-multiplication is the multiplication of one matrix before another matrix. For example, the product AB can also be defined as matrix B pre-multiplied by matrix A.

Not all matrix multiplications can be performed. For a matrix product to be defined, the number of columns in the first matrix must equal the number of rows in the second matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix} \qquad BA = \begin{bmatrix} e \\ f \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Order: } 2 \times \underbrace{2 \quad 2}_{\text{equal}} \times 1 \qquad \text{Order: } 2 \times \underbrace{1 \quad 2}_{\text{not equal}} \times 1$$

The product AB is defined since the number of columns in matrix A equals the number of rows in matrix B. The product BA is not defined since the number of columns in matrix B is not equal to the number of rows in matrix A.

Matrix multiplication is therefore not commutative (reversible): $AB \neq BA$.

If the matrix product is defined, the order of the matrix product will be equal to the number of rows in the first matrix and the number of columns in the second matrix.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\text{Order: } \underbrace{2 \times 2 \quad 2 \times 1}_{2 \times 1}$$

The order of the matrix product AB is 2×1.

Worked example 1

Determine whether the following matrix products are defined. If defined, determine its order.

$K = \begin{bmatrix} 3 & -2 \\ 0 & 5 \end{bmatrix}$ $L = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

a. KL

Explanation

Step 1: Determine the order of each matrix.

Matrix K has an order of 2×2.

Matrix L has an order of 2×1.

Step 2: Determine whether the matrix product is defined.

The number of columns of matrix K must equal the number of rows of matrix L.

$$KL = \begin{bmatrix} 3 & -2 \\ 0 & 5 \end{bmatrix} \times \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\text{Order: } 2 \times \underbrace{2 \quad 2}_{\text{equal}} \times 1$$

The matrix product is defined.

Step 3: Determine the order of the matrix product.

The order of the matrix product is equal to the number of rows of matrix K and the number of columns of matrix L.

$$KL = \begin{bmatrix} 3 & -2 \\ 0 & 5 \end{bmatrix} \times \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\text{Order: } \underbrace{2 \times 2 \quad 2 \times 1}_{2 \times 1}$$

Answer

KL is defined. The order of KL is 2×1.

b. LK

Explanation

Step 1: Determine the order of each of the matrices.

Matrix K has an order of 2×2.

Matrix L has an order of 2×1.

Step 2: Determine whether the matrix product is defined.

The number of columns of matrix L must equal the number of rows of matrix K.

$$LK = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 & -2 \\ 0 & 5 \end{bmatrix}$$

$$\text{Order: } 2 \times \underbrace{1 \quad 2}_{\text{not equal}} \times 2$$

Answer

LK is not defined.

Calculating a matrix product

Multiplying two matrices involves both multiplication and addition of elements. To multiply two matrices together, elements in specific rows and columns must be multiplied and summed together.

$$AB = C$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

To find c_{mn}, multiply each element in the m^{th} row of matrix A by its corresponding element in the n^{th} column of matrix B. Add these products together to find the value of c_{mn}.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \times e + b \times g & a \times f + b \times h \\ c \times e + d \times g & c \times f + d \times h \end{bmatrix}$$

For example, to find c_{12}, the elements in the 1st row of matrix A are multiplied by their corresponding elements in the 2nd column of matrix B. These numbers are then added.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \times e + b \times g & a \times f + b \times h \\ c \times e + d \times g & c \times f + d \times h \end{bmatrix}$$

Worked example 2

Consider the following matrices.

$P = \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix}$ $Q = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Evaluate the matrix product PQ.

Explanation – Method 1: By hand

Step 1: Determine whether the product matrix is defined.

$$PQ = \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Order: $2 \times \underbrace{2 \quad 2}_{\text{equal}} \times 1$

The matrix product is defined.

Step 2: Determine the order of the product matrix.

The order is equal to the number of rows of matrix P and the number of columns of matrix Q.

The order of PQ is 2×1.

Step 3: Set up the equation using matrices.

Ensure PQ has the correct order.

$$PQ = \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

Step 4: Find the elements of PQ.

$$PQ = \begin{bmatrix} -1 & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \times 2 + 3 \times 4 \\ 0 \times 2 + 5 \times 4 \end{bmatrix}$$

Step 5: Sum the products to find the final elements of matrix PQ.

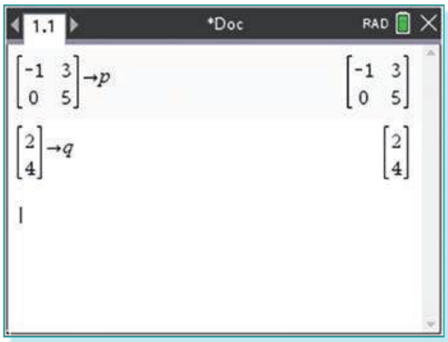
Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

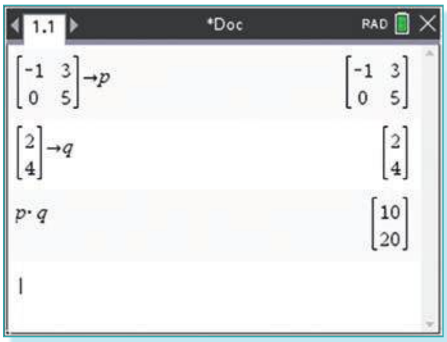
Step 2: Create matrix P and Q.

Create a 2×2 matrix and enter the values for P.
Press ctrl + var + 'p' to store the matrix as 'p'.
Press enter.

Create a 2×1 matrix and enter the values for Q.
Press ctrl + var + 'q' to store the matrix as 'q'.
Press enter.



Step 3: Type '$p \times q$' to calculate PQ and press enter.



Continues →

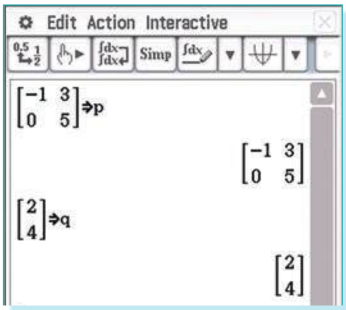
Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

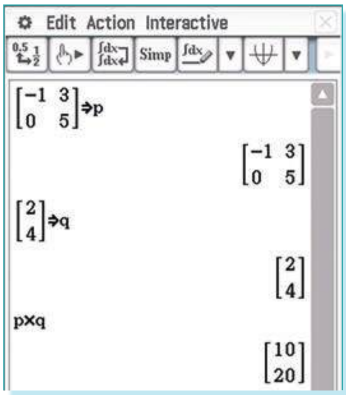
Step 2: Create matrix P and Q.

Create a 2×2 matrix and enter the values for P. On the touch-screen keyboard, go to var and tap ⇒ + 'p' to store the matrix as 'p'. Press EXE.

Create a 2×1 matrix and enter the values for Q. On the touch-screen keyboard, go to var and press ⇒ + 'q' to store the matrix as 'q'. Press EXE.



Step 3: Type '$p \times q$' to calculate PQ.



Answer – Method 1, 2 and 3

$PQ = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

Using a summing matrix

A **summing matrix** is a row or column matrix that consists of only the number 1 for each element, and is used to find the sum of either the rows or columns of another matrix.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

The sum of the rows of a matrix can be found by post-multiplying a column summing matrix. The number of rows in the summing matrix must be equal to the number of columns in the matrix to be summed, otherwise the matrix product will be undefined.

For example, to find the sum of the rows in matrix E, the post-multiplication of a 3×1 column summing matrix would be required, since it has 3 columns.

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 1 + 3 \times 1 \\ 4 \times 1 + 5 \times 1 + 6 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

The sum of the columns of a matrix can be found by pre-multiplying a row summing matrix. The number of columns in the summing matrix must be equal to the number of rows in the matrix to be summed, otherwise the matrix product will be undefined.

For example, to find the sum of the columns in matrix F, the pre-multiplication of a 1×4 row summing matrix would be required, since it has 4 rows.

$$F = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 3 \\ 7 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 3 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 1 \times 4 + 1 \times 3 + 1 \times 7 & 1 \times 2 + 1 \times 5 + 1 \times 3 + 1 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 16 \end{bmatrix}$$

Worked example 3

$S = \begin{bmatrix} -1 & 0 & 6 \\ 2 & -3 & 4 \end{bmatrix}$

Use an appropriate summing matrix to

a. sum the rows of matrix S.

Explanation

Step 1: Construct a summing matrix.

To sum the rows of a matrix, a column summing matrix is required.

The number of rows in the summing matrix should equal the number of columns in matrix S.

Matrix S has 3 columns. The summing matrix must be a 3×1 matrix.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 2: Post-multiply matrix S with the summing matrix.

$$\begin{bmatrix} -1 & 0 & 6 \\ 2 & -3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

b. sum the columns of matrix S.

Explanation

Step 1: Construct a summing matrix.

To sum the columns of a matrix, a row summing matrix is required.

The number of columns in the summing matrix should equal the number of rows in matrix S.

Matrix S has 2 rows. The summing matrix must be a 1×2 matrix.

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

Step 2: Pre-multiply matrix S with the summing matrix.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 6 \\ 2 & -3 & 4 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -3 & 10 \end{bmatrix}$$

Calculating a matrix power

A **matrix power** is the resultant product when a matrix is raised to an index or power. Only square matrices can be raised to a power, since the number of columns in the first matrix must equal the number of rows in the second matrix for a matrix product to be defined. The order of the resultant matrix product will be the same as the original matrix.

It is important to maintain the normal order of operations that are used with standard numbers when working with matrices. For example, if A and B are square matrices, then $(A + B)^2 = (A + B)(A + B)$ and $(AB)^3 = (AB)(AB)(AB)$.

Worked example 4

Consider the following matrices.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$$

a. Calculate B^2.

Explanation – Method 1: By hand

Step 1: Determine if the matrix power is defined.

Matrix B has an order of 2×2. It is a square matrix so it can be raised to a power.

Step 2: Evaluate the matrix power.

$$B^2 = \begin{bmatrix} 2 & 1 \\ 8 & 9 \end{bmatrix}^2$$

$$= \begin{bmatrix} 2 & 1 \\ 8 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 8 & 9 \end{bmatrix}$$

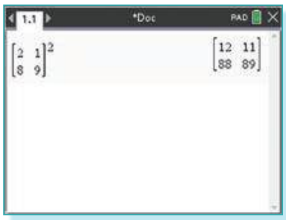
$$= \begin{bmatrix} 2 \times 2 + 1 \times 8 & 2 \times 1 + 1 \times 9 \\ 8 \times 2 + 9 \times 8 & 8 \times 1 + 9 \times 9 \end{bmatrix}$$

Explanation – Method 2: Ti-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

Step 2: Create a 2×2 matrix and enter the values for B.

Step 3: Type ^ + '2' and press enter.

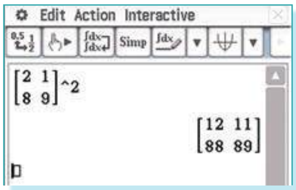


Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Create a 2×2 matrix and enter the values for B.

Step 3: Type ^ + '2' and press EXE.



Answer – Method 1, 2 and 3

$$\begin{bmatrix} 12 & 11 \\ 88 & 89 \end{bmatrix}$$

b. Determine whether $(AC)^2$ is defined.

Explanation

Step 1: Determine if the matrix product is defined.

Matrix A has an order of 3×1 and C an order of 1×3.

The matrix product AC is defined since the number of columns in matrix A is equal to the number of rows in matrix C.

Step 2: Determine if the matrix being raised to a power is a square matrix.

The order of the matrix product AC will be 3×3.

This is a square matrix so it can be raised to a power.

Answer

$(AC)^2$ is defined.

Exam question breakdown

VCAA 2016 Exam 1 Matrices Q4

The table shows the number of each type of coin saved in a money box.

coin	5 cent	10 cent	20 cent	50 cent
number	15	32	48	24

The matrix product that displays the total number of coins and the total value of these coins is

A. $\begin{bmatrix} 5 & 10 & 20 & 50 \end{bmatrix}\begin{bmatrix} 15 \\ 32 \\ 48 \\ 24 \end{bmatrix}$

B. $\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 10 \\ 1 & 20 \\ 1 & 50 \end{bmatrix}$

C. $\begin{bmatrix} 5 & 10 & 20 & 50 \end{bmatrix}\begin{bmatrix} 1 & 15 \\ 1 & 32 \\ 1 & 48 \\ 1 & 24 \end{bmatrix}$

D. $\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 20 \\ 50 \end{bmatrix}$

E. $\begin{bmatrix} 5 & 10 & 20 & 50 \\ 15 & 32 & 48 & 24 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Explanation

Step 1: Determine the information to be displayed in the matrix product.

The matrix product must display the total number of coins and the total value of these coins.

Step 2: Determine the values needed to find the total number of coins.

The number of coins can be represented in the following row matrix.

$\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix}$

The total number of coins is the sum of these values. The sum can be found by post-multiplying the matrix by a 4×1 column summing matrix:

$$\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 3: Determine the values needed to calculate the total value of the coins.

The total value of the coins can be found by multiplying the number of each type of coin by its value and then finding the sum.

This can be represented by the following matrix multiplication:

$$\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 20 \\ 50 \end{bmatrix}$$

Step 4: Combine the matrix multiplications.

Both expressions post-multiply the row matrix of the number of coins by a column matrix.

The column matrices in each expression can be combined into a 4×2 matrix.

$$\begin{bmatrix} 15 & 32 & 48 & 24 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 10 \\ 1 & 20 \\ 1 & 50 \end{bmatrix}$$

Answer

B

34% of students answered this question correctly.

31% of students incorrectly chose option A. Option A multiplied the value of each type of coin by the number of those coins and summed these values, resulting in the total value of the coins. However, the matrix product needed to calculate two quantities: the total value and the total number of the coins. Only option B calculates the required information.

7C Questions

Defining matrix products

1. Which of the following matrix products is defined?

$A = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 8 & 4 \\ 3 & 1 & 6 \\ 9 & 2 & 6 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

A. AB **B.** BC **C.** $(BD)C$ **D.** $A(BC)$

2. Consider the following matrices.

$A = \begin{bmatrix} 9 & 7 & -3 \\ 1 & -6 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 \\ -5 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 8 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 9 & 2 \end{bmatrix}$ $E = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$ $F = \begin{bmatrix} 4 \\ 1 \\ -6 \end{bmatrix}$

Determine if the following matrix products are defined. If defined, state its order.

a. AB **b.** BA **c.** AD **d.** AF

e. CB **f.** EB

3. Using the matrices from question 2, determine the order of the following matrix products.

a. $C(BA)$

A. 1×1
B. 1×3
C. 2×3
D. $C(BA)$ is undefined

b. $(DF)(CA)$

A. 1×1
B. 1×3
C. 2×3
D. $(DF)(CA)$ is undefined

Calculating a matrix product

4. Calculate the following matrix product.

$\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$

A. $\begin{bmatrix} -2 \\ -8 \end{bmatrix}$ **B.** $\begin{bmatrix} -6 & 0 \\ 12 & -6 \end{bmatrix}$ **C.** $\begin{bmatrix} -2 & 0 \\ 8 & -3 \end{bmatrix}$ **D.** $\begin{bmatrix} -2 & -1 \\ -8 & -1 \end{bmatrix}$

5. Calculate the following matrix products.

a. $\begin{bmatrix} 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 0 \\ 5 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 7 & 6 \\ 1 & 5 & -4 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \\ 1 & -3 \\ 2 & 5 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 5 & -9 \\ 3 & 4 & 8 \\ 7 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 6 & 1 & 9 \\ 5 & -2 & 4 \\ 7 & 2 & 0 \end{bmatrix}$

6. In a soccer tournament, team Soccerolo (S) and Edroloball (E) played in 20 games and their wins (W), draws (D) and losses (L) are shown in matrix G. A team received 2 points if they won, 1 point if they drew and 0 points if they lost, as shown in matrix H.

$$G = \begin{array}{c} \begin{array}{ccc} \text{W} & \text{D} & \text{L} \end{array} \\ \begin{bmatrix} 11 & 4 & 5 \\ 8 & 3 & 9 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{S} \\ \text{E} \end{array} \qquad H = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{array}{c} \text{W} \\ \text{D} \\ \text{L} \end{array}$$

a. Calculate GH.

b. What does the matrix GH represent?

c. Which team had the most points at the end of the tournament and how many points did they have?

7. Tess runs a social media page and is paid based on the number of likes and comments she gets on each post. Tess makes 5 cents per like and 10 cents per comment, as shown in matrix J. At the end of the week, the number of likes (L) and comments (C) on each of her four posts is shown in matrix K.

$$J = \begin{bmatrix} 5 & 10 \end{bmatrix} \quad K = \begin{array}{c} \text{post} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 27 & 31 & 56 & 12 \\ 12 & 19 & 11 & 21 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{L} \\ \text{C} \end{array}$$

a. What does element k_{13} represent?

b. Calculate JK.

c. How much money, in dollars, did Tess make in that week?

d. Which post was most profitable for Tess?

---

8. Mac needs three different cheeses, cheddar (C), gruyere (G) and parmesan (P), to cook his famous pasta dish. The price of each cheese pack from two different stores is shown in matrix L.

$$L = \begin{array}{c} \begin{array}{ccc} \text{C} & \text{G} & \text{P} \end{array} \\ \begin{bmatrix} 1.75 & 6.20 & 3.40 \\ 1.30 & 6.95 & 2.80 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Cheeseworld} \\ \text{Cheesetopia} \end{array}$$

a. Mac needs 5 packs of cheddar, 2 packs of gruyere and 3 packs of parmesan for his dish. Construct a matrix, M, to represent this information and calculate LM.

b. Which store should Mac buy his cheese from?

c. Mac finds out that Cheeseworld is having a sale on parmesan cheese and the price has dropped to \$2.70. Mac decides he will buy his cheese from Cheeseworld. Did he make the right decision? Justify.

Using a summing matrix

9. Which of the following is the correct summing matrix to sum the columns of matrix A?

$$A = \begin{bmatrix} 5 & 4 & 5 & 1 \\ 2 & 1 & 9 & 2 \\ 3 & 2 & 2 & 4 \end{bmatrix}$$

A. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

---

10. Construct a summing matrix that

a. sums the columns of $\begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 9 & 3 \\ 7 & 4 \end{bmatrix}$.

b. sums the rows of $\begin{bmatrix} 7 & 0 & 3 & 3 \\ -2 & 6 & 0 & 1 \\ 6 & 1 & 8 & 4 \end{bmatrix}$.

---

11. Heston has four dessert stores and wanted to know which store was doing better in the past month. The matrix shows how many desserts each store sold in the past month.

$$\begin{array}{c} \begin{array}{ccc} \text{strudels} & \text{macaroons} & \text{tarts} \end{array} \\ \begin{bmatrix} 20 & 24 & 24 \\ 16 & 30 & 21 \\ 18 & 15 & 28 \\ 22 & 24 & 19 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{store 1} \\ \text{store 2} \\ \text{store 3} \\ \text{store 4} \end{array}$$

a. Construct a matrix that, when multiplied with the one provided, will find the total number of desserts sold by each store in the past month.

b. Show, using matrix calculations, which store sold the most desserts in the past month.

c. Construct a matrix that, when multiplied with the one provided, will find the total number of each type of dessert sold across all stores.

d. Show, using matrix calculations, which type of dessert is the most popular across all stores.

Calculating a matrix power

12. Which of the following matrices can be raised to a power?

A. $\begin{bmatrix} 4 & 2 & 8 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ **C.** $\begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$ **D.** $\begin{bmatrix} 0 & 2 \end{bmatrix}$

---

13. For the following matrices:

$$A = \begin{bmatrix} 4 & 5 \\ 8 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 6 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 9 & 3 \\ 4 & 7 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 4 \\ 7 & 1 \\ 5 & 3 \end{bmatrix}$$

Determine if each of the following expressions is defined.

a. A^2 b. B^2 c. BD^2 d. $(BD)^2$ e. $(BD)(AC)^3$

---

14. Use the matrices shown to evaluate the following expressions.

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a. C^5 b. $A^2 - 3B$ c. $AC^2 - B^2$

Joining it all together

15. Consider the following matrices.

$$J = \begin{bmatrix} 3 & 1 \\ 7 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 3 & 6 \end{bmatrix} \quad L = \begin{bmatrix} 9 & 3 \\ 2 & 5 \\ 4 & 6 \end{bmatrix}$$

a. Is the matrix product KL defined? If defined, state its order.

b. Which of the matrices can be raised to a power?

c. Is the matrix expression $J^2 - (KL)^2$ defined? If defined, evaluate.

d. Evaluate $(J + KL)^2$.

---

16. Fin Demali is an avid bird watcher. He tracks the population of three different species in two areas. The recorded population (in thousands) of mockingjays (M), finches (F) and woodpeckers (W) in 2022 are shown in matrix B.

$$B = \begin{array}{c} \begin{array}{ccc} \text{M} & \text{F} & \text{W} \end{array} \\ \begin{bmatrix} 23 & 12 & 7 \\ 18 & 14 & 8 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{area 1} \\ \text{area 2} \end{array}$$

a. Construct a summing matrix to sum the number of birds in each area and use this to calculate the total number of birds in each area.

b. Fin predicts the bird population will change in 2023. The population in 2023 can be predicted by post-multiplying matrix C with matrix B. Create the matrix showing the new populations (in thousands) in 2023.

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 1.3 \end{bmatrix}$$

c. What is the total bird population predicted to be in 2023?

d. By what percentage did the bird populations change in each area between 2022 and 2023?

Exam practice

17. A school canteen sells pies (P), rolls (R) and sandwiches (S).

The number of each item sold over three school weeks is shown in matrix M.

$$M = \begin{array}{c} \begin{array}{ccc} P & R & S \end{array} \\ \begin{bmatrix} 35 & 24 & 60 \\ 28 & 32 & 43 \\ 32 & 30 & 56 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{week 1} \\ \text{week 2} \\ \text{week 3} \end{array}$$

In total, how many sandwiches were sold in these three weeks? (1 MARK)

VCAA 2017 Exam 2 Matrices Q1a

89% of students answered this question correctly.

---

18. The matrix product $[4 \quad 2 \quad 0] \times \begin{bmatrix} 4 \\ 12 \\ 8 \end{bmatrix}$ is equal to

A. $[144]$

B. $\begin{bmatrix} 16 \\ 24 \\ 0 \end{bmatrix}$

C. $4 \times [1 \quad 2 \quad 0] \times \begin{bmatrix} 1 \\ 12 \\ 8 \end{bmatrix}$

D. $2 \times [2 \quad 1 \quad 0] \times \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

E. $4 \times [2 \quad 1 \quad 0] \times \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

VCAA 2018 Exam 1 Matrices Q2

68% of students answered this question correctly.

