# AOS 2
# CHAPTER 8
## Networks and decision mathematics

**LESSONS**

# 8

## KEY KNOWLEDGE

- the concepts, conventions and terminology of graphs including planar graphs and Euler's rule, and directed (digraphs) and networks
- use of matrices to represent graphs, digraphs and networks and their application
- the concepts, conventions and notations of walks, trails, paths, cycles and circuits
- Eulerian trails and Eulerian circuits: the conditions for a graph to have an Eulerian trail or an Eulerian circuit, properties and applications
- Hamiltonian paths and cycles: properties and applications
- trees and spanning trees
- minimum spanning trees in a weighted connected graph and their determination by inspection or by Prim's algorithm
- use of minimal spanning trees to solve minimal connector problems
- use of networks to model flow problems: capacity, sinks and sources
- solution of small-scale network flow problems by inspection and the use of the 'maximum-flow minimum-cut' theorem to aid the solution of larger scale problems
- determination of the shortest path between two specified vertices in a graph, digraph or network by inspection
- Dijkstra's algorithm and its use to determine the shortest path between a given vertex and each of the other vertices in a weighted graph or network
- use of a bipartite graph and its tabular or matrix form to represent a matching problem
- determination of the optimum assignment(s) of people or machines to tasks by inspection or by use of the Hungarian algorithm for larger scale problems
- construction of an activity network from a precedence table (or equivalent) including the use of dummy activities where necessary
- use of forward and backward scanning to determine the earliest starting times (EST) and latest starting times (LST) for each activity
- use of earliest starting times and latest starting times to identify the critical path in the network and determine the float times for non-critical activities
- use of crashing to reduce the completion time of the project or task being modelled.

Image: ChristianChan/Shutterstock.com

# 8A Introduction to graphs and networks

**STUDY DESIGN DOT POINT**

- the concepts, conventions and terminology of graphs including planar graphs and Euler's rule, and directed (digraphs) and networks

8A 8B 8C 8D 8E 8F 8G 8H 8I 8J

**KEY SKILLS**

During this lesson, you will be:

- identifying properties of a graph
- identifying and constructing graphs
- using Euler's rule.

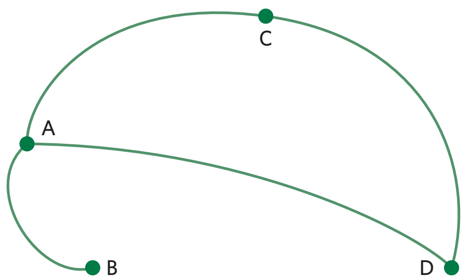
**KEY TERMS**

- Graph
- Vertex
- Edge
- Degree
- Loop
- Face
- Bridge
- Isolated vertex
- Isomorphic graphs
- Planar graph
- Connected graph
- Simple graph
- Complete graph
- Degenerate graph
- Euler's rule

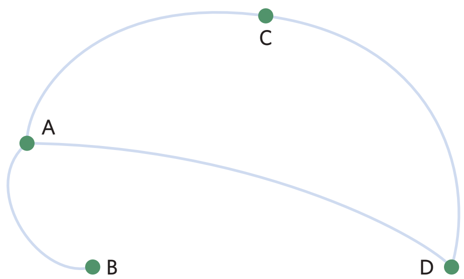
There are many occurences in everyday life that involve connections. In such situations, it can be useful to visually display these connections. This allows them to be analysed mathematically in order to solve problems.

## Identifying properties of a graph

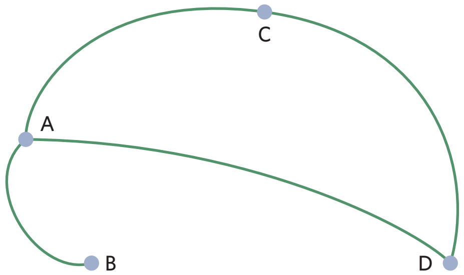
A **graph**, also known as a network, is a diagram that is used to show the connections between a group of common elements, such as objects, locations, people, or activities.