Questions from multiple lessons

Recursion and financial modelling

19. The following graph shows the value, V_n, of the new Pineapple phone as it depreciates over a period of six months. Which one of the following depreciation situations does this graph best represent?

A. Reducing balance depreciation with a decrease in depreciation rate after 4 months.

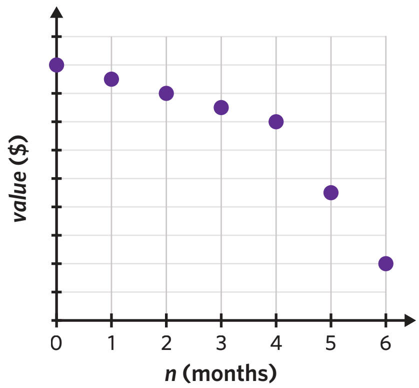
B. Flat rate depreciation with an increase in depreciation rate after 4 months.

C. Unit cost depreciation with constant depreciation rate.

D. Flat rate depreciation with a decrease in depreciation rate after 4 months.

E. Reducing balance depreciation with an increase in depreciation rate after 4 months.

Adapted from VCAA 2018 Exam 1 Recursion and financial modelling Q20



Data analysis

20. Bill has owned and run a small hotel by the beach for 15 years and wants to maximise the number of bookings at his hotel during summer. He collects data on the number of bookings each summer over the last 15 years, which is modelled on the time series plot shown.

The time series plot exhibits

A. seasonality with irregular fluctuations.

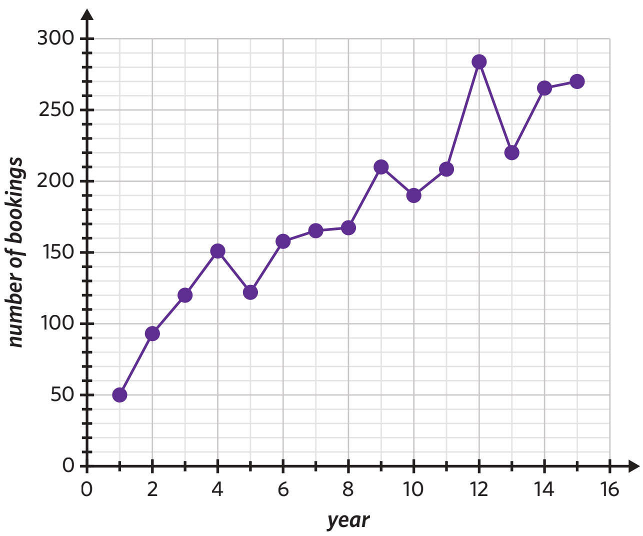
B. seasonality with an increasing trend.

C. seasonality with an increasing trend and irregular fluctuations.

D. irregular fluctuations only.

E. an increasing trend with irregular fluctuations.

Adapted from VCAA 2016 Exam 1 Data analysis Q13



7C QUESTIONS

Recursion and financial modelling

21. Dani withdraws \$20 000 from her account to purchase some dragon eggs. For tax purposes, she plans to depreciate the value of her eggs using the reducing balance method. The value of Dani's eggs, in dollars, after n years, D_n, can be modelled by the recurrence relation shown.

$D_0 = 20\,000, \quad D_{n+1} = R \times D_n$

a. For the first two years of reducing balance depreciation, the value of R is 0.77.

What is the annual rate of depreciation during these two years? (1 MARK)

b. For the next three years of reducing balance depreciation, the annual rate of depreciation of the value of Dani's eggs is changed to 13.5%.

What is the value of the eggs 5 years after they were purchased? Round your answer to the nearest dollar. (2 MARKS)

Adapted from VCAA 2018 Exam 2 Recursion and financial modelling Q5

7D Inverse matrices

STUDY DESIGN DOT POINT

- inverse of a matrix, its determinant, and the condition for a matrix to have an inverse

7A 7B 7C **7D** 7E 7F 7G 7H 7I

KEY SKILLS

During this lesson, you will be:

- calculating the determinant of a matrix
- calculating the inverse of a matrix
- solving simultaneous equations using matrix equations.

KEY TERMS

- Determinant
- Inverse matrix
- Singular matrix

An important feature of matrices is their ability to be incorporated into equations. Matrix operations and inverse matrices can be used to solve for unknown variables. Since matrices cannot be divided, an inverse matrix can instead be found and then used when solving equations.

Calculating the determinant of a matrix

The **determinant** is a number associated with a matrix which determines whether the inverse of a matrix is defined. It can only be calculated for square matrices. For a matrix to have an inverse, its determinant must not equal zero.

The determinant of matrix A is denoted as $\det(A)$.

For 2×2 matrices, a formula can be used to find the determinant.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

While the formula can be used to find the determinant of 2×2 matrices, a CAS can be used to find the determinant of square matrices of any order.

Worked example 1

$$A = \begin{bmatrix} 4 & 2 & 13 \\ 6 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$$

a. Find the determinant of B.

Explanation

Calculate the determinant.

$$\begin{aligned} \det(B) &= ad - bc \\ &= 2 \times 1 - 2 \times 3 \end{aligned}$$

Answer

-4

Continues →

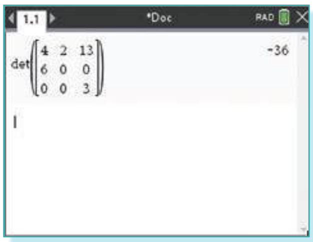
b. Use a CAS to find the determinant of A.

Explanation – Method 1: TI-Nspire

Step 1: From the home screen, select ‘1: New’ → ‘1: Add Calculator’.

Step 2: Press menu. Select ‘7: Matrix & Vector’ → ‘3: Determinant’.

Step 3: Enter matrix A. Press enter.

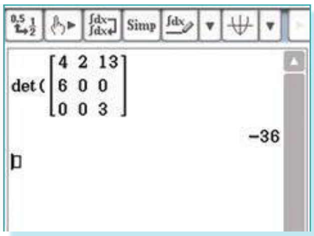


Explanation – Method 2: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Tap ‘Action’ → ‘Matrix’ → ‘Calculation’ → ‘det’.

Step 3: Enter matrix A. Press EXE.



Answer – Method 1 and 2

-36

Calculating the inverse of a matrix

If a matrix’s determinant is not equal to zero, a square matrix will have an inverse matrix. When an **inverse matrix** is pre-multiplied or post-multiplied with its matrix, the result is the identity matrix, I.

The inverse matrix is denoted by A^{-1}.

$A \times A^{-1} = A^{-1} \times A = I$

For example, if

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The identity matrix works in a similar way to the number 1 when multiplied by other matrices. Pre-multiplying or post-multiplying a matrix by the identity matrix results in no change to the original matrix.

$A = AI = IA$

The inverse of a 2×2 matrix in the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ or } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A **singular matrix** is a matrix with no inverse, where the determinant equals zero. For a 2×2 singular matrix with a determinant of zero, the fraction, $\frac{1}{ad - bc}$ will be equal to $\frac{1}{0}$ which is undefined. Hence there is no inverse.

Worked example 2

$A = \begin{bmatrix} 7 & 5 \\ 2 & 3 \end{bmatrix}$

a. Does the inverse of A exist?

Explanation

Step 1: Calculate the determinant.

$$\begin{aligned}\det(A) &= 7 \times 3 - 5 \times 2 \\ &= 21 - 10 \\ &= 11\end{aligned}$$

Step 2: Determine if the inverse is defined.

$\det(A) \neq 0$

Since the determinant is not equal to 0, the inverse matrix is defined.

Answer

Yes

b. Calculate A^{-1}.

Explanation – Method 1: By hand

Find the inverse matrix by using the formula for 2×2 matrices.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

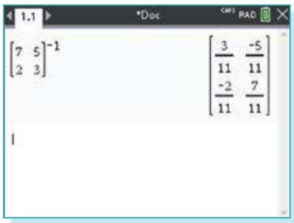
$$A^{-1} = \frac{1}{7 \times 3 - 5 \times 2}\begin{bmatrix} 3 & -5 \\ -2 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 3 & -5 \\ -2 & 7 \end{bmatrix}$$

Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

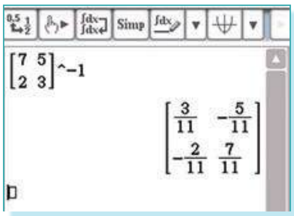
Step 2: Enter matrix A and type '^-1'. Press enter.



Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Enter matrix A and type '^-1'. Press EXE.



Answer – Method 1, 2 and 3

$$\begin{bmatrix} \frac{3}{11} & -\frac{5}{11} \\ -\frac{2}{11} & \frac{7}{11} \end{bmatrix}$$

Solving simultaneous equations using matrix equations

If an equation contains two unknown variables, two equations are required to solve for these values. These equations are called 'simultaneous equations'. It is possible to have more than two simultaneous equations where there are more than two unknown variables.

Matrices can be used to solve simultaneous equations.

The simultaneous equations

$ax + by = e$

$cx + dy = f$

can be converted into a matrix equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

The matrix equation is in the form $AX = B$.

- Matrix A, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, contains information about the coefficients of x and y.
- Matrix X, $\begin{bmatrix} x \\ y \end{bmatrix}$, contains the unknown variables, x and y.
- Matrix B, $\begin{bmatrix} e \\ f \end{bmatrix}$ contains information about what the matrix equations equate to.

Once simultaneous equations have been represented in matrix form, the equations can be rearranged using matrix operations to solve for x and y.

Matrices cannot be divided. Therefore, when solving for matrix X in a matrix equation, the inverse of A is used.

Assuming A^{-1} exists, $AX = B$ can be solved for X by using the inverse matrix.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

The order of multiplication by the inverse matrix needs to be consistent on each side of the equation. If A is pre-multiplied by A^{-1}, then B must also be pre-multiplied by A^{-1}.

Note: Questions involving three or more simultaneous equations should be solved using a CAS instead.

If the determinant of a matrix is zero, its inverse does not exist. This means there is no unique solution for the simultaneous equations.

Worked example 3

Solve the following sets of simultaneous equations.

a. $2x + 3y = 2$

$3x + 4y = 3$

Explanation

Step 1: Set up the matrix equations.

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Step 2: Define the matrices in the matrix equation.

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$

Let $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Step 3: Solve the matrix equation.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Step 4: Calculate A^{-1}.

$$A^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

Continues →

Step 5: Evaluate $X = A^{-1}B$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Answer

$x = 1, \quad y = 0$

---

b. $3x + 2y - 2z = 26$

$4x + 6y + z = 76$

$-2x + y + 4z = 12$

Explanation – Method 1: TI-Nspire

Step 1: Write a matrix equation representing the simultaneous equations in the form $AX = B$.

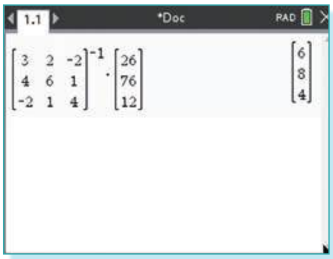
$$\begin{bmatrix} 3 & 2 & -2 \\ 4 & 6 & 1 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 26 \\ 76 \\ 12 \end{bmatrix}$$

Step 2: Rearrange the equation using inverse matrices.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 4 & 6 & 1 \\ -2 & 1 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 26 \\ 76 \\ 12 \end{bmatrix}$$

Step 3: Enter the equation into the calculator.



Explanation – Method 2: Casio ClassPad

Step 1: Write a matrix equation representing the simultaneous equations in the form $AX = B$.

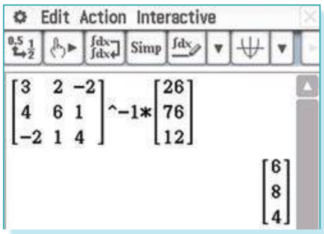
$$\begin{bmatrix} 3 & 2 & -2 \\ 4 & 6 & 1 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 26 \\ 76 \\ 12 \end{bmatrix}$$

Step 2: Rearrange the equation using inverse matrices.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 4 & 6 & 1 \\ -2 & 1 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 26 \\ 76 \\ 12 \end{bmatrix}$$

Step 3: Enter the equation into the calculator.



Answer – Method 1 and 2

$x = 6, \quad y = 8, \quad z = 4$

Exam question breakdown

VCAA 2021 Exam 1 Matrices Q5

A is a 7×7 matrix.

B is a 10×7 matrix.

Which of the following matrix equations is defined?

A. $AB - 2B$ **B.** $A(BA)^{-1}$ **C.** AB^2 **D.** $A^2 - BA$ **E.** $A(B^T)$

Continues →

Explanation

To solve this question, check whether each option is correct or incorrect.

A: This is incorrect. In order for AB to be defined, the number of columns in A must equal the number of rows in B. ✗

B: This is incorrect. The order of BA will be equal to the number of rows in B and the number of columns in A. The order of BA is 10×7. This is not a square matrix so the inverse does not exist. ✗

C: This is incorrect. B cannot be raised to a power because it is not a square matrix. ✗

D: This is incorrect. The order of BA would be 10×7. The order of A^2 is 7×7. $A^2 - BA$ is undefined since both matrices must have the same order. ✗

E: This is correct. The order of B^T will be 7×10. The number of columns in A must equal the number of rows in B in order for AB to be defined. Therefore AB is defined. ✓

Answer

E

45% of students answered this question correctly.

7D Questions

Calculating the determinant of a matrix

1. What is the determinant of matrix A?

$A = \begin{bmatrix} -2 & 2 \\ 3 & 4 \end{bmatrix}$

A. -14 **B.** -2 **C.** 0 **D.** 14

2. Find the determinant for each of the following matrices by hand.

a. $\begin{bmatrix} -3 & 12 \\ -2 & 8 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ **c.** $\begin{bmatrix} \frac{1}{2} & -\frac{2}{3} \\ -3 & -\frac{3}{4} \end{bmatrix}$

3. Find the determinant for each of the following matrices using a CAS.

a. $\begin{bmatrix} 4 & 5 & 1 \\ -2 & 1 & 0 \\ 1 & 3 & -3 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -1 \\ 2 & 0 & 5 \end{bmatrix}$ **c.** $\begin{bmatrix} 2 & 3 & 1 & -2 \\ 6 & 1 & 4 & 1 \\ 0 & 3 & 7 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix}$

4. The determinant of matrix A is -4. Find k.

$A = \begin{bmatrix} \frac{2}{3} & 4 \\ k & 3 \end{bmatrix}$

Calculating the inverse of a matrix

5. What is the inverse of matrix P?

$P = \begin{bmatrix} -2 & 7 \\ 3 & 0 \end{bmatrix}$

A. $\begin{bmatrix} 0 & -7 \\ -3 & -2 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{7} & \frac{2}{21} \end{bmatrix}$ **C.** $\begin{bmatrix} \frac{2}{21} & -\frac{1}{3} \\ -\frac{1}{7} & 0 \end{bmatrix}$ **D.** $\begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{3} & \frac{2}{21} \end{bmatrix}$

6. For each of the following matrices, determine whether the inverse is defined. If it is defined, find the inverse.

a. $A = \begin{bmatrix} 8 & 1 \\ 2 & 3 \end{bmatrix}$

b. $B = \begin{bmatrix} 0 & -6 \\ -2 & -9 \end{bmatrix}$

c. $C = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 7 \\ 0 & 5 & 6 \end{bmatrix}$

d. $D = \begin{bmatrix} 1 & 0 & 4 \\ 8 & 3 & 16 \\ 0 & 0 & 0 \end{bmatrix}$

7. Determine whether the following pairs of matrices are each other's inverses.

a. $\begin{bmatrix} \frac{1}{2} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{8} \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 7 & 3 \\ 6 & 2 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{7} \end{bmatrix}$

c. $\begin{bmatrix} 4 & -\frac{1}{2} \\ 16 & 5 \end{bmatrix}$ and $\begin{bmatrix} 5 & \frac{1}{2} \\ -16 & 4 \end{bmatrix}$

8. $A = \begin{bmatrix} -\frac{1}{3} & u \\ -\frac{3}{2} & 5 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{15}{22} & -\frac{9}{11} \\ \frac{9}{44} & -\frac{1}{22} \end{bmatrix}$

Find the value of u.

Solving simultaneous equations using matrix equations

9. Which of the following pairs of simultaneous equations is represented by this matrix equation?

$\begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

A. $4x - 2y = 3$
$x + 2y = 2$

B. $4x + y = 3$
$-2x + 2y = 2$

C. $y = 3x - 4$
$2y = 2x + 1$

D. $y = -4x + 3$
$y = x + 2$

10. Express the following simultaneous equations in matrix form.

a. $x + 2y = 2$ and $2x + 3y = 6$

b. $3x - 2y = 5$ and $x - 3y = 5$

c. $y = 3x - 3$ and $y = -2x + 1$

d. $2x - 4y + 6z = 12$ and $-x + 3y + 5z = 2$ and $-3x + 2y + 5z = 5$.

11. Solve the following matrix equations for the values of x and y.

a. $\begin{bmatrix} 6 & 8 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 60 \\ 14 \end{bmatrix}$

b. $\begin{bmatrix} -6 & 2 \\ -14 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 58 \\ 137 \end{bmatrix}$

c. $\begin{bmatrix} 8 & 2 \\ 11 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 26 \end{bmatrix}$

12. For each of the following sets of simultaneous equations:

- Express the equations in matrix form.
- Calculate the determinant of the coefficient matrix and state whether a unique solution exists.

a. $3x + 5y = -2$ and $3x + 6y = -2$

b. $-2x + 3y = 0$ and $-3x + 4.5y = 0$

c. $x - 3y = 6$ and $2x + 4y = 3$

d. $-3y + 2x = 6$ and $-3x + 2y = 6$

Joining it all together

13. Pearl loves body products and decides to call her favourite store and ask them to create two gift baskets worth \$120 to sell at an auction. The first basket has 7 bottles of body wash (b), and 6 bottles of moisturiser (m). The second basket has 10 bottles of body wash and 4 bottles of moisturiser. This has been represented by the following simultaneous equations.

$7b + 6m = 120$
$10b + 4m = 120$

a. Represent the simultaneous equations in matrix form.

b. What is the determinant of the coefficient matrix from part **a**?

c. The matrix equation $\begin{bmatrix} b \\ m \end{bmatrix} = \frac{1}{p}\begin{bmatrix} 4 & -6 \\ q & 7 \end{bmatrix}\begin{bmatrix} 120 \\ 120 \end{bmatrix}$ can be used to solve the simultaneous equations.
Determine the values of p and q.

d. How much did each body wash and moisturiser cost?

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14. Yasmine has started a cafe that sells full-cream and soy coffees in both regular and large sizes. The following table contains information about her daily sales.

	full-cream	soy
regular	78	19
large	37	5

a. Which of the following matrices, S, accurately represents the information provided in the table?

A. $$\begin{array}{cc} \begin{matrix}\text{full-cream} & \text{soy}\end{matrix} & \\ \begin{bmatrix}78 & 37\\19 & 5\end{bmatrix} & \begin{matrix}\text{regular}\\\text{large}\end{matrix}\end{array}$$

B. $$\begin{array}{cc} \begin{matrix}\text{regular} & \text{large}\end{matrix} & \\ \begin{bmatrix}78 & 19\\37 & 5\end{bmatrix} & \begin{matrix}\text{full-cream}\\\text{soy}\end{matrix}\end{array}$$

C. $$\begin{array}{cc} \begin{matrix}\text{full-cream} & \text{soy}\end{matrix} & \\ \begin{bmatrix}78 & 37\\5 & 19\end{bmatrix} & \begin{matrix}\text{regular}\\\text{large}\end{matrix}\end{array}$$

D. $$\begin{array}{cc} \begin{matrix}\text{full-cream} & \text{soy}\end{matrix} & \\ \begin{bmatrix}78 & 19\\37 & 5\end{bmatrix} & \begin{matrix}\text{regular}\\\text{large}\end{matrix}\end{array}$$

Yasmine aims to have daily revenue of \$500. She expects regular-sized coffees to account for 70% of this revenue, with the remaining 30% being provided by large coffees.

b. Complete matrix R which shows the revenue, in dollars, earned from sales of each coffee size.

$$R = \begin{bmatrix} \quad \\ \quad \end{bmatrix}\begin{matrix}\text{regular}\\\text{large}\end{matrix}$$

c. Matrix R can be found by post-multiplying matrix S by a 2×1 matrix, C. Matrix C contains the prices of each coffee type (full-cream or soy). Write down a matrix equation showing the calculation for matrix R.

d. In order to achieve her daily revenue goal, how much should Yasmine charge, correct to the nearest 10 cents, for a coffee with full-cream milk and a coffee with soy milk?

Exam practice

15. Which of the following matrices has a determinant of zero?

A. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ **E.** $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$

VCAA 2018 Exam 1 Matrices Q1

78% of students answered this question correctly.

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16. The preferred number of cafes (x) and sandwich bars (y) in Grandmall's food court can be determined by solving the following equations written in matrix form.

$$\begin{bmatrix} 5 & -9 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

a. The value of the determinant of the 2×2 matrix is 1.
Use this information to explain why this matrix has an inverse. (1 MARK)

b. Write the three missing values of the inverse matrix that can be used to solve these equations. (1 MARK)

$$\begin{bmatrix} \quad & 9 \\ \quad & \quad \end{bmatrix}$$

c. Determine the preferred number of sandwich bars for Grandmall's food court. (1 MARK)

VCAA 2020 Exam 2 Matrices Q2

Part **a**: **37%** of students answered this question correctly.
Part **b**: **71%** of students answered this question correctly.
Part **c**: **51%** of students answered this question correctly.

17. The following matrix equation represents a pair of simultaneous linear equations.

$$\begin{bmatrix} 12 & 9 \\ m & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

These simultaneous linear equations have no unique solution when m is equal to

A. -4 **B.** -3 **C.** 0 **D.** 3 **E.** 4

VCAA 2016 Exam 1 Matrices Q3

63% of students answered this question correctly.

Questions from multiple lessons

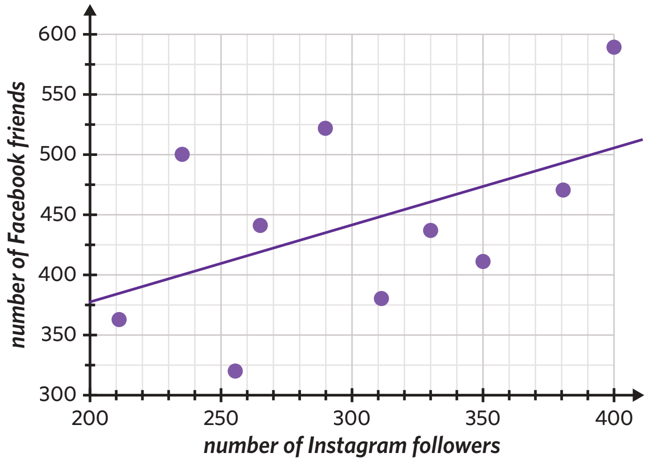
Data analysis

18. The scatterplot shows the *number of Facebook friends* and *number of Instagram followers* for a group of 10 people. A least squares regression line has been fitted to the graph.