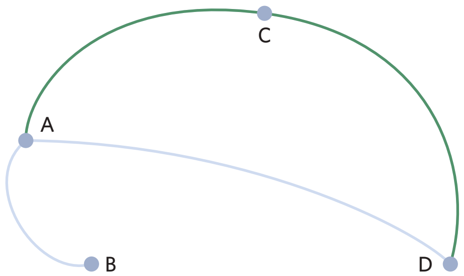

A **vertex** is a point on a graph. Points A, B, C and D are all vertices.



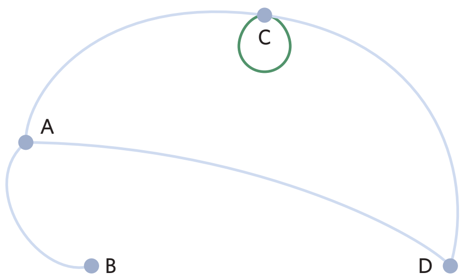
An **edge** is a line connecting one vertex to either another vertex or itself. They represent connections between vertices. Edges can overlap each other.



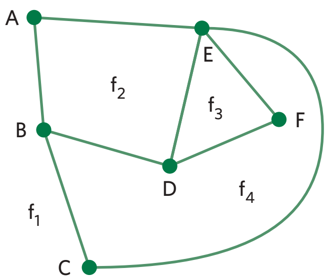

A **degree** of a vertex is the number of times an edge attaches to it. The degree of vertex C is two as there are two edges attached to it.



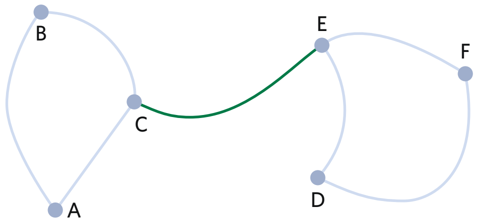

A **loop** is an edge that connects a vertex back to itself without passing through any other vertices. A loop adds two degrees to a vertex as there are two connections, so the degree of vertex C is four.



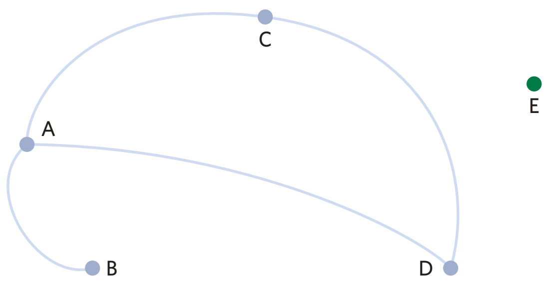

A **face** is a separate area on a graph that is bordered by edges. For the number of faces to be counted directly from a graph, there must be no overlapping edges. The space outside the graph is counted as an additional face. If there are overlapping edges, the graph must be redrawn so that the number of faces can be counted. The following graph has four faces.




In some graphs, all vertices are connected to each other, either directly or indirectly. Every vertex is able to be reached by another vertex. A **bridge** is an edge that, when removed, will prevent this. In the following graph, the edge connecting vertices C and E is a bridge. If it was removed, vertices A, B and C would no longer be able to reach vertices D, E and F and vice versa.

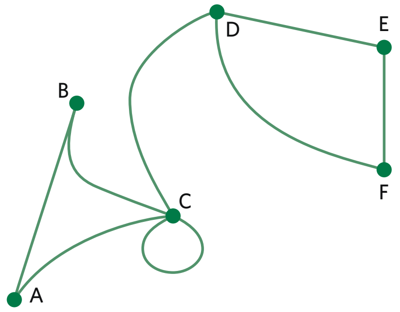


An **isolated vertex** is not connected by an edge to any other vertices on the graph. Vertex E is isolated.




## Worked example 1

Consider the graph.




**a.** How many edges are in this graph?

### Explanation

There are eight edges in the graph.

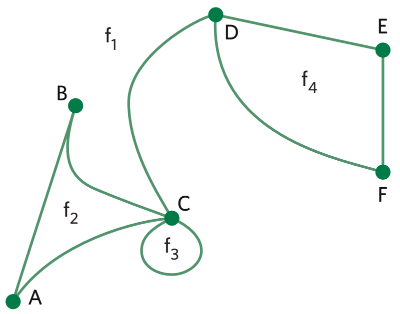
### Answer

8

**b.** How many faces are in this graph?

### Explanation

This graph has no overlapping edges so the number of faces can be counted directly from the graph. The space outside the graph is counted as an additional face.




### Answer

4

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**c.** How many vertices are in this graph?

### Explanation

There are six points that are connected by edges.

### Answer

6

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**d.** Which vertex has a loop?

### Explanation

A loop can be found at vertex C.

### Answer

C

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**e.** Determine the degree of vertex C.

### Explanation

There are four edges connected to vertex C, however a loop counts as two degrees as it connects to the vertex twice.

### Answer

5

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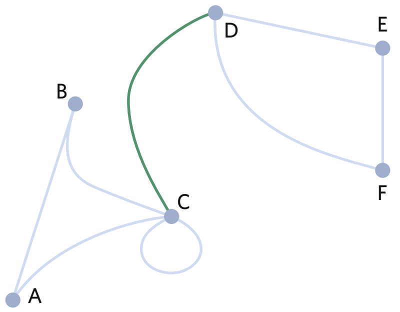
**f.** Between which vertices can a bridge be found?

### Explanation

If the edge between vertices C and D were removed, vertices A, B and C would be unable to reach vertices D, E and F and vice versa.

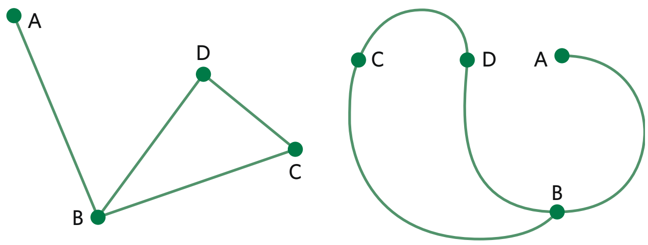



### Answer

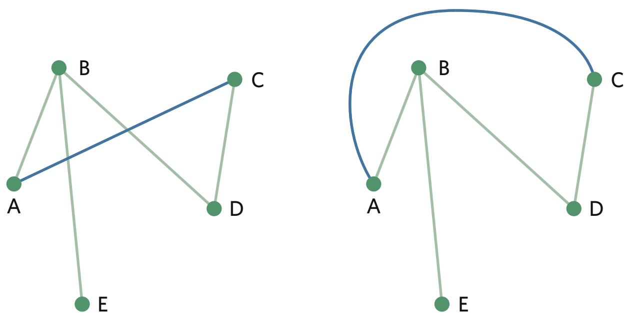
C and D

## Identifying and constructing graphs

**Isomorphic graphs**, or equivalent graphs, are graphs that display the same information as each other. They have the same vertices and connections, but they are drawn differently. The following graphs are isomorphic.

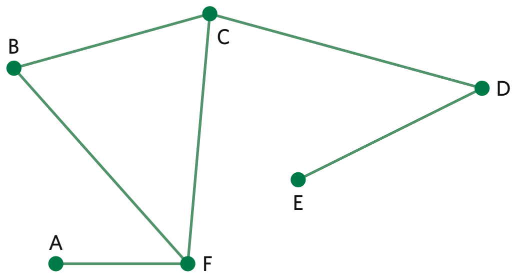



A **planar graph** is a graph that can be drawn with no overlapping edges. If a graph cannot be redrawn this way, it is considered non-planar. The following is an example of a graph being redrawn in planar form with no overlapping edges.