The regression line is used to predict the number of Facebook friends for a person with 350 Instagram followers. The residual is closest to

A. -78
B. -63
C. -36
D. 62
E. 78

Adapted from VCAA 2017 Exam 1 Data analysis Q9



Matrices

19. Last night Lena won the powerball jackpot.

Each week, she allocates some of her winnings, in dollars, to either spending (SP) or savings (SV) for four weeks. This is represented in the following matrix:

$$\begin{array}{cc} \text{SP} & \text{SV} \end{array}$$
$$\begin{bmatrix} 300\,000 & 40 \\ 10\,000 & 100 \\ 98\,000 & 60 \\ 50\,450 & 50 \end{bmatrix} \begin{array}{l} \text{week 1} \\ \text{week 2} \\ \text{week 3} \\ \text{week 4} \end{array}$$

How much did Lena spend in week 4?

A. \$50 **B.** \$10 000 **C.** \$50 450 **D.** \$50 500 **E.** \$98 000

Adapted from VCAA 2017 Exam 1 Matrices Q1

Data analysis

20. Mary has a shopping addiction. She goes shopping every single day. The amount of money she spends each day is approximately normally distributed with a mean of \$130 and a standard deviation of \$40.

a. Using the 68–95–99.7% rule, determine:

i. the percentage of days where she spent more than \$170. (1 MARK)

ii. the expected number of days where Mary spends less than \$250 from a sample of 2000 days. (1 MARK)

b. The standardised amount she spent on Boxing Day is given by $z = 2.7$.

Determine the actual amount she spent on Boxing Day. (1 MARK)

Adapted from VCAA 2017 Exam 2 Data analysis Q1

7E Binary and permutation matrices

STUDY DESIGN DOT POINT

- binary and permutation matrices, and their properties and applications

7A 7B 7C 7D **7E** 7F 7G 7H 7I

KEY SKILLS

During this lesson, you will be:

- applying permutations to matrices
- constructing permutation matrices.

KEY TERMS

- Binary matrix
- Permutation matrix
- Column permutation
- Row permutation

A permutation matrix is a unique way to solve problems with several possible solutions (called permutations). This particular application of the binary matrix allows for the efficient rearrangement of rows or columns in other matrices.

Applying permutations to matrices

A **binary matrix** is a matrix in which all elements are either 0 or 1. A **permutation matrix** is a square binary matrix in which each row and column must contain the number 1 only once.

Binary matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Permutation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When multiplied with another matrix, a permutation matrix has the effect of rearranging either the rows or columns of the other matrix. A permutation matrix can be post-multiplied (multiplied after another matrix) or pre-multiplied (multiplied before another matrix).

If a permutation matrix is defined as matrix P, and another matrix is defined as matrix Q, then:

- $Q \times P$ is a post-multiplication of the permutation matrix, resulting in a column permutation.

  Note: A **column permutation** is a new matrix formed from the rearranged columns of another matrix.

- $P \times Q$ is a pre-multiplication of the permutation matrix, resulting in a row permutation.

  Note: A **row permutation** is a new matrix formed from the rearranged rows of another matrix.

When applying a column permutation, the permutation matrix P must be an $m \times m$ square matrix, where m is equal to the number of columns in matrix Q.

See worked example 1

For example,

$$\underset{\text{matrix } Q}{\begin{bmatrix} 3 & 7 & 2 \\ 4 & 1 & 5 \end{bmatrix}} \times \underset{\text{matrix } P}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

When interpreting a column permutation matrix, the position of each '1' element becomes extremely important.

For the example $Q \times P$, matrix P contains '1' elements in the following positions:

- p_{13} – This indicates that column 1 in matrix Q (elements 3 and 4) will become column 3.
- p_{21} – This indicates that column 2 in matrix Q (elements 7 and 1) will become column 1.
- p_{32} – This indicates that column 3 in matrix Q (elements 2 and 5) will become column 2.

Because of these permutations, the new matrix will be

$$\begin{bmatrix} 7 & 2 & 3 \\ 1 & 5 & 4 \end{bmatrix}$$

When applying a row permutation, the permutation matrix P must be an $n \times n$ square matrix, where n is equal to the number of rows in matrix Q.

See worked example 2

When interpreting a row permutation matrix, the position of each '1' element becomes extremely important. These are read in the reverse order to column permutations.

For example, consider the following matrix multiplication.

$$\underset{\text{matrix } P}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \times \underset{\text{matrix } Q}{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix}}$$

Matrix P contains '1' elements in the following positions:

- p_{14} – This indicates that row 4 in matrix Q (elements j, k and l) will become row 1.
- p_{22} – This indicates that row 2 in matrix Q (elements d, e and f) will stay as row 2 (unchanged).
- p_{31} – This indicates that row 1 in matrix Q (elements a, b and c) will become row 3.
- p_{43} – This indicates that row 3 in matrix Q (elements g, h and i) will become row 4.

Because of these permutations, the new matrix will be

$$\begin{bmatrix} j & k & l \\ d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

Permutations can be more efficiently performed using a calculator.

When multiple iterations of a permutation are to be applied, matrix powers can be used to complete the calculation more efficiently.

For example, a permutation matrix raised to the power of 3 would indicate that this permutation is being applied three times in succession.

Worked example 1

Perform a column permutation of matrix X by using permutation matrix P.

$$X = \begin{bmatrix} u & m & l & p \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Explanation – Method 1: By hand

Step 1: Determine the movement of each column in matrix X using the permutation matrix.

p_{13} – column 1 moves to column 3

p_{24} – column 2 moves to column 4

p_{32} – column 3 moves to column 2

p_{41} – column 4 moves to column 1

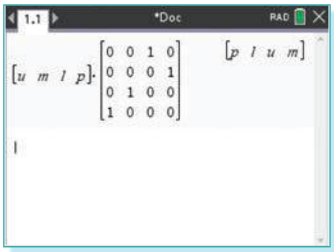
Step 2: Write the new matrix.

Continues →

Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '1: Add Calculator'.

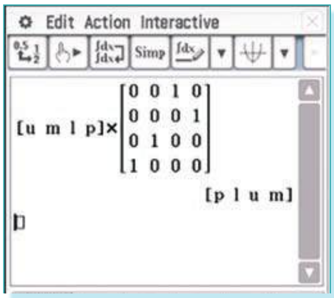
Step 2: Enter the matrix calculation. This is a column permutation, which means the permutation matrix must be post-multiplied to matrix X. Press enter.



Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Enter the matrix calculation. This is a column permutation, which means the permutation matrix must be post-multiplied to matrix X. Press EXE.



Answer – Method 1, 2 and 3

$[p \quad l \quad u \quad m]$

Worked example 2

Perform a row permutation of matrix X by using permutation matrix P.

$$X = \begin{bmatrix} f & o & x \\ f & e & d \\ t & h & e \\ w & a & s \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Explanation – Method 1: By hand

Step 1: Determine the movement of each row in matrix X using the permutation matrix.

p_{13} – row 3 moves to row 1

p_{21} – row 1 moves to row 2

p_{34} – row 4 moves to row 3

p_{42} – row 2 moves to row 4

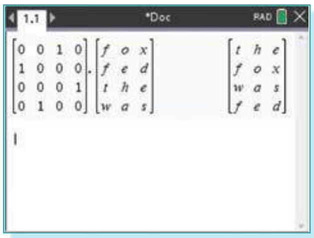
Step 2: Write the new matrix.

Continues →

Explanation – Method 2: TI-Nspire

Step 1: From the home screen, select '1: New' → '4: Add Calculator'.

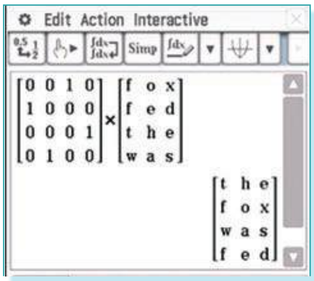
Step 2: Enter the matrix calculation. This is a row permutation, which means the permutation matrix must be pre-multiplied to matrix X. Press enter.



Explanation – Method 3: Casio ClassPad

Step 1: From the main menu, tap √α Main.

Step 2: Enter the matrix calculation. This is a row permutation, which means the permutation matrix must be pre-multiplied to matrix X. Press EXE.



Answer – Method 1, 2 and 3

$$\begin{bmatrix} t & h & e \\ f & o & x \\ w & a & s \\ f & e & d \end{bmatrix}$$

Constructing permutation matrices

When there is a requirement to create a matrix permutation, an understanding of the position of the '1' elements allows for the construction of a permutation matrix.

For example, if column 3 is to be moved to column 1, the resultant permutation matrix P would have a '1' element at p_{31}.

Similarly, if row 3 is to be moved to row 1, the resultant permutation matrix P would have a '1' element at p_{13}.

Worked example 3

Determine the permutation matrix P that would be required to change

$$\begin{bmatrix} 21 & 17 & 42 \\ 31 & 22 & 27 \\ 46 & 33 & 15 \end{bmatrix} \text{ to } \begin{bmatrix} 31 & 22 & 27 \\ 21 & 17 & 42 \\ 46 & 33 & 15 \end{bmatrix}$$

Explanation

Step 1: Determine the type of permutation.

As the rows are staying intact and are being moved up or down, this question represents a row permutation.

Step 2: Determine the order of the permutation matrix.

As the original matrix has 3 rows, the permutation matrix will be a 3×3 square matrix.

Continues →

Step 3: Identify the row changes to determine the '1' elements in the permutation matrix.

Row 1 moves to row 2 – there is a '1' element at p_{21}.

Row 2 moves to row 1 – there is a '1' element at p_{12}.

Row 3 stays as row 3 – there is a '1' element at p_{33}.

Step 4: Construct the permutation matrix.

Write '1's in the required positions, with '0's everywhere else. There should only be one '1' element in each row and column.

Answer

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exam question breakdown

VCAA 2017 Exam 1 Matrices Q4

A permutation matrix, P, can be used to change $\begin{bmatrix} f \\ e \\ a \\ r \\ s \end{bmatrix}$ into $\begin{bmatrix} s \\ a \\ f \\ e \\ r \end{bmatrix}$.

Matrix P is

A. $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ **D.** $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ **E.** $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Explanation

Step 1: Determine the type of permutation.

As there is only one column, this will be a row permutation.

The order of the permutation matrix does not need to be determined, as all solutions are 5×5 matrices.

Step 2: Identify the row changes to determine the '1' elements in the permutation matrix.

Row 1 moves to row 3 – there is a '1' element at p_{31}.

Row 2 moves to row 4 – there is a '1' element at p_{42}.

Row 3 moves to row 2 – there is a '1' element at p_{23}.

Row 4 moves to row 5 – there is a '1' element at p_{54}.

Row 5 moves to row 1 – there is a '1' element at p_{15}.

Answer

C

Step 3: Construct the matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

74% of students answered this question correctly.

This question was generally well answered, but the **12%** of students who chose option A or D did not recall that a permutation matrix can only have one '1' element in every column and row.

7E Questions

Applying permutations to matrices

1. For the following matrices

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\quad D = \begin{bmatrix} 1 & 0 \end{bmatrix},\quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a. Which of these are binary matrices?

b. Which of these are permutation matrices?

---

2. Perform a column permutation on matrix M using permutation matrix P in the following.

a. $M = \begin{bmatrix} 4 & 7 \\ 3 & 9 \end{bmatrix},\quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b. $M = \begin{bmatrix} -3 & 0 & 4 \end{bmatrix},\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $M = \begin{bmatrix} f & m & v \\ c & q & p \end{bmatrix},\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

d. $M = \begin{bmatrix} 15 & 27 & 19 & 12 \\ 7 & 15 & 9 & 11 \\ 14 & 8 & 17 & 15 \end{bmatrix},\quad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

---

3. Perform a row permutation on matrix Q using permutation matrix P in the following.

a. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad Q = \begin{bmatrix} 4 & 9 \\ 3 & 8 \\ 6 & 7 \end{bmatrix}$

b. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad Q = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

c. $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad Q = \begin{bmatrix} a & s & d & f \\ h & j & k & l \end{bmatrix}$

d. $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\quad Q = \begin{bmatrix} 22 & 31 \\ 14 & 17 \\ 25 & 19 \end{bmatrix}$

Constructing permutation matrices

4. Which of the following permutation matrices would be required to change $\begin{bmatrix} s & l & o & t \end{bmatrix}$ to $\begin{bmatrix} l & o & t & s \end{bmatrix}$?

A. Pre-multiplication of $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

B. Pre-multiplication of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

C. Post-multiplication of $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

D. Post-multiplication of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

---

5. For matrix, $A = \begin{bmatrix} s \\ l \\ a \\ p \end{bmatrix}$

a. Determine the permutation matrix P that would be required to change $\begin{bmatrix} s \\ l \\ a \\ p \end{bmatrix}$ to $\begin{bmatrix} l \\ a \\ p \\ s \end{bmatrix}$.

b. Would this be a pre-multiplication or post-multiplication of P?

---

6. Show a matrix calculation that can be used to rearrange the following matrix to spell the word 'MASH':

$\begin{bmatrix} H & S & M & A \end{bmatrix}$

Joining it all together

7. Wordle is a game where players find words inside a grid of jumbled letters. Samantha is playing a variation of the game where she is tasked with finding four letter words running either horizontally or vertically in the following matrix:

$$Q = \begin{bmatrix} i & r & l & s \\ b & u & h & d \\ n & a & e & m \\ t & c & a & t \end{bmatrix}$$

She decides to apply a series of permutation matrices to rearrange the rows and columns so that the words will be easier for her to see.

a. Apply a column permutation to the original letter matrix, Q, so that a word is spelled in one of the rows. Show the calculation as well as the final answer.

b. Apply a row permutation to the original letter matrix, Q, so that a word is spelled in one of the columns. Show the calculation as well as the final answer.

Exam practice

8. The matrix product $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} L \\ E \\ A \\ P \\ S \end{bmatrix}$ is equal to

A. $\begin{bmatrix} L \\ A \\ P \\ S \\ E \end{bmatrix}$ **B.** $\begin{bmatrix} L \\ E \\ A \\ P \\ S \end{bmatrix}$ **C.** $\begin{bmatrix} P \\ L \\ E \\ A \\ S \end{bmatrix}$

D. $\begin{bmatrix} P \\ A \\ L \\ E \\ S \end{bmatrix}$ **E.** $\begin{bmatrix} P \\ E \\ A \\ L \\ S \end{bmatrix}$

VCAA 2016 Exam 1 Matrices Q2

86% of students answered this question correctly.

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9. Matrix P is a 4×4 permutation matrix.

Matrix W is another matrix such that the matrix product $P \times W$ is defined.

This matrix product results in the entire first and third rows of matrix W being swapped.

The permutation matrix P is

A. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **B.** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **E.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

VCAA 2018 Exam 1 Matrices Q4

69% of students answered this question correctly.

10. Matrices P and W are defined as shown.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} A \\ S \\ T \\ O \\ R \end{bmatrix}$$

If $P^n \times W = \begin{bmatrix} A \\ S \\ T \\ O \\ R \end{bmatrix}$, the value of n could be

A. 1 **B.** 2 **C.** 3

D. 4 **E.** 5

VCAA 2020 Exam 1 Matrices Q3

64% of students answered this question correctly.

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11. A theme park has four locations, Air World (A), Food World (F), Ground World (G) and Water World (W). The proportion of visitors moving from one location to another each hour on Saturday is shown in matrix T.

$$\begin{array}{c} \text{this hour} \\ \begin{array}{cc} & \begin{array}{cccc} \text{A} & \text{F} & \text{G} & \text{W} \end{array} \\ T = & \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \begin{array}{c} \text{A} \\ \text{F} \\ \text{G} \\ \text{W} \end{array} \text{ next hour} \end{array} \end{array}$$

The proportion of visitors moving from one location to another each hour on Sunday is different from Saturday, shown in matrix V.

$$\begin{array}{c} \text{this hour} \\ \begin{array}{cc} & \begin{array}{cccc} \text{A} & \text{F} & \text{G} & \text{W} \end{array} \\ V = & \begin{bmatrix} 0.3 & 0.4 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.4 \end{bmatrix} \begin{array}{c} \text{A} \\ \text{F} \\ \text{G} \\ \text{W} \end{array} \text{ next hour} \end{array} \end{array}$$

Matrix V is similar to matrix T but has the first two rows of matrix T interchanged.

The matrix product that will generate matrix V from matrix T is

$V = M \times T$

where matrix M is a binary matrix.

Write down matrix M. (1 MARK)

VCAA 2019 Exam 2 Matrices Q2d

21% of students answered this question correctly.

Questions from multiple lessons

Recursion and financial modelling

12. The value of an annuity investment, in dollars, after n years, V_n, can be modelled by the following recurrence relation.

$V_0 = 58\ 000, \quad V_{n+1} = 1.0027\ V_n + 350$

The increase in value of the investment between the second and third years is closest to

A. \$507 **B.** \$508 **C.** \$509

D. \$1015 **E.** \$1524

Adapted from VCAA 2018 Exam 1 Recursion and financial modelling Q18

Matrices *Year 11 content*

13. The cost of different pastries at a bakery are shown in the table.

croissant	\$5.50
scone	\$3.00
doughnut	\$4.20

Jamie wants to buy two croissants, four scones and three doughnuts.

Which one of the following matrix multiplications will result in a matrix showing the total cost of Jamie's purchase, in dollars?

A. $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \times \begin{bmatrix} 5.50 \\ 3.00 \\ 4.20 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5.50 \\ 3.00 \\ 4.20 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 5.50 \\ 3.00 \\ 4.20 \end{bmatrix}$

D. $\begin{bmatrix} 5.50 \\ 3.00 \\ 4.20 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 3 \end{bmatrix}$

E. $\begin{bmatrix} 2 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 5.50 & 3.00 & 4.20 \end{bmatrix}$

Adapted from VCAA 2017NH Exam 1 Matrices Q2

Matrices *Year 11 content*

14. Three of the most popular streaming websites are Netflix, Hulu and Stan.

The cost, in dollars, for a one month subscription to each of these streaming websites is shown in matrix C.

$$C = \begin{array}{c} \begin{bmatrix} 12.49 \\ 5.99 \\ 10 \end{bmatrix} \end{array} \begin{array}{l} \text{Netflix} \\ \text{Hulu} \\ \text{Stan} \end{array}$$

a. What is the cost of a one month subscription to Stan? (1 MARK)

b. Write down the order of matrix C. (1 MARK)

c. Oscar bought three months of a Netflix subscription and two months of a Hulu subscription.

The total amount of money he spent on subscriptions can be found by the matrix product $S \times C$.

Determine matrix S. (1 MARK)

Adapted from VCAA 2018 Exam 2 Matrices Q1

7F Communication and dominance matrices

STUDY DESIGN DOT POINT

- communication and dominance matrices and their use in analysing communication systems and ranking players in round-robin tournaments

7A 7B 7C 7D 7E **7F** 7G 7H 7I

KEY SKILLS

During this lesson, you will be:

- interpreting and constructing communication matrices
- interpreting and constructing dominance matrices.

KEY TERMS

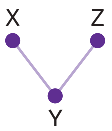
- Communication matrix
- One-step communication links
- Two-step communication links
- Redundant communication links
- Dominance matrix
- One-step dominance
- Two-step dominance

Binary matrices can be used to model everyday life situations. One of these examples is identifying direct and indirect communication pathways between individuals. Additionally, they can be used to analyse and rank teams or individuals in tournaments. In these situations, they are particularly useful when not every team or individual has played each other.

Interpreting and constructing communication matrices

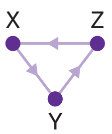
A **communication matrix** is a square binary matrix where the 1's represent the connections in a communication system. In an undirected communication matrix, the connections go in both directions. As a result, undirected communication matrices are always symmetrical about the leading diagonal. In a directed communication matrix, the connections can go in both, or one direction.

The following example is an undirected communication matrix, constructed from the network shown. Element c_{12} indicates a connection from X to Y, as well as from Y to X.



$$C = \begin{array}{c} \begin{matrix} \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{matrix}$$

The following example is a directed communication matrix, constructed from the network shown. Element c_{23} indicates that Y can communicate to Z. Element c_{32}, however, indicates that Z cannot communicate to Y.



$$C = \begin{array}{c} \text{receiver} \\ \begin{matrix} \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{matrix} \ \text{sender}$$

One-step communication links are direct connections between two points. They are demonstrated by a 1 in a communication matrix. Element c_{12} demonstrates a one-step communication link from X to Y.

$$C = \begin{array}{c} \text{receiver} \\ \begin{matrix} \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} \\ \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{matrix} \ \text{sender}$$

Two-step communication links are connections between two points via another point. Squaring a one-step communication matrix will generate a matrix that shows all two-step communication links.

The original matrix C shows that Z cannot communicate directly to Y. However, Z can communicate to Y via X. Element $(c^2)_{32}$ from matrix C^2 shows this two-step communication link from Z to Y.

$$C^2 = \begin{array}{c} \text{receiver} \\ \begin{array}{ccc} \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{array} \ \text{sender}$$

Summing one and two-step communication matrices gives the total number of communication links between two points which use up to two steps.

Matrix T shows the total number of one-step and two-step communication links between the points by summing the one and two-step communication matrices.

$$T = C + C^2 = \begin{array}{c} \text{receiver} \\ \begin{array}{ccc} \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{array} \ \text{sender}$$

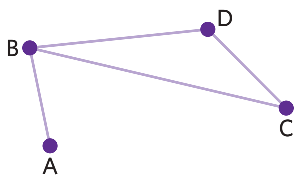
Redundant communication links are links which start and finish at the same point, and are represented by non-zero elements in the leading diagonal. They are 'redundant' as they indicate unchanged information that has returned to the starting point.

Consider a new undirected two-step communication matrix. The element in row 1, column 1, indicates that there is a two-step communication link between J and J. This is redundant as it is sending the information from J back to J.

$$C^2 = \begin{array}{c} \begin{array}{ccc} \mathrm{J} & \mathrm{K} & \mathrm{L} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \mathrm{J} \\ \mathrm{K} \\ \mathrm{L} \end{array}$$

Worked example 1

The following network diagram shows communication lines between four lighthouses, A, B, C and D.



a. Construct an undirected communication matrix, C, that indicates all one-step communication links between the lighthouses.

Explanation

Step 1: Set up an appropriate square matrix.

As there are four lighthouses, it will be a 4×4 matrix.

Label the rows and columns A to D to represent the lighthouses.

As this network is undirected, each element in the matrix represents a two-way connection, so *sender* and *receiver* labels are not required.

$$C = \begin{array}{c} \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \begin{array}{l} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array}$$

Step 2: Fill in the first row of the matrix.