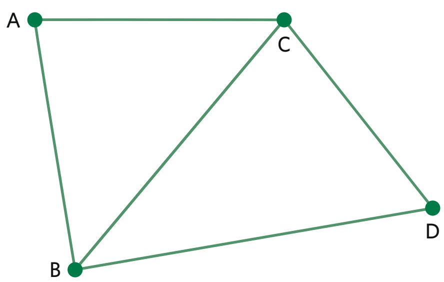



In addition to redrawing edges, vertices can also be moved around to make it easier to redraw in planar form. However, a non-planar graph will always remain non-planar, regardless of how the vertices are rearranged.

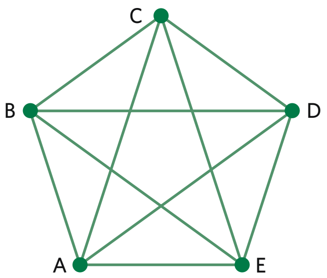
A **connected graph** is a graph in which every vertex is connected to each other, either directly or indirectly. Every vertex is able to be reached by any other vertex by travelling along the edges.



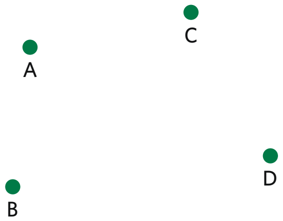

A **simple graph** does not contain any loops or duplicate edges.



A **complete graph** directly connects every vertex to every other vertex.

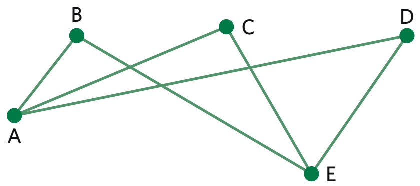



A **degenerate graph** is a graph with no edges. All of the vertices are isolated.




## Worked example 2

Consider the following graph.




**a.** Redraw the graph in planar form.

### Explanation

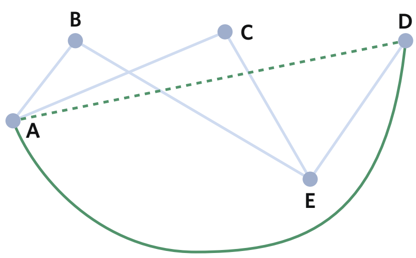
**Step 1:** Identify instances of overlapping edges.

The A-C edge is overlapping with the B-E edge.

The A-D edge is overlapping with the B-E edge and the C-E edge.

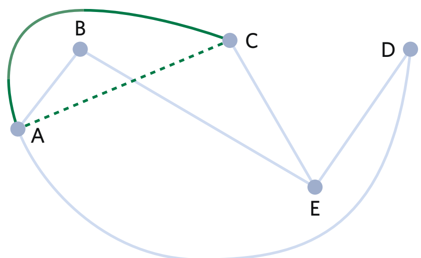
**Step 2:** Remove an overlapping edge and redraw it.

The A-D edge can be redrawn on the outside so it is not crossing any other edges.




**Step 3:** Repeat for the remaining overlapping edges.

The A-C edge can be redrawn on the outside so it is not crossing any other edges.

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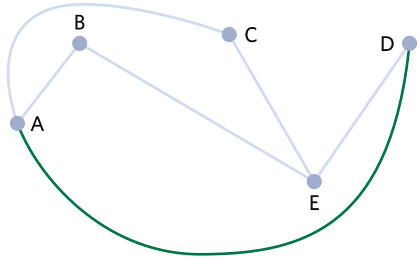
**Answer**




Note: This is just one example of the graph in planar form. There are other possible solutions.

**b.** Is this a connected graph?

**Explanation**

This graph is considered connected as vertices A, B, C, D and E can all reach each other using the edges.

**Answer**

Yes

**c.** Is this a complete graph?

**Explanation**

In this graph, there are multiple instances of vertices not being directly connected to all other vertices. For example, vertex A is not directly connected to vertices C or E.

**Answer**

No

## Using Euler's rule

**Euler's rule** describes the relationship between the number of vertices, edges and faces for all connected planar graphs. This relationship can be written as

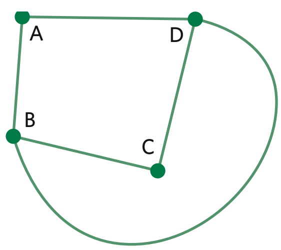
$v - e + f = 2$, where

- $v$ is the number of vertices
- $e$ is the number of edges
- $f$ is the number of faces

### Worked example 3

Use Euler's rule to solve the following questions.

**a.** Show that the graph is a connected planar graph.

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### Explanation

**Step 1:** Identify the number of vertices, edges and faces.

There are four vertices, so $v = 4$.

There are five edges, so $e = 5$.

There are three faces, so $f = 3$.

**Step 2:** Apply these values to Euler's rule, ensuring to show calculations in the final answer.

### Answer

$$v - e + f = 4 - 5 + 3$$
$$= 2$$

Therefore, the graph is planar.

**b.** If a connected planar graph has five vertices and five faces, how many edges does the graph have?

### Explanation

**Step 1:** Identify the known values.

There are five vertices, so $v = 5$.

There are five faces, so $f = 5$.

**Step 2:** Substitute into Euler's rule.

$$v - e + f = 2$$
$$5 - e + 5 = 2$$
$$10 - e = 2$$
$$e = 8$$

### Answer

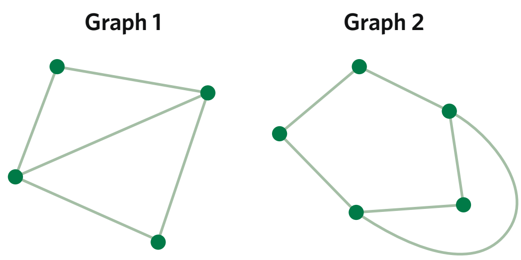
8 edges

## Exam question breakdown

*VCAA 2019 Exam 1 Networks and decision mathematics Q4*

Two graphs, labelled Graph 1 and Graph 2, are shown.

Which of the following statements is **not** true?




**A.** Graph 1 and Graph 2 are isomorphic.

**B.** Graph 1 has five edges and Graph 2 has six edges.

**C.** Both Graph 1 and Graph 2 are connected graphs.

**D.** Both Graph 1 and Graph 2 have three faces each.

**E.** Neither Graph 1 nor Graph 2 are complete graphs.

### Explanation

To solve this question, determine whether each is true or not.

A: This is not true. Graph 1 and Graph 2 are not isomorphic, as they have a different number of vertices. Graph 1 has four vertices and five edges whereas Graph 2 has five vertices and six edges. ✓

B: This is true. Graph 1 has five edges and Graph 2 has six edges. ✗

C: This is true. All vertices in both graphs are able to be reached by all other vertices. ✗

D: This is true. Both graphs contain three faces. ✗

E: This is true. All vertices in a graph must be directly connected to every other vertex in a graph for it to be considered complete. ✗

**71%** of students answered this question correctly.

**18%** of students incorrectly chose option E. It is likely that students were confused between the definitions of connected and complete graphs, leading them to answer incorrectly.

### Answer

A

# 8A Questions

## Identifying properties of a graph

**1.** The following is a connected graph with four vertices.

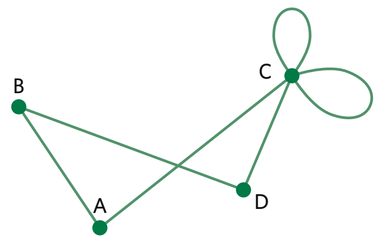
What is the degree of vertex C?