There is a communication line between A and B. There are no communication lines between A and A, A and C or A and D.

$$C = \begin{array}{c} \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \begin{array}{l} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array}$$

Step 3: Repeat this process for the remaining rows.

Continues →

Answer

$$C = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

b. Construct a communication matrix, T, that indicates all one-step and two-step communication links between the lighthouses.

Explanation

Step 1: Determine the two-step communication links.

Calculate matrix C^2.

$$C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Step 2: Calculate matrix T.

Sum the one-step and two-step communication matrices.

$$T = C + C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Answer

$$T = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

c. Using matrix T from part **b**, in how many ways can lighthouse B communicate with lighthouse D?

Explanation

Identify the element from matrix T that represents the communication between lighthouses B and D.

As this is an undirected matrix, either t_{24} or t_{42} are suitable.

$$T = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

This means there are two ways in which lighthouse B can communicate with lighthouse D.

Answer

2

d. Using matrix T from part **b**, how many redundant communication links are there?

Explanation

Sum the elements in the leading diagonal.

$$T = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

$1 + 3 + 2 + 2 = 8$

Answer

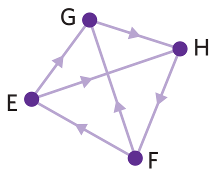
8

Interpreting and constructing dominance matrices

A **dominance matrix** is a square binary matrix where the 1's represent one-step dominances within the associated network. They are generally used for displaying and interpreting competitive matchups between two individuals or teams, where there is a winner and a loser.

Dominance matrices are similar to communication matrices in that they demonstrate links between two points. Unlike communication matrices however, dominance matrices are always directed.

The following network diagram displays the results of a tournament. An arrow from E to G indicates that team E defeated team G. This is represented in the dominance matrix, D, by the element d_{13}.



$$D = \begin{array}{c} \text{loser} \\ \begin{array}{cccc} \text{E} & \text{F} & \text{G} & \text{H} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{E} \\ \text{F} \\ \text{G} \\ \text{H} \end{array} \text{winner}$$

One-step dominance refers to the direct dominance links from one point to another. Summing each row of the dominance matrix will give the one-step dominance scores.

Teams E and F both have dominance scores of 2. This means that both teams defeated 2 other teams in direct matchups.

$$D = \begin{array}{c} \text{loser} \\ \begin{array}{cccc} \text{E} & \text{F} & \text{G} & \text{H} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{E} \\ \text{F} \\ \text{G} \\ \text{H} \end{array} \text{winner} \qquad \begin{array}{c} \text{dominance} \\ 2 \\ 2 \\ 1 \\ 1 \end{array}$$

Two-step dominance occurs when one point has dominance over a second point, and the second point has dominance over a third point. In this case, the first point has two-step dominance over the third point. Squaring the dominance matrix will generate a matrix that indicates all two-step dominances.

In matrix D^2, element $(d^2)_{12}$ indicates that team E has two-step dominance over team F. Team E and F didn't play each other, however team E defeated team H, who in turn defeated team F.

$$D^2 = \begin{array}{c} \text{loser} \\ \begin{array}{cccc} \text{E} & \text{F} & \text{G} & \text{H} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{E} \\ \text{F} \\ \text{G} \\ \text{H} \end{array} \text{winner} \qquad \begin{array}{c} \text{dominance} \\ 2 \\ 3 \\ 1 \\ 2 \end{array}$$

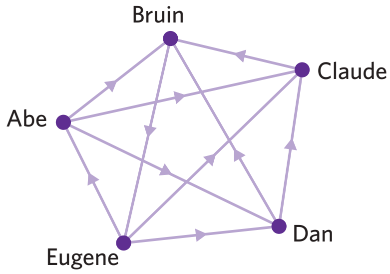
Similar to combining communication matrices, summing one and two-step dominance matrices gives the total number of dominances one point has over each other point. Summing the rows of these total dominances will give a total dominance score, which allows each point to be ranked.

In the total dominance matrix T, team F has a total dominance score of 5. This makes them the top ranked team in the tournament.

$$T = D + D^2 = \begin{array}{c} \text{loser} \\ \begin{array}{cccc} \text{E} & \text{F} & \text{G} & \text{H} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{E} \\ \text{F} \\ \text{G} \\ \text{H} \end{array} \text{winner} \qquad \begin{array}{c} \text{total} \\ \text{dominance} \\ 4 \\ 5 \\ 2 \\ 3 \end{array}$$

Worked example 2

Five players participated in a round-robin jousting tournament. The results are shown in the following network.
An arrow from Abe to Bruin indicates that Abe defeated Bruin.



a. Construct a dominance matrix, D, that indicates all one-step dominances.

Explanation

Step 1: Set up an appropriate square matrix.

As there are five players, it will be a 5×5 matrix.

Label the rows and columns A to E to represent each player.

Label the rows *winner* and the columns *loser*.

$$D = \begin{array}{c} \text{loser} \\ \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\ \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array}\right] \end{array} \begin{array}{l} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} \text{winner}$$

Step 2: Fill in the first row of the matrix.

Abe defeated Bruin, Claude and Dan.

$$D = \begin{array}{c} \text{loser} \\ \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array}\right] \end{array} \begin{array}{l} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} \text{winner}$$

Step 3: Repeat this process for the remaining rows.

Answer

$$D = \begin{array}{c} \text{loser} \\ \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} \text{winner}$$

b. Rank the players from most to least dominant.

Explanation

Step 1: Determine the two-step dominances.

Calculate matrix D^2.

$$D^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Step 2: Calculate matrix T.

Sum the one-step and two-step dominance matrices.

$$T = D + D^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 3 & 3 & 2 & 0 \end{bmatrix}$$

Continues →

Step 3: Sum the rows in matrix T.

$$T = \begin{array}{c} \\ \end{array}\begin{array}{ccccc} & & \text{loser} & & \\ A & B & C & D & E \end{array}$$

$$T = \begin{bmatrix} 0 & 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 3 & 3 & 2 & 0 \end{bmatrix} \begin{array}{l} A \\ B \\ C \\ D \\ E \end{array} \text{ winner} \quad \begin{array}{c} \text{total dominance} \\ 7 \\ 4 \\ 2 \\ 4 \\ 9 \end{array}$$

Step 4: Rank the players in order from most dominant to least dominant according to their total dominance score.

Answer

Eugene, Abe, Bruin/Dan (tied), Claude

Exam question breakdown

VCAA 2019 Exam 1 Matrices Q7

The following communication matrix shows the direct paths by which messages can be sent between two people in a group of six people, U to Z.

$$\begin{array}{c} \text{receiver} \\ \begin{array}{cccccc} U & V & W & X & Y & Z \end{array} \end{array}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} U \\ V \\ W \\ X \\ Y \\ Z \end{array} \text{ sender}$$

A '1' in the matrix shows that the person named in that row can send a message directly to the person named in that column.

For example, the '1' in row 4, column 2 shows that X can send a message directly to V.

In how many ways can Y get a message to W by sending it directly to one other person?

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

Explanation

Step 1: Interpret the question.

The question asks for the number of ways Y can send a message to W via a third person. This is a two-step communication.

Step 2: Determine the two-step communication links for the entire matrix.

This is done by squaring the matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{array}{c} \text{receiver} \\ \begin{array}{cccccc} U & V & W & X & Y & Z \end{array} \\ \begin{bmatrix} 3 & 2 & 2 & 3 & 2 & 2 \\ 1 & 2 & 2 & 2 & 1 & 3 \\ 2 & 3 & 2 & 1 & 4 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 3 & 0 & 2 & 2 & 2 \\ 1 & 2 & 3 & 1 & 3 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ U \\ V \\ W \\ X \\ Y \\ Z \end{array} \text{ sender}$$

Step 3: Identify the number of two-step communication links from Y to W.

This is represented by the element in row 5, column 3, which is 0.

Answer

A

54% of students answered this question correctly.

19% of students incorrectly answered B. This is the number of one-step communication links from Y to W. These students likely didn't recognise that the question required them to find the number of communication links from Y to W via a third person.

7F Questions

Interpreting and constructing communication matrices

1. Which of the following can be classified as a communication matrix?

A.
$$\begin{array}{c} \text{receiver} \\ \begin{array}{cc} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array} \end{array} \text{ sender}$$

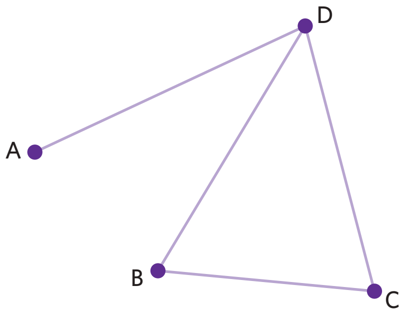
B.
$$\begin{array}{c} \text{receiver} \\ \begin{array}{cc} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 0 & 8 & 0 & 2 \\ 4 & 1 & 5 & 5 \\ 3 & 1 & 2 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array} \end{array} \text{ sender}$$

C.
$$\begin{array}{c} \text{receiver} \\ \begin{array}{cc} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \end{array} \end{array} \text{ sender}$$

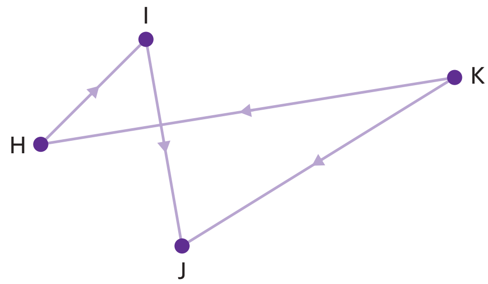
D.
$$\begin{array}{c} \text{receiver} \\ \begin{array}{cc} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 2 & 4 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 1 & 0 & 8 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \end{array} \end{array} \text{ sender}$$

2. Construct a communication matrix that shows all one-step communication links for each of the following diagrams.

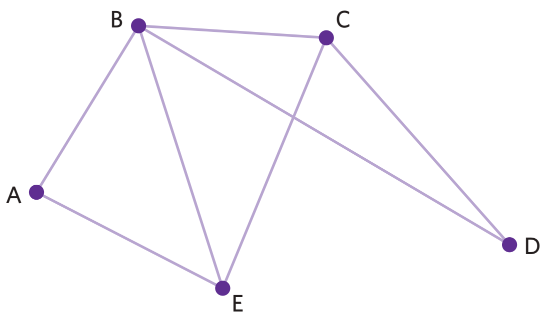
a.



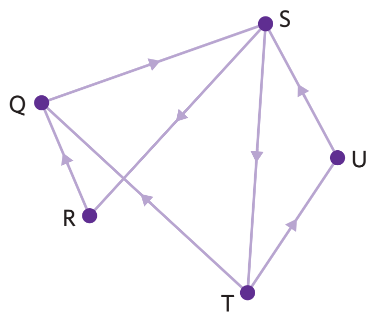
b.



c.



d.



3. The following communication matrix C, shows how four aeroplanes can communicate with each other.

$$C = \begin{array}{cc} \begin{array}{cccc} P1 & P2 & P3 & P4 \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} P1 \\ P2 \\ P3 \\ P4 \end{array} \end{array}$$

a. Interpret c_{13}.

b. Which aeroplane(s) can P4 directly communicate with?

c. Is this a directed or undirected communication matrix?

4. Ariana, Beyonce, Chance, Drake and Eminem are all very careful about who they share their phone number with. The following communication matrix shows which celebrities can directly call each other.

$$\begin{array}{c c} \begin{matrix} A & B & C & D & E \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \end{array}$$

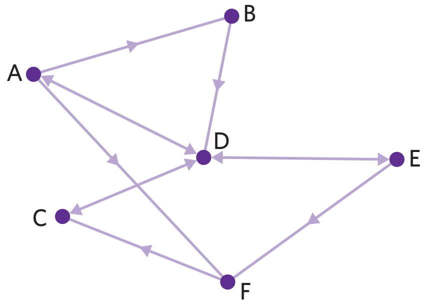
a. Who can Beyonce directly call?

b. Construct a matrix that will represent all the calls that can be made either directly or via a third person.

c. How many communication paths in the matrix found in part **b** are redundant?

---

5. The mayors of six towns, labelled A to F, decided to set up a communication network between the towns. They created a diagram to map out the existing communication network.



a. Construct a total communication matrix, T, that shows all one-step and two-step communication links between the towns.

The mayors decided it would be too expensive to set up a direct communication line between each town. Mayor Abrams, from town A, proposed that they add links to ensure each town had at least a two-step communication link to every other town.

b. Does the existing network already fulfil Mayor Abrams' proposal?

c. Determine the smallest number of one-way communication links that have to be added to fulfil Mayor Abrams' proposal and list each link.

Interpreting and constructing dominance matrices

6. Consider the following dominance matrix.

$$\begin{array}{c c c} \text{loser} & & \\ \begin{matrix} A & B & C & D \end{matrix} & & \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \text{winner} \end{array}$$

Which of the following statements is false?

A. Team A defeated team B.

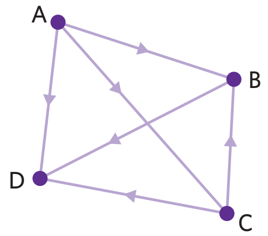
B. Team B lost to three teams.

C. Team C defeated two teams.

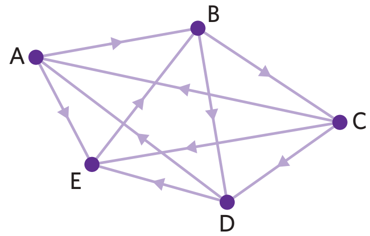
D. Team D lost to less teams than team A.

7. Construct a dominance matrix that shows all one-step dominances in the following networks.

a.



b.



---

8. Four players, ai.zhao999, rainbowmuffin, xthundercatx and felipe.sanchez. competed in a round-robin style League of the Ancients (LOTA) tournament. The following matrix shows the results of the tournament.

$$D = \begin{array}{c} \text{loser} \\ \begin{array}{cc} \begin{array}{cccc} A & R & X & F \end{array} & \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} A \\ R \\ X \\ F \end{array} \end{array} \end{array} \text{ winner}$$

a. Interpret d_{23}.

b. Which player(s) did ai.zhao999 lose to?

c. Who is the top-ranking player according to one-step dominances?

---

9. Six teams competed in a round-robin beach volleyball tournament. After round 4, a tsunami warning was issued and the tournament was put on hold. A dominance matrix was constructed based on the four rounds played.

$$D = \begin{array}{c} \text{loser} \\ \begin{array}{cc} \begin{array}{cccccc} A & B & C & D & E & F \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} \end{array} \end{array} \text{ winner}$$

a. Construct a dominance matrix that shows both one and two-step dominances.

b. Rank the teams from most to least dominant.

The tsunami warning turned out to be a false alarm and the tournament resumed the next day.
In round 5, team C wins against team B, team E wins against team A, and team F wins against team D.

c. Taking into account both one and two-step dominances, rank the teams from most to least dominant after round 5.

---

10. Five members of a chess club played in a round-robin chess tournament, where each person played each of the other people once. None of the games ended in a stalemate.

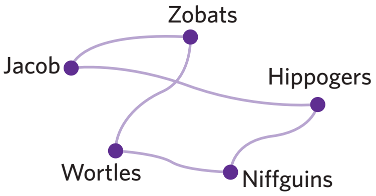
After all the matches were played, the following table of their one-step and two-step dominances was prepared to summarise the results.

member	**one-step dominance**	**two-step dominance**
Fred (F)	3	6
Georgia (G)	3	4
Harry (H)	2	2
Indiya (I)	1	3
Jax (J)	1	1

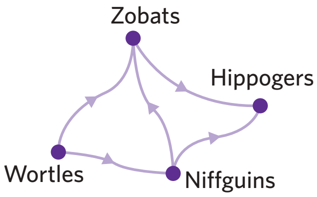
Use the information in the table to construct a one-step dominance matrix.

Joining it all together

11. Jacob stumbled upon a mysterious jungle and was curious to learn more about the inhabitants. After observing them for a couple of days, he realised that whilst he could understand some of them, not all inhabitants communicated to one another. He drew a diagram based on his observations. A line connecting Jacob and Zobats indicates that they directly communicated with one another.



He also discovered a hierarchy amongst predators and prey. The following network diagram illustrates that hierarchy. An arrow from Wortles to Zobats indicates that Wortles eat Zobats.



a. Complete the following matrix to represent all one-step and two-step communications links that Jacob observed.

$$\begin{array}{c c} \begin{array}{ccccc} H & J & N & W & Z \end{array} & \\ \begin{bmatrix} & & & & \\ \\ \\ \\ \\ \end{bmatrix} & \begin{array}{c} H \\ J \\ N \\ W \\ Z \end{array} \end{array}$$

b. Jacob believes that if some of the species translated for the others, Jacob will be able to communicate with all the species in the jungle. Is Jacob right? Justify your answer.

c. Jacob declares the leader of the most dominant species, based on the hierarchy amongst predators and prey, to be King of the Jungle. Taking into account one-step and two-step dominances, of which species does the King belong to?

Exam practice

12. Four teams, A, B, C and D, competed in a round-robin competition where each team played each of the other teams once. There were no draws.

The results are shown in the following matrix.

$$\begin{array}{c c c} \text{loser} & & \\ \begin{array}{cccc} A & B & C & D \end{array} & & \\ \begin{bmatrix} 0 & 0 & f & 1 \\ 1 & 0 & 0 & 0 \\ 1 & g & 0 & 1 \\ 0 & 1 & 0 & h \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \text{winner} \end{array}$$

A ‘1’ in the matrix shows that the team named in that row defeated the team named in that column.

For example, the ‘1’ in row 2 shows that team B defeated team A.

In this matrix, the values of f, g and h are

A. $f = 0, \quad g = 1, \quad h = 0$

B. $f = 0, \quad g = 1, \quad h = 1$

C. $f = 1, \quad g = 0, \quad h = 0$

D. $f = 1, \quad g = 1, \quad h = 0$

E. $f = 1, \quad g = 1, \quad h = 1$

VCAA 2017 Exam 1 Matrices Q5

74% of students answered this question correctly.

13. The main computer system in Elena's office has broken down.

The five staff members, Alex (A), Brie (B), Chai (C), Dex (D) and Elena (E), are having problems sending information to each other.

Matrix M shows the available communication links between the staff members.

$$\begin{array}{c} \text{receiver} \\ M = \begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \text{ sender} \end{array} \end{array}$$

In this matrix:

- the '1' in row A, column B indicates that Alex can send information to Brie.
- the '0' in row D, column C indicates that Dex cannot send information to Chai.

a. Which two staff members can send information directly to each other? (1 MARK)

b. Elena needs to send documents to Chai.
What is the sequence of communication links that will successfully get the information from Elena to Chai? (1 MARK)

c. Matrix M^2 is the square of matrix M and shows the number of two-step communication links between each pair of staff members.

$$\begin{array}{c} \text{receiver} \\ M^2 = \begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ & \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \text{ sender} \end{array} \end{array}$$

Only one pair of individuals has two different two-step communication links.
List each two-step communication link for this pair. (1 MARK)

VCAA 2021 Exam 2 Matrices Q2

Part **a**: **92%** of students answered this question correctly.
Part **b**: **83%** of students answered this question correctly.
Part **c**: **21%** of students answered this question correctly.

Questions from multiple lessons

Data analysis

14. The amount of time, in minutes, that a population of Year 12 students spent watching MeTube over the weekend is approximately normally distributed with a mean of 310 minutes and a standard deviation of 40 minutes.

A student selected at random from this population has a standardised watch time of $z = 2.5$.

The actual amount of time that this student spent watching MeTube in minutes is

A. 160 **B.** 310 **C.** 330 **D.** 410 **E.** 775

Adapted from VCAA 2018 Exam 1 Data analysis Q3

Matrices *Year 11 content*

15. Consider the following matrix equation.

$$3 \times \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} + A = \begin{bmatrix} 4 & -11 \\ 6 & 9 \end{bmatrix}$$

Matrix A is equal to

A. $\begin{bmatrix} 2 & -4 \\ 6 & 6 \end{bmatrix}$ **B.** $\begin{bmatrix} 10 & -26 \\ -6 & 12 \end{bmatrix}$ **C.** $\begin{bmatrix} 2 & -4 \\ -6 & -6 \end{bmatrix}$ **D.** $\begin{bmatrix} 10 & -26 \\ 6 & 12 \end{bmatrix}$ **E.** $\begin{bmatrix} -2 & 4 \\ 6 & 6 \end{bmatrix}$

Adapted from VCAA 2018NH Exam 1 Matrices Q2

Recursion and financial modelling

16. Pedro is a violin virtuoso and recently won $200 000 in a prestigious international chamber music competition. He decided to invest his entire winnings into a perpetuity.

The perpetuity earns interest at a rate of 6.6% per annum, with interest being calculated and paid monthly.

a. What is the value of the monthly payment that Pedro will receive? (1 MARK)

b. Pedro later decides to convert the perpetuity to an annuity investment. The annuity earns interest at a rate of 5.5% per annum, compounding monthly.

For the first three years of the investment, Pedro makes an extra payment of $600 every month to the annuity investment. This payment is made immediately after the interest has been added.

After three years of these payments, Pedro decides to increase his monthly payment into the investment. After two further years of this new monthly payment, Pedro's annuity has a balance of $320 000.

What is the value of Pedro's new monthly payment, correct to the nearest cent? (2 MARKS)

Adapted from VCAA 2017 Exam 2 Recursion and financial modelling Q7

7G Introduction to transition matrices

STUDY DESIGN DOT POINT

- use of the matrix recurrence relation: $S_0 =$ initial state matrix, $S_{n+1} = TS_n$ or $S_{n+1} = LS_n$ where T is a transition matrix, L is a Leslie matrix, and S_n is a column state matrix, to generate a sequence of state matrices (assuming the next state only relies on the current state)

7A 7B 7C 7D 7E 7F **7G** 7H 7I

KEY SKILLS

During this lesson, you will be:

- interpreting and constructing state matrices
- interpreting and constructing transition matrices
- calculating state matrices
- using Leslie matrices in applied scenarios.

KEY TERMS

- State matrix
- Initial state matrix
- Transition matrix
- Leslie matrix

A system consists of two or more interconnected parts working together. Examples of systems include the weather system, the ecosystem and computer systems. Systems are in a constant state of flux, with components of the system ebbing and flowing over time. Transition matrices help to define these changes and allow for the analysis of changes to the system. They can be designed to model real-life situations such as animal population mapping, election poll predictions, and even music composition.

Interpreting and constructing state matrices

A **state matrix** is a column matrix that is a snapshot of a system at a point in time. In a given scenario, there will be regular time intervals between these snapshots, such as days or weeks. The state matrix at time period n is denoted S_n.

For example, if state matrices are used to capture the populations of cities in different years, S_4 would show the populations after 4 years.

An **initial state matrix** is a state matrix that represents the initial, or starting, state of a system. It is most commonly denoted as S_0, but can also be represented as S_1.