**A.** 3

**B.** 4

**C.** 5

**D.** 6

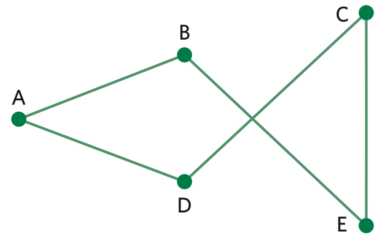



**2.** Consider the following graph.

**a.** How many vertices does it contain?

**b.** All vertices in this graph have the same degree.
What is the degree of each vertex?

**c.** How many faces are there in this graph?




**3.** Consider the following graph.

**a.** How many vertices does it contain?

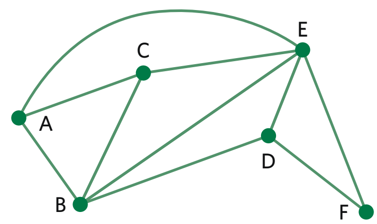
**b.** What is the degree of vertex C?

**c.** What is the degree of vertex D?

**d.** What is the degree of vertex E?

**e.** How many faces are there in this graph?

**f.** Is there a bridge in this graph? If so, which vertices does it connect to?




**4.** The following graph shows roads connecting countries in Europe.

**a.** How many vertices are present?

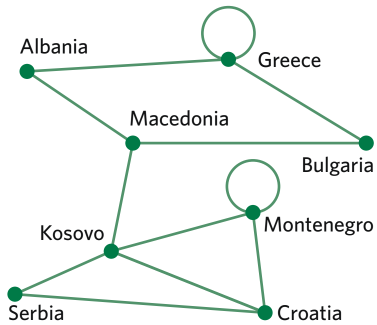
**b.** What is the degree of Bulgaria?

**c.** What is the degree of Greece?

**d.** The degree of Croatia is three. What does this mean in this context?

**e.** How many faces are there in this graph?

**f.** Is there a bridge in this graph? If so, which vertices does it connect to?




## Identifying and constructing graphs

**5.** Consider the following graph.

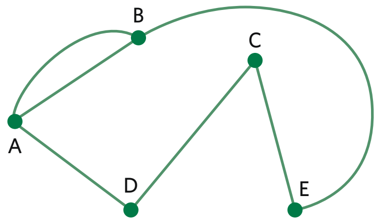
What term can be used to describe this graph?

**A.** Simple

**B.** Complete

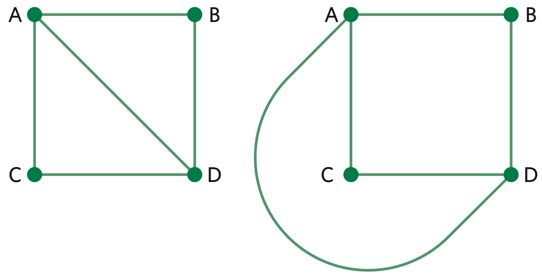
**C.** Planar

**D.** Degenerate

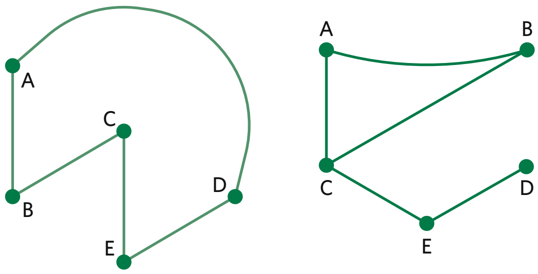


**6.** Which of the following graphs are isomorphic?

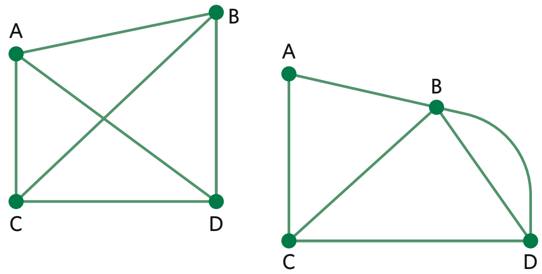
**A.**




**B.**



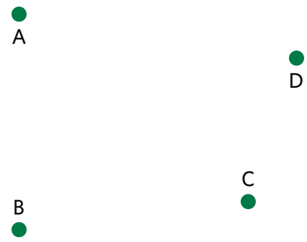

**C.**




**D.** None of these.

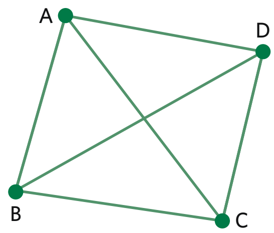
**7.** Consider the graph.

Is this a connected graph or a degenerate graph? Explain briefly.



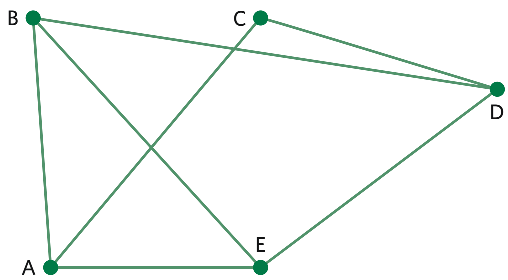

**8.** Consider the graph.

What type of graph is this? There may be more than one type of graph.

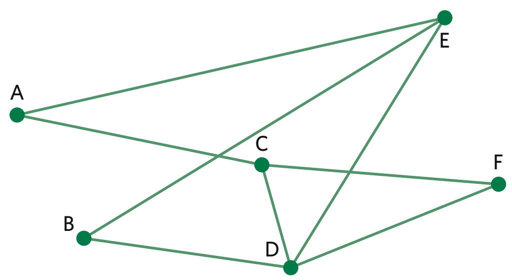



**9.** Redraw the following graphs in planar form.

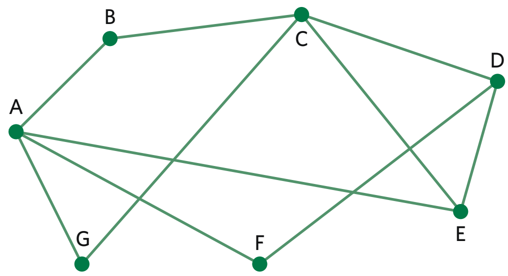
**a.**



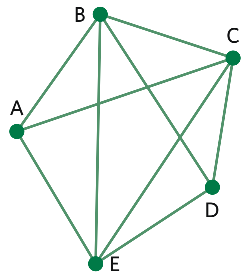

**b.**




**c.**




**d.**



10. Create a graph that represents the following information.
- There are five towns connected by roads: Melbourne, Ballarat, Geelong, Bendigo and Mildura.
- There is a road between Melbourne and Ballarat.
- There is a road between Ballarat and Mildura.
- Bendigo is directly connected to Geelong and Mildura.
- There are two roads connecting Melbourne and Geelong.
- None of the roads cross over each other.

## Using Euler's rule

11. Consider a graph that has five vertices and eight edges. If the graph was connected and planar, how many faces would it have?

**A.** 3

**B.** 4

**C.** 5

**D.** 6

12. For a connected planar graph,

**a.** find $v$ if $e = 6$ and $f = 5$.

**b.** find $e$ if $v = 4$ and $f = 6$.

**c.** find $f$ if $v = 7$ and $e = 9$.

**d.** find $e$ if $v = 5$ and $f = 8$.

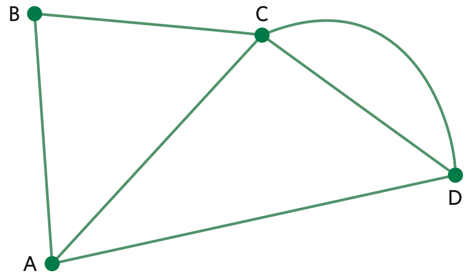
**e.** find $f$ if $v = 10$ and $e = 13$.