When interpreting the elements of a state matrix, the context needs to be considered. For example, the following initial state matrix shows the number of people in a school who prefer apples, bananas, or pears.

See worked example 1

$$S_0 = \begin{bmatrix} 130 \\ 180 \\ 145 \end{bmatrix} \begin{array}{l} \text{apples} \\ \text{bananas} \\ \text{pears} \end{array}$$

It can be determined that, initially, 130 people prefer apples, 180 people prefer bananas, and 145 people prefer pears.

When constructing a state matrix, make sure to label the rows to make it clear what each element represents.

See worked example 2

Worked example 1

The number of Year 10, 11 and 12 students who chose to study maths this year is shown in the initial state matrix, S_0.

$$S_0 = \begin{bmatrix} 137 \\ 112 \\ 82 \end{bmatrix} \begin{array}{l} \text{Year 10} \\ \text{Year 11} \\ \text{Year 12} \end{array}$$

How many Year 11 students chose to study maths this year?

Explanation

Interpret the state matrix in the context of the question.

The second row represents Year 11 students.

Answer

112 students

Worked example 2

In a warehouse, employees are allocated to one of two possible work stations; the picking or packing station. Initially, there were 20 people at the picking station, and 35 people at the packing station. Construct an initial state matrix, S_0, for the information provided.

Explanation

Construct the initial state matrix, making sure to label the rows.

There were initially 20 people at the picking station.

There were initially 35 people at the packing station.

Answer

$$S_0 = \begin{bmatrix} 20 \\ 35 \end{bmatrix} \begin{array}{l} \text{picking} \\ \text{packing} \end{array}$$

Interpreting and constructing transition matrices

See worked example 3

A **transition matrix**, denoted T, is a square matrix that is used to represent the movement or changes of a system between states. It provides information about how much of the data from the system remains the same, or changes. Transition matrices follow the understanding that the value of one state is dependent on the value of a previous state. This helps predict the movement from one time period to another, and determine future and past states.

For example, the following transition matrix can be used to predict the number of workers allocated to each station (A or B) tomorrow based on their designated station today.

$$\begin{array}{c} \text{today} \\ \begin{array}{cc} \text{A} & \text{B} \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \begin{array}{l} \text{A} \\ \text{B} \end{array} \ \text{tomorrow}$$

To interpret the movement of workers between stations from one day to the next, it is important to understand that each element represents a proportion of the column that it is in. For example, element t_{11} of the transition matrix indicates that 20% of people working at station A on one day will continue to work at station A the following day. In contrast, element t_{21} indicates that the remaining 80% of people working at station A one day will instead be working at station B the following day.

See worked example 4

When constructing a transition matrix, it is critical to label the direction of movement with respect to columns and rows. Common words used to label the columns are 'from', 'today', and 'start', whilst common words used to label rows are 'to', 'tomorrow', and 'end'.

All values must be represented as decimal proportions. As the movement of each individual in a system needs to be accounted for, each column will add up to 1.

Worked example 3

The transition matrix, T, is used to predict the number of people who drive, bus, or train to work on a day-to-day basis.

$$\begin{array}{cc} & \text{today} \\ & \begin{array}{ccc} \text{drive} & \text{bus} & \text{train} \end{array} \\ T = & \begin{bmatrix} 0.68 & 0.34 & 0.58 \\ 0.22 & 0.17 & 0.36 \\ 0.10 & 0.49 & 0.06 \end{bmatrix} \end{array} \begin{array}{l} \text{drive} \\ \text{bus} \\ \text{train} \end{array} \text{ tomorrow}$$

What percentage of people who caught the train today will choose to drive tomorrow?

Explanation

Step 1: Identify the relevant element.

The element that corresponds to the proportion of people who caught a train today and will drive tomorrow is located in column 3 (train today), and row 1 (drive tomorrow).

$$\begin{array}{cc} & \text{today} \\ & \begin{array}{ccc} \text{drive} & \text{bus} & \text{train} \end{array} \\ T = & \begin{bmatrix} 0.68 & 0.34 & 0.58 \\ 0.22 & 0.17 & 0.36 \\ 0.10 & 0.49 & 0.06 \end{bmatrix} \end{array} \begin{array}{l} \text{drive} \\ \text{bus} \\ \text{train} \end{array} \text{ tomorrow}$$

Step 2: Convert the element value to a percentage.

$0.58 = 58\%$

Answer

58%

Worked example 4

Johnny is an avid fan of fish, and has two different ponds (A and B) for his Koi fish.

The number of Koi fish in each pond can change on a daily basis.

- 10% of the Koi fish in pond A today will remain in pond A tomorrow.
- 15% of the Koi fish in pond B today will be moved to pond A tomorrow.

Construct a transition matrix, T, to represent the movement of Koi fish in each pond from one day to the next.

Explanation

Step 1: Set up a blank matrix, labelled T.

Since there are two possible states, pond A and pond B, a 2×2 matrix is required.

Label the rows and columns 'A' and 'B' to represent the two different ponds.

The columns represent today and the rows represent tomorrow.

$$\begin{array}{cc} & \text{today} \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} \\ T = & \begin{bmatrix} & \\ & \end{bmatrix} \end{array} \begin{array}{l} \text{A} \\ \text{B} \end{array} \text{ tomorrow}$$

Step 2: Fill in the matrix with the information provided, converting percentages into decimals.

10% of Koi fish in pond A today will remain in pond A tomorrow.

$t_{11} = 0.10$

15% of Koi fish in pond B today will be moved to pond A tomorrow.

$t_{12} = 0.15$

$$\begin{array}{cc} & \text{today} \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} \\ T = & \begin{bmatrix} 0.10 & 0.15 \\ & \end{bmatrix} \end{array} \begin{array}{l} \text{A} \\ \text{B} \end{array} \text{ tomorrow}$$

Continues →

Step 3: Calculate the remaining elements in the transition matrix.

Since each column must add up to 1, the unknown elements can be identified.

Column A: $1 - 0.10 = 0.90$

Column B: $1 - 0.15 = 0.85$

Answer

$$T = \begin{array}{c} \text{today} \\ \begin{array}{cc} \mathrm{A} & \mathrm{B} \end{array} \\ \begin{bmatrix} 0.10 & 0.15 \\ 0.90 & 0.85 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \mathrm{A} \\ \mathrm{B} \end{array} \text{ tomorrow}$$

Calculating state matrices

The next state matrix, denoted S_{n+1}, can be calculated recursively by multiplying the transition matrix, T, with the current state matrix, S_n.

See worked example 5

This can be modelled using a recurrence relation of the form

$S_0 = \textit{initial state matrix}, \quad S_{n+1} = T \times S_n$, where

- S_n is the current state matrix
- S_{n+1} is the next state matrix
- T is the transition matrix.

It is also possible to calculate previous state matrices using inverse transition matrices.

See worked example 6

When the value of n is large, calculating a state matrix recursively is time-consuming. In these instances, the following rule can be used to calculate state matrices for any value of n.

See worked example 7

$S_n = T^n \times S_0$, where

- S_n is the current state matrix
- T is the transition matrix
- S_0 is the initial state matrix.

Worked example 5

Every night, a colony of seals can settle on either island A or island B. On Sunday, there were 130 seals on island A and 180 seals on island B.

The initial state and transition matrices are provided.

$$S_0 = \begin{bmatrix} 130 \\ 180 \end{bmatrix} \begin{array}{l} \mathrm{A} \\ \mathrm{B} \end{array} \qquad T = \begin{array}{c} \text{today} \\ \begin{array}{cc} \mathrm{A} & \mathrm{B} \end{array} \\ \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \mathrm{A} \\ \mathrm{B} \end{array} \text{ tomorrow}$$

a. Determine the matrix recurrence relation that can be used to calculate the number of seals on each island every day.

Explanation

Substitute S_0 and T into the recurrence relation form.

$S_0 = \textit{initial state matrix}, \quad S_{n+1} = T \times S_n$

Answer

$$S_0 = \begin{bmatrix} 130 \\ 180 \end{bmatrix}, \quad S_{n+1} = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \times S_n$$

Continues →

b. Use recursion to calculate the number of seals on each island on Tuesday, rounded to the nearest whole number.

Explanation

Step 1: Use the recurrence relation to calculate the number of seals on Monday, S_1.

$S_1 = T \times S_0$

$S_1 = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \times \begin{bmatrix} 130 \\ 180 \end{bmatrix}$

$S_1 = \begin{bmatrix} 134 \\ 176 \end{bmatrix}$

Step 2: Use the recurrence relation to calculate the number of seals on Tuesday, S_2.

$S_2 = T \times S_1$

$S_2 = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix} \times \begin{bmatrix} 134 \\ 176 \end{bmatrix}$

$S_2 = \begin{bmatrix} 132.4 \\ 177.6 \end{bmatrix}$

Answer

Island A: 132 seals

Island B: 178 seals

Worked example 6

Consider the following state matrix after 3 periods, S_3, and transition matrix, T.

$S_3 = \begin{bmatrix} 120 \\ 150 \end{bmatrix} \quad T = \begin{bmatrix} 0.84 & 0.21 \\ 0.16 & 0.79 \end{bmatrix}$

Determine S_2, rounding the matrix elements to the nearest whole number.

Explanation

Step 1: Substitute S_2 and S_3 into the equation $S_{n+1} = T \times S_n$.

$S_3 = T \times S_2$

Step 2: Solve for S_2.

Pre-multiply both sides by T^{-1}.

$T^{-1} \times S_3 = T^{-1} \times T \times S_2$

$T^{-1} \times S_3 = S_2$

$S_2 = \begin{bmatrix} 0.84 & 0.21 \\ 0.16 & 0.79 \end{bmatrix}^{-1} \times \begin{bmatrix} 120 \\ 150 \end{bmatrix}$

$S_2 = \begin{bmatrix} 100.47... \\ 169.52... \end{bmatrix}$

Answer

$\begin{bmatrix} 100 \\ 170 \end{bmatrix}$

Worked example 7

Consider the following initial state matrix and transition matrix.

$S_0 = \begin{bmatrix} 213 \\ 142 \end{bmatrix} \quad T = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$

Determine S_{10}, rounding the matrix elements to the nearest whole number.

Explanation

Step 1: Substitute the known matrices into the rule $S_n = T^n \times S_0$.

$S_n = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}^n \times \begin{bmatrix} 213 \\ 142 \end{bmatrix}$

Step 2: Calculate S_{10}.

The n value required is $n = 10$.

$S_{10} = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}^{10} \times \begin{bmatrix} 213 \\ 142 \end{bmatrix}$

$S_{10} = \begin{bmatrix} 199.76... \\ 155.23... \end{bmatrix}$

Continues →

Answer

$$\begin{bmatrix} 200 \\ 155 \end{bmatrix}$$

Using Leslie matrices in applied scenarios

A **Leslie matrix**, denoted L, is a unique application of transition matrices that can be used to model the growth of a population and its age distribution over time. Generally, population growth is modelled year-to-year.

When applying a Leslie matrix to a population, only the females in the population are considered. In contrast to a standard transition matrix, the columns of a Leslie matrix do not add up to 1.

Leslie matrices are square matrices of size n and are presented in the following form.

$$L = \begin{array}{c} \text{age} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} F_0 & F_1 & F_2 \\ P_0 & 0 & 0 \\ 0 & P_1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{fertility rate} \\ \text{survival rate from age 0 – age 1} \\ \text{survival rate from age 1 – age 2} \end{array}$$

In a Leslie matrix, F represents fertility rates.

- F_0 is the average number of females born to each female that is less than 1 year old.
- F_1 is the average number of females born to each 1-year-old female.
- F_2 is the average number of females born to each 2-year-old female.

As such, F_n is the average number of females born to each female that is n years old.

In a Leslie matrix, P represents survival rates.

- P_0 is the average survival rate of females that are less than 1 year old.
- P_1 is the average survival rate of females that are 1 year old.
- The remaining elements contain zeros (0), allowing the Leslie matrix to calculate the number of females that survive at each age group, when multiplied with an initial state matrix.

As such, P_n is the average survival rate of females that are n years old.

As this is a 3×3 matrix, there is no P_2 value. This means that all females aged 2 years and over will not survive into the following year.

A state matrix S_n, is used to represent the breakdown of female age groups in the population. Future state matrices can be calculated by using the Leslie matrix as a transition matrix.

Worked example 8

Scientists have collected the yearly fertility and survival rates for siamese fighting fish. A small sample of the species is under observation. The following Leslie matrix for the survival rate of siamese fighting fish can be used to model the growth of the sample population.

$$L = \begin{array}{c} \text{age} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix} \end{array}$$

The following initial state matrix represents the female population of the sample being observed at the start of the study.

$$S_0 = \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix} \begin{array}{l} 0 \\ 1 \\ 2 \end{array} \text{ age}$$

Continues →

a. Interpret the element in row 2 and column 1 of the Leslie matrix.

Explanation

Step 1: Locate the relevant element on the Leslie matrix.

$$\begin{array}{c} \text{age} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \end{array}$$

$$L = \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix}$$

Step 2: Interpret the element.

0.85 is the average survival rate of females that are less than 1 year old.

Answer

85% of the female population that is less than 1 year old will survive into the next year.

b. Interpret the element in row 1 and column 3 of the Leslie matrix.

Explanation

Step 1: Locate the relevant element on the matrix.

$$\begin{array}{c} \text{age} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \end{array}$$

$$L = \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix}$$

Step 2: Interpret the element.

0.90 is the average number of females born to each female that is 2 years old.

Answer

On average, there will be 0.90 females born to each 2-year-old female.

c. What is the estimated total number of female siamese fighting fish after two years?

Explanation

Step 1: Substitute S_0 and L into the recurrence relation.

$S_0 = \textit{initial state matrix}, \quad S_{n+1} = L \times S_n$

$$S_0 = \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix}, \quad S_{n+1} = \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix} \times S_n$$

Step 2: Use the recurrence relation to calculate S_1.

$S_1 = L \times S_0$

$$S_1 \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix} \times \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 61.50 \\ 42.50 \\ 30.00 \end{bmatrix}$$

Step 3: Use the recurrence relation to calculate S_2.

$S_2 = L \times S_1$

$$S_2 = \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix} \times \begin{bmatrix} 61.50 \\ 42.50 \\ 30.00 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 64.075 \\ 52.275 \\ 31.875 \end{bmatrix}$$

Step 4: Round the elements in S_2 to the nearest whole number.

There can only be a whole number of siamese fighting fish at each age.

$$S_2 \approx \begin{bmatrix} 64 \\ 52 \\ 32 \end{bmatrix}$$

Step 5: Sum the elements in S_2.

$64 + 52 + 32 = 148$

Answer

148 female siamese fighting fish

Continues →

d. Determine S_8, the expected female population in the study after 8 years. Round the matrix elements to the nearest whole number.

Explanation

Step 1: Substitute S_0 and L into the rule $S_n = L^n \times S_0$.

$$S_n = \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix}^n \times \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix}$$

Step 2: Calculate S_8.

The n value required is $n = 8$.

$$S_8 = \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix}^8 \times \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix}$$

$$S_8 = \begin{bmatrix} 127.95... \\ 97.23... \\ 65.64... \end{bmatrix}$$

Answer

$$\begin{array}{c c l} \begin{bmatrix} 128 \\ 97 \\ 66 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \text{age} \end{array}$$

Exam question breakdown

VCAA 2019 Exam 1 Matrices Q4

Stella completed a multiple-choice test that had 10 questions.

Each question had five possible answers, A, B, C, D and E.

For question number one, Stella chose the answer E.

Stella chose each of the nine remaining answers, in order, by following the transition matrix, T.

$$T = \begin{array}{c} \text{this question} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ A \\ B \\ C \quad \text{next question} \\ D \\ E \end{array}$$

What answer did Stella choose for question number six?

A. A **B.** B **C.** C **D.** D **E.** E

Explanation

Step 1: Identify S_0, the initial state matrix.

S_0 would be a 5×1 matrix due to the five options (A, B, C, D, E). Since Stella chose E for the first question, the elements in the first 4 rows are 0, and the element in the fifth row is 1.

$$S_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 2: Substitute S_0 and T into the rule $S_n = T^n \times S_0$.

$$S_n = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^n \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 3: Calculate S_5.

The n value required is $n = 5$. This is because the initial state matrix, with $n = 0$, represents Stella's answer to question 1.

$$S_5 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^5 \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Answer

E

56% of students answered this question correctly.

24% of students incorrectly answered C. This may be because some students raised their transition matrix to a power of 6 instead of 5 when using a rule to answer this question. Students likely overlooked the fact that their initial state matrix corresponds to Stella's answer to question 1, so only 5 more iterations of the matrix multiplication were required.

7G Questions

Interpreting and constructing state matrices

1. A transition model follows the recurrence relation.

$$S_0 = \begin{bmatrix} 27 \\ 45 \\ 15 \end{bmatrix}, \quad S_{n+1} = \begin{bmatrix} 0.5 & 0.3 & 0.8 \\ 0.3 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0 \end{bmatrix} \times S_n$$

What is the initial state matrix?

A. $S_0 = \begin{bmatrix} 27 \\ 45 \\ 15 \end{bmatrix}$

B. $S_0 = \begin{bmatrix} 0.5 & 0.3 & 0.8 \\ 0.3 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0 \end{bmatrix}$

C. $S_{n+1} = \begin{bmatrix} 0.5 & 0.3 & 0.8 \\ 0.3 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0 \end{bmatrix} \times S_n$

D. $T = \begin{bmatrix} 0.5 & 0.3 & 0.8 \\ 0.3 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0 \end{bmatrix}$

---

2. People attending a local carnival have the choice of enjoying 3 different rides: 'Timid Tiger' (T), 'Wild Warrior' (W) or 'The Devastator' (D), or can choose not to take a ride at all (N). After a ride, attendees can choose to take the same ride, take a different ride, or stop taking rides altogether.

The ride that attendees first chose, on a particular day, is shown in the initial state matrix, S_0. The way in which attendees are predicted to move between rides is modelled by the transition matrix, T.

$$S_0 = \begin{array}{c} \begin{bmatrix} 32 \\ 85 \\ 57 \\ 24 \end{bmatrix} \end{array} \begin{array}{l} \text{T} \\ \text{W} \\ \text{D} \\ \text{N} \end{array} \qquad T = \begin{array}{c} \text{this ride} \\ \begin{array}{cccc} \text{T} & \text{W} & \text{D} & \text{N} \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.2 & 0 \\ 0.6 & 0.3 & 0.3 & 0 \\ 0.1 & 0.5 & 0.1 & 0 \\ 0.1 & 0.1 & 0.4 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{T} \\ \text{W} \\ \text{D} \\ \text{N} \end{array} \text{ next ride}$$

a. How many people rode 'The Devastator' as their first ride of the day?

b. How many people attended the carnival on the given day?

c. What percentage of people chose not to take a ride at all when attending the carnival, rounded to the nearest percent?

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3. The food company 'Barnets' releases 3 all-new flavours of their popular savoury biscuit brand: sweet-and-sour (S), hot-and-spicy (H), and American mustard (A).

a. When the flavours were first released, 47 270 boxes of sweet-and-sour, 39 231 boxes of hot-and-spicy, and 56 159 boxes of American mustard were sold in the first week.

Construct an initial state matrix, S_0, to represent this information.

b. At the end of the 10th week, Barnets noticed that their new flavours weren't selling as well as they hoped, with significant declines in sales. Only 21 452 boxes of sweet-and-sour, 23 527 boxes of hot-and-spicy, and 12 339 boxes of American mustard were sold in that week.

Construct a state matrix to represent this information.

Interpreting and constructing transition matrices

4. A transition model follows the recurrence relation

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}, \quad S_{n+1} = \begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix} \times S_n$$

What is the transition matrix?

A. $S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$

B. $S_0 = \begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix}$

C. $S_{n+1} = \begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix} \times S_n$

D. $T = \begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix}$

5. When recycling, people either fold (F) or scrunch (S) their paper. A researcher finds that she can predict a person's technique each time they recycle paper using the following transition matrix.

$$T = \begin{array}{c} \text{this time} \\ \begin{array}{cc} \text{F} & \text{S} \end{array} \\ \begin{bmatrix} 0.91 & 0.23 \\ 0.09 & 0.77 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{F} \\ \text{S} \end{array} \text{ next time}$$

a. What percentage of people who fold their paper for recycling this time are expected to scrunch next time?

b. What percentage of people who scrunch their paper for recycling this time are expected to fold next time?

c. What percentage of people who fold their paper for recycling this time are expected to fold again next time?

d. What percentage of people who scrunch their paper for recycling this time are expected to scrunch again next time?

6. A teacher finds that 35% of students who buy lunch today (B) will buy lunch tomorrow, and 50% of students who do not buy lunch today (N) will buy lunch tomorrow.

The teacher also recorded that 36 students bought lunch today, whilst 15 students did not.

Use this information to construct a transition matrix, T.

Calculating state matrices

7. Consider the following initial state matrix and transition matrix.

$$S_0 = \begin{bmatrix} 20 \\ 10 \end{bmatrix} \quad T = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$$

a. Which calculation can be used to determine S_1?

A. $S_1 = \begin{bmatrix} 20 \\ 10 \end{bmatrix} \times \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

B. $S_1 = \begin{bmatrix} 20 \\ 10 \end{bmatrix}^{-1} \times \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

C. $S_1 = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix} \times \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

D. $S_1 = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}^{-1} \times \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

b. Calculate S_2, rounding the matrix elements to one decimal place.

c. Calculate S_3, rounding the matrix elements to one decimal place.

8. Consider the following initial state matrix and transition matrix.

$$S_0 = \begin{bmatrix} 90 \\ 1968 \end{bmatrix} \quad T = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$$

a. Which calculation can be used to determine S_7?

A. $S_7 = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}^7 \times \begin{bmatrix} 90 \\ 1968 \end{bmatrix}$

B. $S_7 = \begin{bmatrix} 90 \\ 1968 \end{bmatrix}^7 \times \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

C. $S_7 = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix} \times \begin{bmatrix} 90 \\ 1968 \end{bmatrix}^7$

D. $S_7 = \begin{bmatrix} 90 \\ 1968 \end{bmatrix} \times \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}^7$

b. Calculate S_5, rounding the matrix elements to two decimal places.

c. Calculate S_{40}, rounding the matrix elements to two decimal places.

9. Consider the following state matrix after one period, S_1, and transition matrix, T.

$$S_1 = \begin{bmatrix} 169 \\ 116 \\ 125 \end{bmatrix} \quad T = \begin{bmatrix} 0.5 & 0.2 & 0.5 \\ 0.2 & 0.6 & 0.1 \\ 0.3 & 0.2 & 0.4 \end{bmatrix}$$

Calculate S_0.

10. A psychology experiment is conducted where 50 participants decide to eat either a red or blue gummy every hour. A transition matrix, T, has been constructed to try and predict the number of participants who will eat each gummy each hour. The number of participants who decided to eat each of the gummies at the start of the experiment is shown in matrix S_0.

$$S_0 = \begin{array}{c} \begin{bmatrix} 40 \\ 10 \end{bmatrix} \end{array}\begin{array}{l} \text{red} \\ \text{blue} \end{array} \qquad T = \begin{array}{c} \text{this hour} \\ \begin{array}{cc} \text{red} & \text{blue} \end{array} \\ \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix} \end{array}\begin{array}{l} \\ \\ \text{red} \\ \text{blue} \end{array} \ \text{next hour}$$

a. How many participants are predicted to eat the red gummy in the third hour?

b. How many participants are predicted to eat the blue gummy in the sixth hour?

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11. Superstar travel agent Tal has 300 regular customers who often go on trips to Mars. Based on previous years, Tal has come up with a transition matrix, T, to help him predict whether his customers will go on holiday or not. In 2050, 167 customers went to Mars and 133 customers did not.

$$T = \begin{array}{c} \text{this year} \\ \begin{array}{cc} \text{trip} & \text{no trip} \end{array} \\ \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix} \end{array}\begin{array}{l} \\ \\ \text{trip} \\ \text{no trip} \end{array} \ \text{next year} \qquad S_0 = \begin{bmatrix} 167 \\ 133 \end{bmatrix}\begin{array}{l} \text{trip} \\ \text{no trip} \end{array}$$

a. Calculate how many customers are predicted to go to Mars in 2060, rounded to the nearest whole number.

b. To make a profit, Tal must send at least 150 holiday goers to Mars every year. Determine if he will be profitable or not in 2063.

Using Leslie matrices in applied scenarios

12. The following Leslie matrix can be used to model the growth of a population of hedgehogs.

$$L = \begin{array}{c} \text{age} \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.5 & 0.8 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix} \end{array}$$

The female population of a sample of hedgehogs at the start of a study is shown in the initial state matrix, S_0.

$$S_0 = \begin{bmatrix} 120 \\ 100 \\ 85 \\ 65 \end{bmatrix}\begin{array}{l} 0 \\ 1 \\ 2 \\ 3 \end{array} \ \text{age}$$

a. Interpret the element in row 4 and column 3 of the Leslie matrix.

A. On average, 60% of females will be born to a 2-year-old female.

B. On average, there will be 0.6 females born to each 2-year-old female.

C. 0.6% of the female population that is 2 years old will survive into the next year.

D. 60% of the female population that is 2 years old will survive into the next year.

b. Interpret the element in row 1 and column 4 of the Leslie matrix.

A. On average, 80% of females will be born to a 3-year-old female.

B. On average, there will be 0.8 females born to each 3-year-old female.

C. 0.8% of the female population that is 3 years old will survive into the next year.

D. 80% of the female population that is 3 years old will survive into the next year.

13. The Leslie matrix for a critically endangered species is given.

$$L = \begin{array}{c} \text{age} \\ \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.8 & 0.6 \\ 0.4 & 0 \end{bmatrix} \end{array}$$

The initial female population has also been provided.

$$S_0 = \begin{bmatrix} 45 \\ 20 \end{bmatrix} \begin{array}{c} 0 \\ 1 \end{array} \ \text{age}$$

a. Using a matrix recurrence relation, determine the expected female population after 2 years.

b. Determine the state matrix that represents the expected female population after 10 years.

c. Has the female population of the critically endangered species increased after 10 years? If so, by how much?

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14. Scientists have been monitoring the survival and reproductive rates of a rare species of chameleon on the island of Madagascar. They've collated their findings in the following Leslie matrix.

$$L = \begin{array}{c} \text{age} \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.7 & 0.8 & 0.6 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix} \end{array}$$

The female population of the species at the start of the study is shown by the initial state matrix, S_0.

$$S_0 = \begin{bmatrix} 59 \\ 64 \\ 48 \\ 32 \end{bmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \ \text{age}$$

If 45% of the population of the species of chameleons are female, what is the expected total population of the species after 15 years?

Joining it all together

15. In Circle Hill, every household has tacos for Mexican night once a week. They either have soft (S) or hard (H) shelled tacos; they cannot have both.

A survey of the neighbourhood found that 40% of families that eat soft shell tacos (S) for Mexican night this week will eat hard shell tacos next week. 65% of families that eat hard shell tacos (H) for Mexican night this week will eat hard shell tacos next week.

a. Use this information to construct a transition matrix, T.

b. What percentage of families that eat soft shell tacos this week are expected to eat soft shell tacos next week?

c. What percentage of families that eat hard shell tacos this week are expected to eat soft shell tacos next week?

There are 400 families in this neighbourhood.

d. If 200 families eat soft shell tacos this week, how many families will eat hard shell tacos next week?

e. In a particular week, 120 families eat soft shell tacos. How many families are expected to change taco shells the following week?

16. A new cafeteria is opening at Edrolo High and the principal wants to know how many students will be buying food at the cafeteria each day. Mr Barry is a matrices enthusiast and says he can predict the number of students that will buy food from the cafeteria each day. By surveying the students, he constructs a transition matrix, T, and an initial state matrix, S_0, based on the first Monday of school.

$$S_0 = \begin{array}{c} \begin{bmatrix} 122 \\ 78 \end{bmatrix} \end{array} \begin{array}{l} \text{buy} \\ \text{not buy} \end{array} \qquad T = \begin{array}{c} \text{today} \\ \begin{array}{cc} \text{buy} & \text{not buy} \end{array} \\ \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{buy} \\ \text{not buy} \end{array} \ \text{tomorrow}$$

a. How many students will be expected to buy food on the first Tuesday?

b. How many students will be expected to buy food on the first Friday?

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17. In the lead up to exams, teachers are offering study sessions in English (E), maths (M) and history (H). To ensure that there are enough teachers to help the students, they construct a transition matrix, T, which will help to predict the next study session a student will attend.

$$T = \begin{array}{c} \text{this period} \\ \begin{array}{ccc} \text{E} & \text{M} & \text{H} \end{array} \\ \begin{bmatrix} 0.29 & 0.32 & 0.29 \\ 0.32 & 0.62 & 0.12 \\ 0.39 & 0.06 & 0.59 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{E} \\ \text{M} \\ \text{H} \end{array} \ \text{next period}$$

- 14 students attended the first English study session.
- 23 students attended the first maths study session.
- 13 students attended the first history study session.

a. Construct the initial state matrix, S_0.

b. How many students are predicted to attend the third maths study session?

c. There must be at least one teacher for every 5 students.
How many teachers will be required for the sixth history study session?

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18. Scientists have been investigating the population changes of green tree frogs. They've collected their findings on the yearly survival and fertility rates in the following Leslie matrix.

$$L = \begin{array}{c} \text{age} \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix} \end{array}$$

A study is being conducted on an isolated group of green tree frogs to verify if their findings are accurate. Initially, there were

- 10 females under 1 year old
- 12 female one-year-olds
- 14 female two-year-olds
- 20 female three-year-olds.

a. Construct the initial state matrix, S_0.

b. What is the average number of female babies that are expected to be born to each two-year-old female green tree frog?

c. What is the expected number of three-year-old female green tree frogs after three years of observations?

d. What is the expected number of female green tree frogs after 10 years of observations?

e. If 60% of the population are female, determine the total population of green tree frogs after 2 years of observations.

Exam practice

19. A travel company is studying the choice between air (A), land (L), and sea (S) or no (N) travel by some of its customers each year.

Matrix T contains the percentages of customers and their choice of travel from year to year.

$$T = \begin{array}{c} \text{this year} \\ \begin{array}{cccc} \text{A} & \text{L} & \text{S} & \text{N} \end{array} \\ \begin{bmatrix} 0.65 & 0.25 & 0.25 & 0.50 \\ 0.15 & 0.60 & 0.20 & 0.15 \\ 0.05 & 0.10 & 0.25 & 0.20 \\ 0.15 & 0.05 & 0.30 & 0.15 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \text{A} \\ \text{L} \\ \text{S} \\ \text{N} \end{array} \quad \text{next year}$$

Let S_n be the matrix that shows the number of customers who choose each type of travel n years after 2014. Matrix S_0 shows the number of customers who chose each type of travel in 2014.

$$S_0 = \begin{bmatrix} 520 \\ 320 \\ 80 \\ 80 \end{bmatrix} \begin{array}{c} \text{A} \\ \text{L} \\ \text{S} \\ \text{N} \end{array}$$

Matrix S_1 shows the number of customers who chose each type of travel in 2015.

$$S_1 = TS_0 = \begin{bmatrix} 478 \\ d \\ e \\ f \end{bmatrix} \begin{array}{c} \text{A} \\ \text{L} \\ \text{S} \\ \text{N} \end{array}$$

Write the values missing from matrix S_1 (d, e, f) in the boxes provided. (1 MARK)

$d =$ ☐ $e =$ ☐ $f =$ ☐

VCAA 2016 Exam 2 Matrices Q3a

85% of students answered this question correctly.

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20. Senior students at a high school must choose one elective activity in each of the four terms in 2018.

Their choices are communication (C), investigation (I), problem-solving (P) and services (S).

The transition matrix, T, shows the way in which senior students are expected to change their choice of elective activity from term to term.

$$T = \begin{array}{c} \text{this term} \\ \begin{array}{cccc} \text{C} & \text{I} & \text{P} & \text{S} \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.3 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \text{C} \\ \text{I} \\ \text{P} \\ \text{S} \end{array} \quad \text{next term}$$

Let S_n be the state matrix for the number of senior students expected to choose each elective activity in term n.

For the given matrix S_1, a matrix rule that can be used to predict the number of senior students in each elective activity in terms 2, 3 and 4 is

$$S_1 = \begin{bmatrix} 300 \\ 200 \\ 200 \\ 300 \end{bmatrix}, \quad S_{n+1} = TS_n$$

a. How many senior students will not change their elective activity from term 1 to term 2? (1 MARK)

b. Complete S_2, the state matrix for term 2. (1 MARK)

$$S_2 = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \begin{array}{c} \text{C} \\ \text{I} \\ \text{P} \\ \text{S} \end{array}$$

c. Of the senior students expected to choose investigation (I) in term 3, what percentage choose service (S) in term 2? (2 MARKS)

VCAA 2017 Exam 2 Matrices Q3

Part **a**: **47%** of students answered this question correctly.

Part **b**: **73%** of students answered this question correctly.

Part **c**: The average mark on this question was **0.3**.

Questions from multiple lessons

Data analysis

21. The Year 12 cohort at Edrolo High just sat their end-of-year Psychology exam. Their exam scores were approximately normally distributed with a mean of 70.2% and a standard deviation of 10.1%. 278 students sat the exam. The number of students expected to have passed the exam (received a mark over 50%) is closest to

A. 7 **B.** 14 **C.** 236 **D.** 264 **E.** 271

Adapted from VCAA 2016 Exam 1 Data analysis Q4

Matrices *Year 11 content*

22. The elements in matrix M are determined by the rule $m_{ij} = 3i + 2j$.

Which of the following **cannot** be matrix M?

A. $\begin{bmatrix} 5 & 7 \\ 8 & 10 \end{bmatrix}$ **B.** $[5]$ **C.** $[5 \quad 8 \quad 11]$ **D.** $\begin{bmatrix} 5 \\ 8 \\ 11 \\ 14 \end{bmatrix}$ **E.** $\begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{bmatrix}$

Adapted from VCAA 2017NH Exam 1 Matrices Q4

Matrices *Year 11 content*

23. An op-shop sells tops (T), pants (P), and dresses (D).

The number of each sold on Monday, Tuesday and Wednesday is shown in matrix N.

$$N = \begin{array}{c} \begin{array}{ccc} \text{T} & \text{P} & \text{D} \end{array} \\ \begin{bmatrix} 14 & 10 & 21 \\ 23 & 16 & 27 \\ 28 & 15 & 22 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Monday} \\ \text{Tuesday} \\ \text{Wednesday} \end{array}$$

a. What was the total number of tops sold over the three days? (1 MARK)

b. Interpret element n_{32}. (1 MARK)

Consider the following matrix equation.

$$\begin{bmatrix} 14 & 10 & 21 \\ 23 & 16 & 27 \\ 28 & 15 & 22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 225 \\ 322.50 \\ 305 \end{bmatrix}$$

- x is the cost of a top.
- y is the cost of one pair of pants.
- z is the cost of a dress.

c. What is the cost of a dress? (1 MARK)

d. The following matrix equation shows that the total value of all clothing sold over Monday and Tuesday is $547.50.

$$M \times \begin{bmatrix} 225 \\ 322.50 \\ 305 \end{bmatrix} = [547.50]$$

Given that the order of matrix M is 1×3, write down matrix M. (1 MARK)

Adapted from VCAA 2017 Exam 2 Matrices Q1a-c

7H The equilibrium state matrix

STUDY DESIGN DOT POINT

- informal identification of the equilibrium state matrix in the case of regular transition matrices (no noticeable change from one state matrix to the next state matrix)

7A 7B 7C 7D 7E 7F 7G **7H** 7I

KEY SKILLS

During this lesson, you will be:

- calculating the equilibrium state matrix
- interpreting the equilibrium state matrix.

KEY TERMS

- Equilibrium state matrix
- Steady state matrix

When state matrices are used to model data that transforms over time, the data may eventually settle to a point where there is no visible change between different states, even though the transition matrix is still functioning. This can provide useful insight into the long-term projections of data.

Calculating the equilibrium state matrix

The **equilibrium state matrix** (often called the **steady state matrix**) is the state matrix which has no difference compared to the matrix occurring after it.

From the recurrence relation

$S_0 = \textit{initial state matrix}, \quad S_{n+1} = T \times S_n,$

the equilibrium state matrix is the matrix in which $S_n = S_{n+1}$.

Although each data point is still affected by the transition matrix, the changes that occur ultimately cancel out, resulting in the previous state matrix. In other words, there is no net change of the elements within the state matrix. Once the equilibrium state has been reached, it will remain that way for all future states.

As large values of n are being used, the equilibrium state matrix should be calculated using a rule, $S_n = T^n \times S_0$, rather than a recurrence relation.

As T^n cannot be calculated for when n is infinite, an approximated value of the equilibrium can be found by calculating S_n for large values of n. Usually, the equilibrium state matrix will be found somewhere between $n = 15$ and $n = 30$.

Worked example 1

If $S_0 = \begin{bmatrix} 29 \\ 32 \end{bmatrix}$ and $T = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$, calculate the equilibrium state matrix.

Explanation

Step 1: Calculate the state matrix for a large n.

$S_n = T^n \times S_0$

Usually the equilibrium state matrix will have occurred by the time $n = 30$.

$S_{30} = T^{30} \times S_0$

$= \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^{30} \times \begin{bmatrix} 29 \\ 32 \end{bmatrix}$

$= \begin{bmatrix} 24.4 \\ 36.6 \end{bmatrix}$

Step 2: Verify the equilibrium state matrix by comparing with S_{n+1}.

$S_{31} = T^{31} \times S_0$

$= \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^{31} \times \begin{bmatrix} 29 \\ 32 \end{bmatrix}$

$= \begin{bmatrix} 24.4 \\ 36.6 \end{bmatrix}$

Since S_{30} and S_{31} are equal, this is the equilibrium state matrix.

If S_{30} and S_{31} were not equal, steps 1 and 2 would have to be repeated by substituting in a higher value of n.

Answer

$\begin{bmatrix} 24.4 \\ 36.6 \end{bmatrix}$

Interpreting the equilibrium state matrix

An equilibrium state matrix contains information about the long term expectations for a given scenario. Using this, the equilibrium state matrix and its elements can be interpreted in relation to the context.

Worked example 2

The school canteen planned to offer two new menu options: spring rolls (S), and avo toast (A). All 100 students order one of the options each day. Their first ever orders are represented in matrix S_0. In order to predict how many of each option will be required, the canteen staff made a transition matrix, T.

$$S_0 = \begin{array}{c} \begin{bmatrix} 75 \\ 25 \end{bmatrix} \end{array} \begin{array}{c} S \\ A \end{array} \qquad T = \begin{array}{c} \text{today} \\ \begin{array}{cc} S & A \end{array} \\ \begin{bmatrix} 0.54 & 0.71 \\ 0.46 & 0.29 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ S \\ A \end{array} \text{ tomorrow}$$

From the matrix equation $S_n = T^n \times S_0$, the equilibrium state matrix is approximately:

$$\begin{bmatrix} 60.68 \\ 39.32 \end{bmatrix} \begin{array}{c} S \\ A \end{array}$$

Using the equilibrium state matrix values, rounded to the nearest whole number, what can the cafe expect in the long term?

Explanation

Interpret the equilibrium state matrix.

The elements in the equilibrium state matrix describe the expected daily sales in the long-term.

The first row corresponds to spring rolls, and the second row corresponds to avo toasts.

Answer

In the long term, the cafe can expect to sell 61 spring rolls and 39 avo toasts daily.

Exam question breakdown

VCAA 2016 Exam 1 Matrices Q7

Each week, 300 students at a primary school choose art (A), music (M) or sport (S) as an afternoon activity.

The following transition matrix shows how the students' choices change from week to week.

$$\begin{array}{c} \text{this week} \\ \begin{array}{ccc} \text{A} & \text{M} & \text{S} \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix} \begin{array}{l} \text{A} \\ \text{M} \\ \text{S} \end{array} \quad \text{next week}$$

Based on this information, it can be concluded that, in the long term

A. no student will choose sport.

B. all students will choose to stay in the same activity each week.

C. all students will have chosen to change their activity at least once.

D. more students will choose to do music than sport.

E. the number of students choosing to do art and music will be the same.

Explanation

Step 1: Calculate the equilibrium state matrix.

As the question specifies 'in the long term', the equilibrium state matrix will be useful for verifying each option. Although the question doesn't specify S_0, it does state that there are 300 students. From this, an initial state matrix can be approximated, as the equilibrium state solution will be the same regardless.

$$S_0 = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} \begin{array}{l} \text{A} \\ \text{M} \\ \text{S} \end{array}$$

$$S_{30} = T^{30} \times S_0$$

$$= \begin{bmatrix} 104.76... \\ 109.52... \\ 85.71... \end{bmatrix} \begin{array}{l} \text{A} \\ \text{M} \\ \text{S} \end{array}$$

Step 2: Verify the equilibrium state matrix by comparing with S_{n+1}.

$$S_{31} = T^{31} \times S_0$$

$$= \begin{bmatrix} 104.76... \\ 109.52... \\ 85.71... \end{bmatrix} \begin{array}{l} \text{A} \\ \text{M} \\ \text{S} \end{array}$$

Step 3: Check whether each option is correct or incorrect.

A: This is incorrect because the equilibrium state matrix suggests that approximately 86 students will choose sport in the long term. ✗

B: This is incorrect because the transition matrix suggests that many students change their activity each week. ✗

C: This cannot be concluded from the information provided. While the transition matrix suggests that some students change their activity from week to week, there is no way of keeping track of which students change and which don't. This means there could be some students picking the same activity every week. ✗

D: This is correct, as $109.52 > 85.71$. ✓

E: This is incorrect, as $109.52 \neq 104.76$. ✗

Answer

D

52% of students answered this question correctly.

30% of students incorrectly answered option C, likely on the basis of reasoning rather than mathematics. While it is likely true that every student will change their activity at least once in the long term, this isn't guaranteed from the provided information.

7H Questions

Calculating the equilibrium state matrix

1. For the state matrix equation $S_n = T^n \times S_0$, which of the following expressions would be most likely to correctly calculate the equilibrium state matrix?

A. $T^5 \times S_0$ **B.** $T^{10} \times S_0$ **C.** $T^{20} \times S_0$ **D.** $T^{30} \times S_0$

2. If $T = \begin{bmatrix} 0.23 & 0.54 \\ 0.77 & 0.46 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 98 \\ 42 \end{bmatrix}$:

a. Write down an expression, in terms of T and S_0, that can be used to calculate the equilibrium state matrix.

b. Calculate and verify the equilibrium state matrix. Round values to two decimal places.

3. Calculate the equilibrium state matrix for the following pairs of T and S_0. Round values to two decimal places.

a. $T = \begin{bmatrix} 0.5 & 0.9 \\ 0.5 & 0.1 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 50 \\ 70 \end{bmatrix}$

b. $T = \begin{bmatrix} 0.19 & 0.44 \\ 0.81 & 0.56 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 401 \\ 225 \end{bmatrix}$

c. $T = \begin{bmatrix} 0.7 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.4 \\ 0.1 & 0.0 & 0.4 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 90 \\ 47 \\ 211 \end{bmatrix}$

4. The transition matrix and initial state matrix used to model the changing preferences of 100 mice in a laboratory test, from hour to hour, are provided.

$$\begin{array}{cc} & \text{current hour} \\ & \begin{array}{cc} \text{eat} & \text{stay} \end{array} \\ T = & \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix} \end{array} \begin{array}{l} \text{eat} \\ \text{stay} \end{array} \text{ next hour} \qquad S_0 = \begin{bmatrix} 84 \\ 16 \end{bmatrix} \begin{array}{l} \text{eat} \\ \text{stay} \end{array}$$

Calculate the equilibrium state matrix, giving values as whole numbers.

Interpreting the equilibrium state matrix

5. Matrix F represents an initial state matrix for the daily selection of different fruits, and matrix E represents the corresponding equilibrium state matrix.

If matrix element f_{mn} represents the number of apples selected on the first day, then matrix element e_{mn} represents

A. the amount of fruit selected on the first day.

B. the daily number of apples selected in the long term.

C. the total number of apples selected in the long term.

D. the daily amount of fruit selected in the long term.

6. The musical instruments tried by 50 primary school students from day to day can be modelled using state matrices. From this, the equilibrium state matrix, M, has been calculated.

$$M = \begin{bmatrix} 9.2 \\ 15.0 \\ 13.9 \\ 7.6 \\ 4.3 \end{bmatrix} \begin{array}{l} \text{keyboard} \\ \text{percussion} \\ \text{guitar} \\ \text{boomwhackers} \\ \text{other} \end{array}$$

In the long term, which type of instrument will students pick most often?

7. The migration of penguins from two local regions, A and B, can be modelled by the following rule:

$S_n = T^n \times S_0$, where

$$S_0 = \begin{array}{c}\\ \begin{bmatrix} 759 \\ 529 \end{bmatrix}\end{array}\begin{array}{l}\\ A \\ B\end{array}, \quad T = \begin{array}{cc} A & B \\ \multicolumn{2}{c}{\begin{bmatrix} 0.7 & 0.9 \\ 0.3 & 0.1 \end{bmatrix}} \end{array}\begin{array}{l}\\ A \\ B\end{array}$$

The equilibrium state matrix for this situation is:

$$\begin{bmatrix} 966 \\ 322 \end{bmatrix}\begin{array}{l} A \\ B\end{array}$$

From this matrix, comment on the long-term projections of penguin migration occurring between the two regions.

Joining it all together

8. A study was conducted on the weekly assignment submissions for 60 university students. The transition matrix and initial state matrix are provided.

$$T = \begin{array}{cc} \multicolumn{2}{c}{\text{this week}} \\ \text{on time} & \text{late} \\ \multicolumn{2}{c}{\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}} \end{array}\begin{array}{l}\\ \\ \text{on time} \\ \text{late}\end{array} \text{ next week} \qquad S_0 = \begin{bmatrix} 53 \\ 7 \end{bmatrix}\begin{array}{l} \text{on time} \\ \text{late}\end{array}$$

a. Write down an expression, in terms of the matrices provided, that can be used to calculate the equilibrium state matrix.

b. Calculate the equilibrium state matrix, rounding all values to whole numbers.

c. In the long term, how many students will hand in their weekly assignments on time each week?

---

9. The chances of a driver reaching a red light at an intersection depend on whether the driver came across a red or green light at the previous intersection. The following transition matrix demonstrates this.

$$T = \begin{array}{cc} \multicolumn{2}{c}{\text{current light}} \\ \text{red} & \text{green} \\ \multicolumn{2}{c}{\begin{bmatrix} 0.15 & 0.25 \\ 0.85 & 0.75 \end{bmatrix}} \end{array}\begin{array}{l}\\ \\ \text{red} \\ \text{green}\end{array} \text{ next light}$$

38 drivers start a journey with a red light at their first intersection, and 50 drivers start with a green light. In the long term, how many drivers would be expected to meet a red light at the intersection?

---

10. Scientists have discovered that turtles tend to migrate monthly between three islands: Amnio, Belix, and Chel. The number of turtles observed at the three islands in January is represented by matrix S_0, and the subsequent migration patterns are represented by the transition matrix, T.

$$T = \begin{array}{ccc} \multicolumn{3}{c}{\text{this month}} \\ \text{Amnio} & \text{Belix} & \text{Chel} \\ \multicolumn{3}{c}{\begin{bmatrix} 0.55 & 0.27 & 0.23 \\ 0.20 & 0.49 & 0.10 \\ 0.25 & 0.24 & 0.67 \end{bmatrix}} \end{array}\begin{array}{l}\\ \\ \text{Amnio} \\ \text{Belix} \\ \text{Chel}\end{array} \text{ next month} \qquad S_0 = \begin{bmatrix} 514 \\ 276 \\ 410 \end{bmatrix}\begin{array}{l} \text{Amnio} \\ \text{Belix} \\ \text{Chel}\end{array}$$

The scientists believe that, eventually, the turtle population will stabilise at all three islands. If the population at any one island exceeds 500 turtles in the long term, the scientists must then relocate some of the turtles to the less populated islands.

Will the scientists need to relocate any turtles? If so, from which island(s)?

11. Steve is a weatherman and wants to boost his ratings for 35 towns in regional Victoria. His aim is to improve his predictions for the upcoming autumn season. From previous years, he concluded that the chance that it will rain in a town on a given day will depend on whether it rained the previous day. The transition matrix, T, represents this information.

$$\begin{array}{c} \text{today} \\ \begin{array}{cc} \text{rain} & \text{no rain} \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.65 & 0.87 \\ 0.35 & 0.13 \end{bmatrix} \begin{array}{l} \text{rain} \\ \text{no rain} \end{array} \quad \text{tomorrow}$$

Using the transition matrix, how many of the 35 towns can be expected to rain each day in the long term? Round the answer to the nearest whole number.

Exam practice

12. A public library organised 500 of its members into five categories according to the number of books each member borrows each month.

These categories are

- J = no books borrowed per month
- K = one book borrowed per month
- L = two books borrowed per month
- M = three books borrowed per month
- N = four or more books borrowed per month

The transition matrix provided, T, shows how the number of books borrowed per month by the members is expected to change from month to month.

$$\begin{array}{c} \text{this month} \\ \begin{array}{ccccc} \text{J} & \text{K} & \text{L} & \text{M} & \text{N} \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0 & 0 \\ 0.5 & 0.2 & 0.3 & 0.1 & 0 \\ 0.3 & 0.3 & 0.4 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.1 & 0.6 & 0.3 \\ 0 & 0.1 & 0 & 0.2 & 0.5 \end{bmatrix} \begin{array}{l} \text{J} \\ \text{K} \\ \text{L} \\ \text{M} \\ \text{N} \end{array} \quad \text{next month}$$

In the long term, which category is expected to have approximately 96 members each month?

A. J **B.** K **C.** L
D. M **E.** N

VCAA 2018 Exam 1 Matrices Q8

46% of students answered this question correctly.

---

13. The Hiroads company has a contract to maintain and improve 2700 km of highway.

Each year, sections of highway must be graded (G), resurfaced (R) or sealed (S).

The remaining highway will need no maintenance (N) that year.

Let S_n be the state matrix that shows the highway maintenance schedule for the nth year after 2018.

The maintenance schedule for 2018 is shown in matrix S_0.

$$S_0 = \begin{bmatrix} 700 \\ 400 \\ 200 \\ 1400 \end{bmatrix} \begin{array}{l} \text{G} \\ \text{R} \\ \text{S} \\ \text{N} \end{array}$$

The type of maintenance in sections of highway varies from year to year, as shown in the transition matrix, T.

$$\begin{array}{c} \text{this year} \\ \begin{array}{cccc} \text{G} & \text{R} & \text{S} & \text{N} \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.2 & 0.1 & 0.0 & 0.2 \\ 0.1 & 0.1 & 0.0 & 0.2 \\ 0.2 & 0.1 & 0.2 & 0.1 \\ 0.5 & 0.7 & 0.8 & 0.5 \end{bmatrix} \begin{array}{l} \text{G} \\ \text{R} \\ \text{S} \\ \text{N} \end{array} \quad \text{next year}$$

In the long term, what percentage of highway each year is expected to have no maintenance (N)? Round the answer to one decimal place. (1 MARK)

VCAA 2018 Exam 2 Matrices Q3e

39% of students answered this question correctly.

14. The three major shopping centres in a large city, Eastmall (E), Grandmall (G) and Westmall (W), are owned by the same company.

An offer to buy the Westmall shopping centre was made by a competitor.

One market research project suggested that if the Westmall shopping centre were sold, each of the three centres (Westmall, Grandmall and Eastmall) would continue to have regular shoppers but would attract and lose shoppers on a weekly basis.

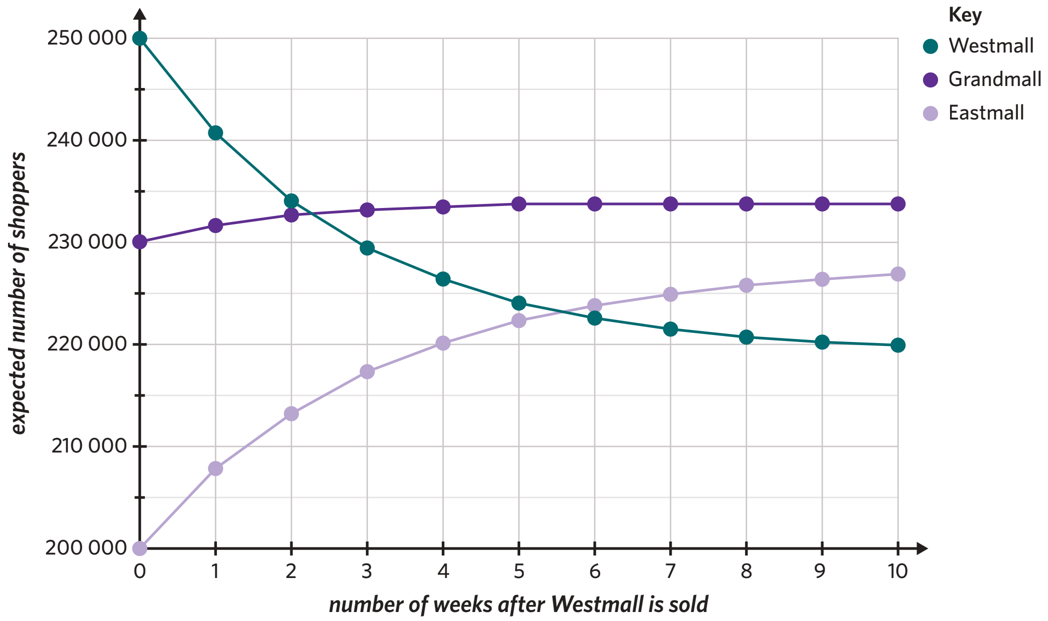
Let S_n be the state matrix that shows the expected number of shoppers at each of the three centres n weeks after Westmall is sold.

A matrix recurrence relation that generates values of S_n is

$S_{n+1} = T \times S_n$, where

$$T = \begin{array}{c} \text{this week} \\ \begin{array}{ccc} W & G & E \end{array} \\ \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.79 & 0.10 \\ 0.08 & 0.12 & 0.80 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ W \\ G \\ E \end{array} \text{ next week} \qquad S_0 = \begin{bmatrix} 250\,000 \\ 230\,000 \\ 200\,000 \end{bmatrix} \begin{array}{c} W \\ G \\ E \end{array}$$

Using values from the recurrence relation, the graph provided displays the expected number of shoppers at Westmall, Grandmall and Eastmall for each of the 10 weeks after Westmall is sold.



In the long term, what is the expected weekly number of shoppers at Westmall? Round your answer to the nearest whole number. (1 MARK)

VCAA 2020 Exam 2 Matrices Q3d

39% of students answered this question correctly.

Questions from multiple lessons

Matrices

15. Matrix P is a 3×3 permutation matrix.

Matrix Q is another matrix such that the matrix product $Q \times P$ is defined.

This matrix product results in the entire first and second columns of matrix Q being swapped.

The permutation matrix P is

A. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **C.** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **D.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **E.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Adapted from VCAA 2018 Exam 1 Matrices Q4

Recursion and financial modelling

16. Nathan is looking to purchase a new Mercedes. He will take out a loan for \$80 000 with interest charged at a rate of 3.9% per annum, compounding fortnightly.

Each fortnight, Nathan pays back the exact amount of interest that is charged for that fortnight.

Let V_n be the value of Nathan's loan, in dollars, after n fortnights.

Which of the following recurrence relations correctly models the value of Nathan's loan?

A. $V_0 = 80\,000, \quad V_{n+1} = 1.0015V_n$

B. $V_0 = 80\,000, \quad V_{n+1} = 1.039V_n - 120$

C. $V_0 = 80\,000, \quad V_{n+1} = 1.0015V_n - 3120$

D. $V_0 = 80\,000, \quad V_{n+1} = 1.039V_n$

E. $V_0 = 80\,000, \quad V_{n+1} = 1.0015V_n - 120$

Adapted from VCAA 2017 Exam 1 Recursion and financial modelling Q20

Matrices

17. The matrix C represents the way in which five friends, Voula (V), Will (W), Xavier (X), Yasmin (Y), and Zoe (Z) interact on Instagram.

The matrix C^2 is also shown.

$$\begin{array}{cc} & \text{followed} \\ & \begin{array}{ccccc} V & W & X & Y & Z \end{array} \\ C = & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ V \\ W \\ X \quad \text{follower} \\ Y \\ Z \end{array} \qquad \begin{array}{cc} & \text{followed} \\ & \begin{array}{ccccc} V & W & X & Y & Z \end{array} \\ C^2 = & \begin{bmatrix} 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 2 & 1 & 3 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ V \\ W \\ X \quad \text{follower} \\ Y \\ Z \end{array}$$

The '1' in row V, column W of matrix C indicates that Voula follows Will on Instagram.

The '0' in row Z, column Y of matrix C indicates that Zoe does not follow Yasmin on Instagram.

a. Who does Will follow? (1 MARK)

b. Yasmin wants to see a photo Voula posted, but cannot do this as she does not follow Voula. She plans to send a message over Instagram to the friend(s) that she follows, who follow Voula themselves, asking for a screenshot of Voula's post. Which friend(s) could she ask? (1 MARK)

Adapted from VCAA 2016 Exam 2 Matrices Q2

7I Applications of transition matrices

STUDY DESIGN DOT POINTS

- use of transition diagrams, their associated transition matrices and state matrices to model the transitions between states in discrete dynamical situations and their application to model and analyse practical situations such as the modelling and analysis of an insect population comprising eggs, juveniles and adults
- use of the matrix recurrence relation S_0 = initial state matrix, $S_{n+1} = TS_n + B$ to extend modelling to populations that include culling and restocking

7A 7B 7C 7D 7E 7F 7G 7H 7I

KEY SKILLS

During this lesson, you will be:

- constructing and interpreting transition diagrams
- using transition matrices to model situations involving culling and restocking.

KEY TERMS

- Transition diagram
- Culling
- Restocking

Transition matrices can be used to model various practical situations such as growing and changing populations, storage levels, rotating activities, menu options and more. Transition diagrams are often used in applications to visually represent the information contained in transition matrices.

Constructing and interpreting transition diagrams

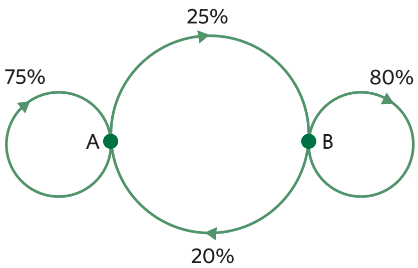
See worked example 1

A **transition diagram** is a visual representation of how a transition matrix functions. Each state is represented by a point on the diagram, and the transitions between states are represented by lines with arrows, connecting all of the points together (and connecting each point to itself).

For example, the transition matrix

$$T = \begin{array}{c} \text{today} \\ \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{A} \\ \text{B} \end{array} \text{ tomorrow}$$

can be represented by the following transition diagram.



In a transition diagram, the sum of the percentages moving away from a given point adds up to 100%. This includes the parts of the diagram where a point loops on itself.

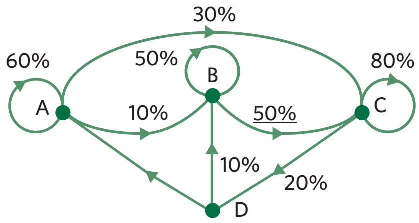
A transition diagram can also be used to construct a transition matrix, and vice versa.

See worked example 2

Worked example 1

Trams operate between four depots, A, B, C, and D. The transition diagram provided represents where the trams end up at the end of each week.

a. What does the underlined value in the diagram represent?



Explanation

The 50% corresponds to the portion of the transition diagram that is flowing from point B to point C.

Answer

50% of trams operating at depot B end up at depot C at the end of each week.

b. Determine the missing percentage value on the transition diagram.

Explanation

Step 1: Identify the location of the missing percentage.

The missing percentage is between points D and A.

Step 2: Calculate the missing percentage.

The sum of the percentages moving away from point D should add up to 100%.

$100\% - 10\% = 90\%$

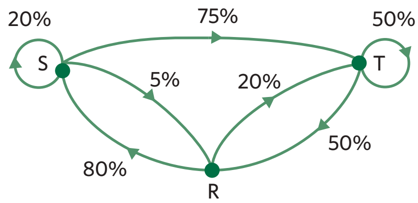
Answer

90%

Worked example 2

Complete the following conversions.

a. Convert the following transition diagram into a transition matrix.



Explanation

Step 1: Set up a square matrix.

As there are three points, the order of the matrix must be 3×3.

Since the points of the transition diagram are S, R and T, these will also be the row and column labels of the transition matrix.

$$\begin{array}{cc} \begin{array}{ccc} R & S & T \end{array} & \\ \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} & \begin{array}{c} R \\ S \\ T \end{array} \end{array}$$

Step 2: Fill in the first column, R.

Column R will represent the transitions from R to each state.

There is no transition from R to itself (0%).

The transition from R to S is 80%.

The transition from R to T is 20%.

Converted to decimals, to two decimal places, these are 0.00, 0.80 and 0.20 respectively.

$$\begin{array}{cc} \begin{array}{ccc} R & S & T \end{array} & \\ \begin{bmatrix} 0.00 & & \\ 0.80 & & \\ 0.20 & & \end{bmatrix} & \begin{array}{c} R \\ S \\ T \end{array} \end{array}$$

Continues →

Step 3: Repeat for every other column.

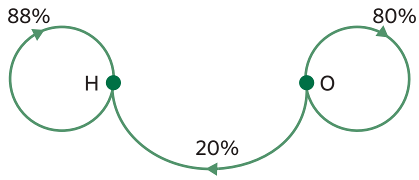
Answer

$$\begin{array}{cc} & \begin{array}{ccc} R & S & T \end{array} \\ & \begin{bmatrix} 0.00 & 0.05 & 0.50 \\ 0.80 & 0.20 & 0.00 \\ 0.20 & 0.75 & 0.50 \end{bmatrix} \begin{array}{c} R \\ S \\ T \end{array} \end{array}$$

---

b. Use the information in the transition matrix T to complete the corresponding transition diagram.

$$\begin{array}{cc} & \text{this month} \\ & \begin{array}{cc} H & O \end{array} \\ T = & \begin{bmatrix} 0.88 & 0.20 \\ 0.12 & 0.80 \end{bmatrix} \begin{array}{c} H \\ O \end{array} \text{ next month} \end{array}$$



Explanation

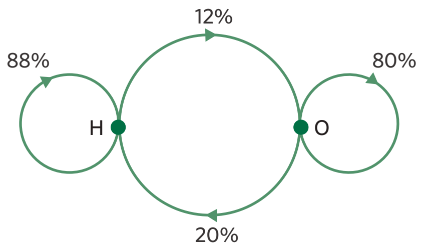
Step 1: Identify which value(s) are missing.

All values except 0.12 (12%) are included in the diagram.

Step 2: Annotate the transition diagram.

Since 0.12 corresponds to the movement from H to O, the line should be drawn between those points, with the arrowhead pointing towards O. Label the line with '12%'.

Answer



Using transition matrices to model situations involving culling and restocking

Situations that can be modelled using the recurrence relation $S_{n+1} = TS_n$ generally assume that all factors are accounted for from one state to the next, represented in the transition matrix, T.

In other situations, this assumption is not valid. In scenarios where the total population is regularly increased or decreased by a set quantity, for a reason that is not already accounted for, a new model is required.

When these types of changes are made to the total population from one state to the next, a more suitable model is

$S_{n+1} = TS_n + B$,

where B is a matrix of the same order as S_n and S_{n+1}.

This model is particularly useful for modelling the population of animals, where humans may choose to add or remove a set number of animals at a regular interval.

Culling is the reduction of an animal population by slaughter. In a matrix recurrence relation scenario, culling represents a subtraction to the population from one state to the next.

Restocking typically refers to replacing stock with a new supply. In a matrix recurrence relation scenario, restocking represents an addition to the population from one state to the next.

For example, the recurrence relation $S_{n+1} = TS_n + B$ can be used to model the number of cows, goats and sheep, respectively, on a farm each month. Matrix $B = \begin{bmatrix} -5 \\ 0 \\ 10 \end{bmatrix}$ indicates that 5 cows are removed (culled), no goats are added (restocked) or removed, and 10 sheep are added each month. This is separate to any information provided in the transition matrix, T.

Worked example 3

Eddortsford High School runs an optional extra-curricular arts program. Each year, students can choose from Music (M), Drama (D), Visual Arts (V), or to opt out (O) of the program. The movement of students from one year to the next is summarised in the transition matrix, T.

$$T = \begin{array}{c} \text{this year} \\ \begin{array}{cccc} M & D & V & O \end{array} \\ \begin{bmatrix} 0.73 & 0.07 & 0.11 & 0 \\ 0.10 & 0.71 & 0.04 & 0 \\ 0.06 & 0.08 & 0.72 & 0 \\ 0.11 & 0.14 & 0.13 & 1 \end{bmatrix} \end{array} \begin{array}{c} M \\ D \\ V \\ O \end{array} \text{ next year}$$

The matrix S_0 represents the number of students enrolled in each course at the beginning of 2023.

$$S_0 = \begin{bmatrix} 73 \\ 36 \\ 52 \\ 0 \end{bmatrix} \begin{array}{c} M \\ D \\ V \\ O \end{array}$$

As the program expands, 30 new students are added to the program each year. 13 of these students are expected to be added to the music course, 6 to the drama course and 11 to the visual arts course.

a. Construct matrix B to represent the predicted new enrolments and their preferences.

Explanation

Step 1: Set up a blank matrix.

Since matrix B is the same order as S_0, its order will be 4×1.

$$B = \begin{bmatrix} \; \\ \; \\ \; \\ \; \end{bmatrix} \begin{array}{c} M \\ D \\ V \\ O \end{array}$$

Step 2: Fill in the missing values.

The missing values are given by the number of new enrolments for each course.

Answer

$$B = \begin{bmatrix} 13 \\ 6 \\ 11 \\ 0 \end{bmatrix} \begin{array}{c} M \\ D \\ V \\ O \end{array}$$

b. How many students are expected to be enrolled in the music course at the beginning of 2025? Round to the nearest whole number.

Explanation

Step 1: Calculate S_1, the state matrix at the start of 2024.

$$S_1 = TS_0 + B$$

$$= \begin{bmatrix} 0.73 & 0.07 & 0.11 & 0 \\ 0.10 & 0.71 & 0.04 & 0 \\ 0.06 & 0.08 & 0.72 & 0 \\ 0.11 & 0.14 & 0.13 & 1 \end{bmatrix} \begin{bmatrix} 73 \\ 36 \\ 52 \\ 0 \end{bmatrix} + \begin{bmatrix} 13 \\ 6 \\ 11 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 74.53 \\ 40.94 \\ 55.70 \\ 19.83 \end{bmatrix} \begin{array}{c} M \\ D \\ V \\ O \end{array}$$

Step 2: Calculate S_2, the state matrix at the start of 2025.

$$S_2 = TS_1 + B$$

$$= \begin{bmatrix} 0.73 & 0.07 & 0.11 & 0 \\ 0.10 & 0.71 & 0.04 & 0 \\ 0.06 & 0.08 & 0.72 & 0 \\ 0.11 & 0.14 & 0.13 & 1 \end{bmatrix} \begin{bmatrix} 74.53 \\ 40.94 \\ 55.70 \\ 19.83 \end{bmatrix} + \begin{bmatrix} 13 \\ 6 \\ 11 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 76.3997 \\ 44.7484 \\ 58.8510 \\ 41.0009 \end{bmatrix} \begin{array}{c} M \\ D \\ V \\ O \end{array}$$

Step 3: Identify the element that corresponds with music in matrix S_2.

The value for music in S_2 is 76.3997.

Answer

76 students

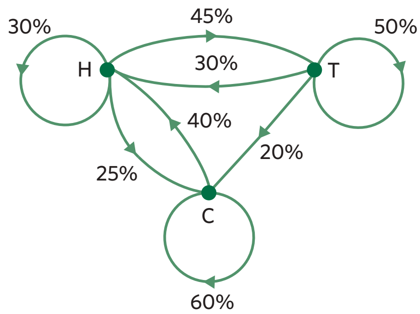
Exam question breakdown

VCAA 2016 Exam 1 Matrices Q6

Families in a country town were asked about their annual holidays.

Every year, these families choose between staying at home (H), travelling (T) and camping (C).

The following transition diagram shows the way families in the town change their holiday preferences from year to year.



A transition matrix that provides the same information as the transition diagram is

A.
$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}0.30&0.75&0.65\\0.75&0.50&0.20\\0.65&0.20&0.60\end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

B.
$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}0.30&0.30&0.40\\0.45&0.50&0\\0.25&0.20&0.60\end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

C.
$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}0.30&0.30&0.40\\0.45&0.50&0.20\\0.25&0.20&0.60\end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

D.
$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}0.30&0.30&0.40\\0.45&0.50&0.20\\0.25&0.20&0.40\end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

E.
$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}0.30&0.45&0.25\\0.30&0.50&0.20\\0.40&0&0.60\end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

Explanation

Step 1: Set up a square matrix.

As there are three points, the matrix size needs to be 3×3.

$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}&&\\ & & \\ & & \end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

Step 2: Fill in the first column, H.

Column H will represent the transitions from H to each state.

The transition from H to itself is 30%.

The transition from H to T is 45%.

The transition from H to C is 25%.

As decimals, these are 0.30, 0.45 and 0.25 respectively.

$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}0.30&&\\0.45& & \\0.25& & \end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

Step 3: Repeat for every other column.

$$\begin{array}{c}\text{from}\\ \begin{array}{ccc}\text{H}&\text{T}&\text{C}\end{array}\\ \left[\begin{array}{ccc}0.30&0.30&0.40\\0.45&0.50&0\\0.25&0.20&0.60\end{array}\right]\end{array}\begin{array}{l}\text{H}\\\text{T to}\\\text{C}\end{array}$$

Answer

B

86% of students answered this question correctly.

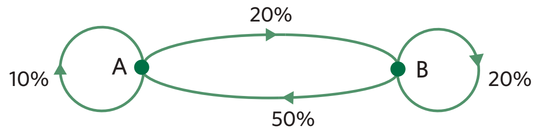
For this question, the key is to understand how the missing connection between T and C on the transition diagram translates to a transition matrix. B is the only answer which displays this feature with a correctly placed 0.

7I Questions

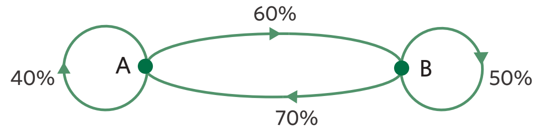
Constructing and interpreting transition diagrams

1. Which of the following is a valid transition diagram?

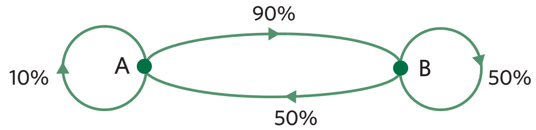
A.



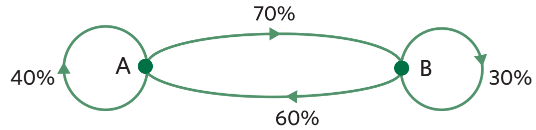
B.



C.



D.



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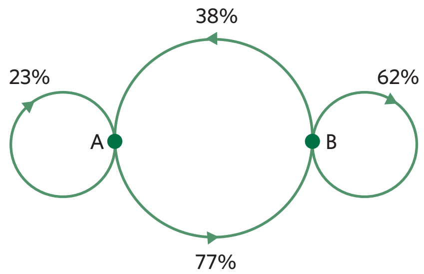
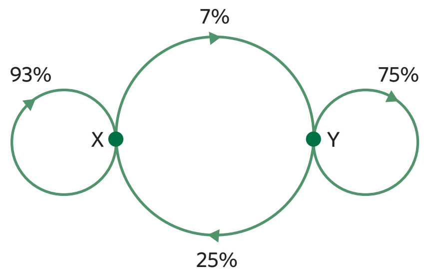
2. Construct a transition matrix from each of the following transition diagrams.
Convert all percentages to decimals.

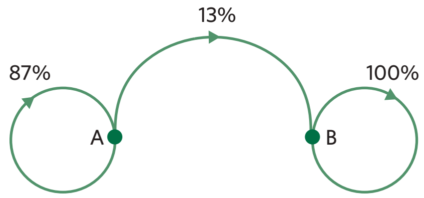
a.



b.



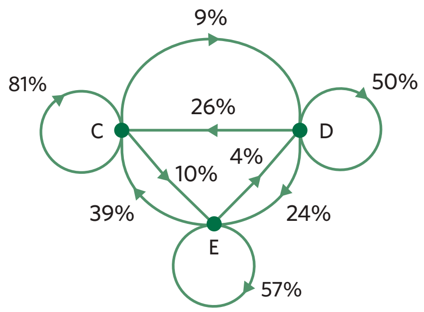
c.



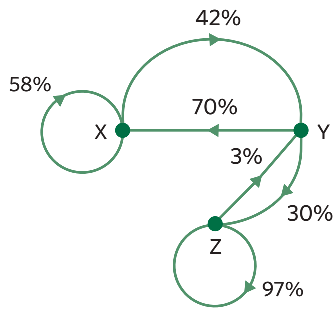
---

3. Construct a transition matrix from each of the following transition diagrams.
Convert all percentages to decimals.

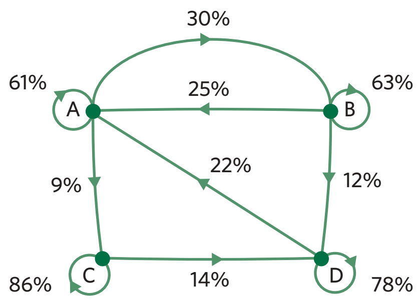
a.



b.



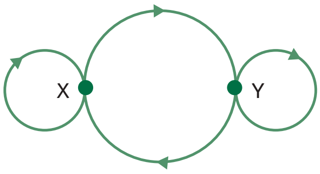
c.



4. Construct a transition diagram from each of the following transition matrices. Convert all decimals to percentages.

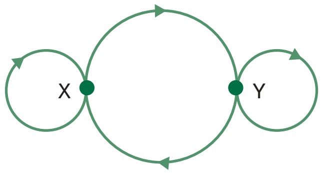
a.

$$\begin{array}{c} \text{from} \\ \begin{array}{cc} X & Y \end{array} \end{array}$$
$$\begin{bmatrix} 0.7 & 0.55 \\ 0.3 & 0.45 \end{bmatrix} \begin{matrix} X \\ Y \end{matrix} \text{ to}$$



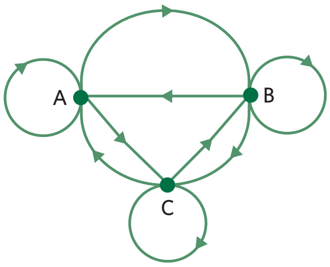
b.

$$\begin{array}{c} \text{from} \\ \begin{array}{cc} X & Y \end{array} \end{array}$$
$$\begin{bmatrix} 0.35 & 0.21 \\ 0.65 & 0.79 \end{bmatrix} \begin{matrix} X \\ Y \end{matrix} \text{ to}$$



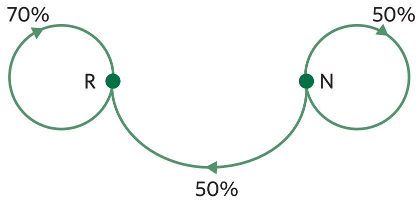
c.

$$\begin{array}{c} \text{from} \\ \begin{array}{ccc} A & B & C \end{array} \end{array}$$
$$\begin{bmatrix} 0.33 & 0.4 & 0.05 \\ 0.34 & 0.42 & 0.67 \\ 0.33 & 0.18 & 0.28 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix} \text{ to}$$



5. An Indian food appreciation group likes to order takeaway each week. Members can either order roti (R) or naan bread (N). It is discovered that the transition matrix, T, can be used to predict each member's choice from week to week.

$$\begin{array}{c} \text{this week} \\ \begin{array}{cc} R & N \end{array} \end{array}$$
$$\begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix} \begin{matrix} R \\ N \end{matrix} \text{ next week}$$



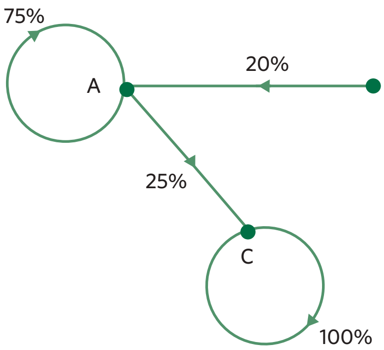
a. Use the information in the transition matrix, T, to complete the transition diagram.

b. What percentage of members are expected to change their order from naan bread to roti each week?

c. If there are 120 members in this club and 80 decide to order roti this week, how many members are expected to order roti next week?

6. Aaron, Brett and Charlie are brothers who like to collect rocks. At the end of each week, Aaron and Brett always give the worst 25% of their rocks to Charlie. Charlie keeps all his rocks. Brett is very picky about his collection and also gives 20% of his rocks to Aaron each week.

a. Using this information, complete the transition diagram shown, which describes the movement of rocks week-to-week between the brothers.



b. Use the answer from part **a** to construct a transition matrix.

c. At the beginning of this week, Aaron has 40 rocks. How many of these rocks will he still have next week?

d. If Aaron and Brett each currently have 24 rocks, how many rocks will be given to Charlie at the end of the week?

e. Charlie received 90 rocks this week. If each of the brothers had an equal number of rocks last week, how many rocks did the brothers have in total?

Using transition matrices to model situations involving culling and restocking

7. Consider the matrix recurrence relation $S_{n+1} = TS_n + B$

If T is a 3×3 matrix, then

A. S_n and B will be 1×1 matrices.

B. S_n and B will be 1×3 matrices.

C. S_n and B will be 3×1 matrices.

D. S_n and B will be 3×3 matrices.

---

8. Consider the following matrix recurrence relation.

$$S_0 = \begin{bmatrix} 38 \\ 64 \end{bmatrix} \quad S_{n+1} = \begin{bmatrix} 0.7 & 0.46 \\ 0.3 & 0.54 \end{bmatrix} S_n + B$$

For each value of B provided, determine S_3. Round to 3 significant figures.

a. $B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} -13 \\ 5 \end{bmatrix}$ **c.** $B = \begin{bmatrix} 17 \\ -12 \end{bmatrix}$

---

9. Consider the following recurrence relation.

$$S_0 = \begin{bmatrix} 200 \\ 300 \end{bmatrix} \quad S_{n+1} = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} S_n + B$$

a. Which of the following must be true in order to keep the sum of the elements in each state matrix, S_n, constant?

A. Matrix B can only be the zero matrix.

B. The sum of elements in matrix B must equal 0.

C. The sum of elements in matrix B must be equal to the sum of elements in the initial state matrix S_0.

D. Matrix B must be equal to the initial state matrix S_0.

b. Determine matrix B for which $S_n = S_{n+1}$.

---

10. Two cosmetics companies, Edphora and L'Órolo, are in direct competition. Each month, 7% of those who bought cosmetics from Edphora last month are expected to buy from L'Órolo this month, and 12% of those who bought cosmetics from L'Órolo last month are expected to buy from Edphora this month.

a. Complete the following transition matrix using the provided information.

$$\begin{array}{cc} & \text{last month} \\ & \begin{array}{cc} \text{Edphora} & \text{L' Órolo} \end{array} \\ T = & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{array}{l} \text{Edphora} \\ \text{L' Órolo} \end{array} \text{this month} \end{array}$$

L'Órolo suddenly realise that they are losing their customers to Edphora and run a targeted advertising campaign. They project that the campaign will result in 32 000 current Edphora customers swapping to L'Órolo next month.

b. Create a matrix which represents this change. It should be of an appropriate order so that it can be used in a matrix equation with the transition matrix, T.

c. Last month, 49 000 people bought cosmetics from Edphora and 56 000 people bought cosmetics from L'Órolo. Including changes due to the successful advertising campaign, how many people will buy from L'Órolo this month?

11. John, Leila, and Lo all have way too much money and decide to start their own business. They decide to buy a fleet of 20 Teslolo self-driving cars and start an automated taxi service. Customers around Greater Melbourne will be able to book trips via an app and are then driven by a Teslolo car to their destinations.

John, Leila, and Lo also establish 4 depots at which the cars can refuel. They are located in Doncaster (D), Frankston (F), Reservoir (R), and Werribee (W).

At 3:30 am every morning, the Teslolo cars will drive themselves to the nearest depot to refuel.

The transition matrix, T, describes the proportion of cars at each depot based on the distribution of cars the previous morning.

$$\begin{array}{c} \text{this morning} \\ \begin{array}{cccc} \text{D} & \text{F} & \text{R} & \text{W} \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.23 & 0.42 & 0.18 & 0.10 \\ 0.36 & 0.25 & 0.10 & 0.05 \\ 0.34 & 0.28 & 0.50 & 0.23 \\ 0.07 & 0.05 & 0.22 & 0.62 \end{bmatrix} \begin{array}{l} \text{D} \\ \text{F} \\ \text{R} \\ \text{W} \end{array} \text{ tomorrow morning}$$

a. Given that, initially, all the cars are evenly distributed across the four depots, find the state matrix that describes the distribution of the Teslolo cars in 2 mornings' time. Round to the nearest whole number.

After the two days, John, Leila, and Lo decide that there are too many Teslolo cars at the Reservoir depot. Each morning they redirect one car from the Reservoir depot to the Frankston depot.

b. Write down a matrix representing this redistribution.

c. Find the state matrix that describes the distribution of the Teslolo cars two days later. Round to the nearest whole number. Use the answer from part **a** for the initial state matrix.

After these 2 days, in a sneaky way to make some more cash, John decides to start selling one car a day for 4 days from the reservoir depot, hoping that his colleagues won't notice.

d. Find the state matrix that describes the distribution of the Teslolo cars 4 days later. Round to the nearest whole number. Use the answer from part **c** for the initial state matrix.

Joining it all together

12. On a crowded bus, people will either be standing or sitting on a seat. On one bus journey, the number of sitting and standing passengers after n stops can be modelled using the following recurrence relation.

$$B_0 = \begin{bmatrix} 3 \\ 15 \end{bmatrix} \begin{array}{l} \text{stand} \\ \text{sit} \end{array} \qquad B_{n+1} = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} B_n + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

a. What does the matrix $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ represent?

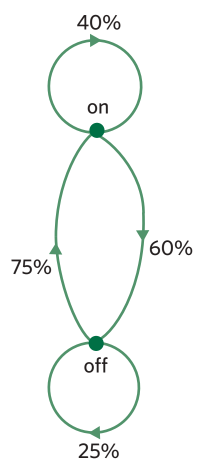
b. Calculate B_3, the matrix displaying the standing and sitting passengers on the bus after 3 stops. Round values to the nearest whole number.

After a while, the bus reaches its maximum capacity, and cannot take any more passengers.

c. Represent the transitioning of sitting and standing passengers, after the bus stops taking passengers, in a transition diagram.

---

13. In basketball, each team always has 5 players on the court, and other players off court on standby. Each time a break is called, some players are taken off the court, and some players on standby are put onto the court to play. The transition diagram provided demonstrates the transitioning pattern for a particular team within a game.



a. Each break, 3 players are taken off the court, and 3 players are put back on the court. How many players are on this team in total?

b. Convert the transition diagram into a transition matrix, T, and construct matrix P_0, which shows the initial number of players on and off court.

c. Let P_n be the state matrix representing the players on and off court after n breaks. Construct a matrix recurrence relation in terms of P_n, P_{n+1}, P_0 and T.

d. How does P_2 and P_5 compare to P_0?

During a basketball game, the audience members are either in the stands (S), standing near the court (C), or in the lobby (L). Additionally, some people may arrive or leave during each break. This information is summarised by the following recurrence relation.

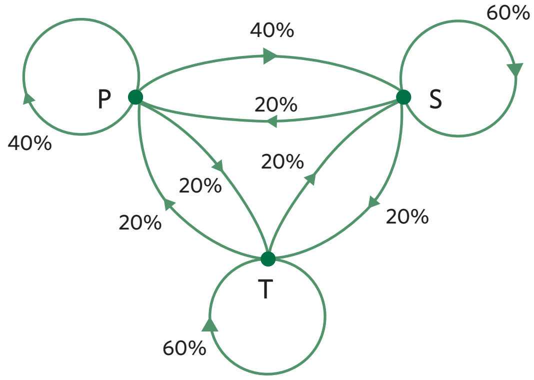
$$A_0 = \begin{array}{c} \begin{bmatrix} 56 \\ 24 \\ 21 \end{bmatrix} \end{array} \begin{array}{l} S \\ C \\ L \end{array} \qquad A_{n+1} = \begin{bmatrix} 0.85 & 0.30 & 0.25 \\ 0.05 & 0.50 & 0.25 \\ 0.10 & 0.20 & 0.5 \end{bmatrix} A_n + \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$$

e. Use the recurrence relation to determine the expected number of people in the stands after 1 break, to the nearest whole number.

f. Use the recurrence relation to determine the expected number of people in the lobby after 2 breaks, to the nearest whole number.

Exam practice

14. Junior students at this school must choose one elective activity in each of the four terms in 2018. Students can choose from the areas of performance (P), sport (S) and technology (T). The transition diagram provided shows the way in which junior students are expected to change their choice of elective activity from term to term.



a. Of the junior students who choose performance (P) in one term, what percentage are expected to choose sport (S) the next term? (1 MARK)

b. Matrix J_1 lists the number of junior students who will be in each elective activity in Term 1.

$$J_1 = \begin{bmatrix} 300 \\ 240 \\ 210 \end{bmatrix} \begin{array}{l} P \\ S \\ T \end{array}$$

306 junior students are expected to choose sport (S) in Term 2.

Complete the following calculation to show this. (1 MARK)

$$300 \times \boxed{\quad} + 240 \times \boxed{\quad} + 210 \times \boxed{\quad} = 306$$

VCAA 2017 Exam 2 Matrices Q2a,b

Part **a**: **95%** of students answered this question correctly.

Part **b**: **69%** of students answered this question correctly.

---

15. At a fish farm:

- young fish (Y) may eventually grow into juveniles (J) or they may die (D)
- juveniles (J) may eventually grow into adults (A) or they may die (D)
- adults (A) eventually die (D).

The initial state of this population, F_0, is shown.

$$F_0 = \begin{bmatrix} 50\,000 \\ 10\,000 \\ 7\,000 \\ 0 \end{bmatrix} \begin{array}{l} Y \\ J \\ A \\ D \end{array}$$

Every month, fish are either sold or bought so that the number of young, juvenile and adult fish in the farm remains constant.

The population of fish in the fish farm after n months, F_n, can be determined by the recurrence relation

$$F_{n+1} = \begin{bmatrix} 0.65 & 0 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0.20 & 0.95 & 0 \\ 0.10 & 0.05 & 0.05 & 1 \end{bmatrix} F_n + B$$

where B is a column matrix that shows the number of young, juvenile and adult fish bought or sold each month and the number of dead fish that are removed.

Each month, the fish farm will

A. sell 1650 adult fish.

B. buy 1750 adult fish.

C. sell 17 500 young fish.

D. buy 50 000 young fish.

E. buy 10 000 juvenile fish.

VCAA 2017 Exam 1 Matrices Q7

36% of students answered this question correctly.

---

16. An airline parks all of its planes at Sydney airport or Melbourne airport overnight.

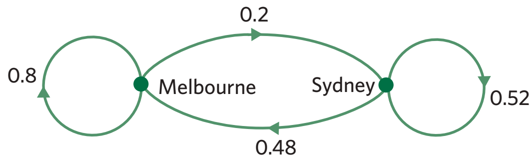
The transition diagram provided shows the change in the location of the planes from night to night.

There are always m planes parked at Melbourne airport.

There are always s planes parked at Sydney airport.

Of the planes parked at Melbourne airport on Tuesday night, 12 had been parked at Sydney airport on Monday night.

How many planes does the airline have?



A. 25　　**B.** 37　　**C.** 62

D. 65　　**E.** 85

VCAA 2019 Exam 1 Matrices Q8

15% of students answered this question correctly.

Questions from multiple lessons

Data analysis

17. The heights of a population of high school students are approximately normally distributed with a mean of 172 cm and a standard deviation of 6 cm.

From a random sample of 400 students, how many students are between 166 cm and 184 cm tall?

A. 68　　**B.** 82　　**C.** 272　　**D.** 326　　**E.** 380

Adapted from VCAA 2018 Exam 1 Data analysis Q5

Matrices

18. A local gym offers five different classes: Boxing (B), Pilates (P), Spin (S), Yoga (Y), and Zumba (Z). The gym has 150 members who attend one of the five classes each week.

A transition matrix, T, shows how the class attended by members is expected to change from week to week.

$$\begin{array}{c} \\ \\ T = \end{array} \begin{array}{c} \text{this week} \\ \begin{array}{ccccc} \mathrm{B} & \mathrm{P} & \mathrm{S} & \mathrm{Y} & \mathrm{Z} \end{array} \\ \begin{bmatrix} 0.3 & 0 & 0.2 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.3 & 0.5 & 0 \\ 0.1 & 0.2 & 0 & 0.1 & 0.2 \\ 0.1 & 0.5 & 0.3 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.2 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \mathrm{B} \\ \mathrm{P} \\ \mathrm{S}\ \text{next week} \\ \mathrm{Y} \\ \mathrm{Z} \end{array}$$

Which gym class is expected to have approximately 43 attendees each week in the long run?

A. Boxing　　**B.** Pilates　　**C.** Spin　　**D.** Yoga　　**E.** Zumba

Adapted from VCAA 2018 Exam 1 Matrices Q8

Matrices

19. The population, in millions, of three East Asian countries, in 2019, is shown in matrix P_{2019}.

$$P_{2019} = \begin{bmatrix} 51 \\ 1419 \\ 127 \end{bmatrix} \begin{array}{l} \text{South Korea} \\ \text{China} \\ \text{Japan} \end{array}$$

The expected percentage annual growth of each of the populations is shown in the following table.

Country	South Korea	China	Japan
Annual Change	0.30% increase	0.35% increase	0.25% decrease

a. Find matrix P_{2020}, which shows the expected population, in millions, to two decimal places, of each country in 2020. (1 MARK)

b. The expected population of each of the countries in 2020 can be determined by the matrix product

$P_{2020} = G \times P_{2019}$

where G is a diagonal matrix.

Find matrix G. (1 MARK)

Adapted from VCAA 2018 Exam 2 Matrices Q2