**f.** find $v$ if $e = 14$ and $f = 7$.

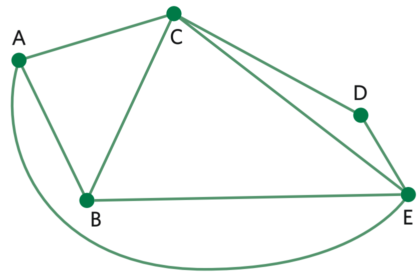
13. The city council has decided they want to plant grass in the areas between and around some roads. There are seven roads in the area that connect four landmarks and none of them intersect. It is only possible to travel along these seven roads to all of the landmarks. How many grassy areas will there be?

14. Determine the values of $v$, $e$ and $f$ for the following graphs and use Euler's rule to show that these graphs are connected and planar.

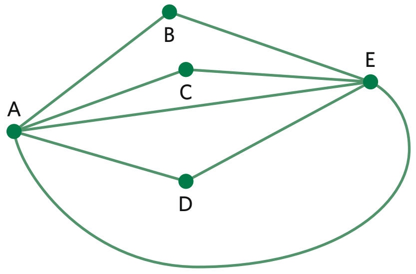
**a.**



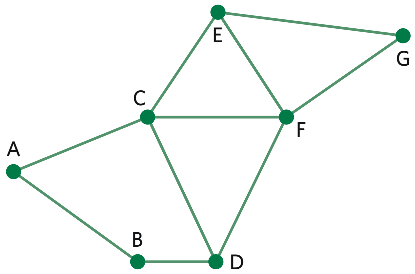

**b.**




**c.**




**d.**

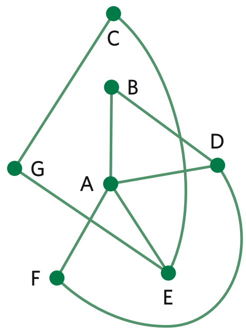


## Joining it all together

**15.** Consider the following graph.

**a.** Between which vertices does a bridge exist?

**b.** Redraw the graph in planar form.




**16.** Consider the following graph.

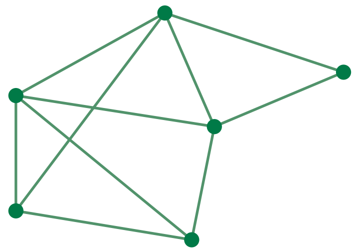
**a.** What type of graph is this? There may be more than one type of graph.

**b.** Redraw this graph without any edges overlapping.

**c.** How many faces are in this graph?

**d.** Calculate the sum of the degrees in this graph.

**e.** How many vertices have an odd degree?



**17.** A group of seven people all give handshakes to each other.

**a.** Create a graph that represents this.

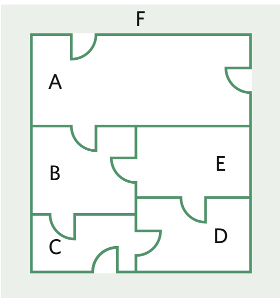
**b.** What type of graph is this? There may be more than one type of graph.

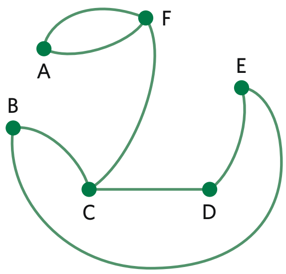
**c.** All vertices have the same degree. What is the degree of each vertex?

## Exam practice

**18.** Maggie's house has five rooms, A, B, C, D and E, and eight doors. The floor plan of these rooms and doors is shown. The outside area, F, is shown shaded on the floor plan.

The floor plan is represented by the following graph. On this graph, vertices represent the rooms and the outside area. Edges represent direct access to the rooms through the doors. One edge is missing from the graph.




**a.** Which vertices should the edge be drawn between? (1 MARK)

**b.** What is the degree of vertex E? (1 MARK)

*VCAA 2021 Exam 2 Networks and decision mathematics Q1a,b*

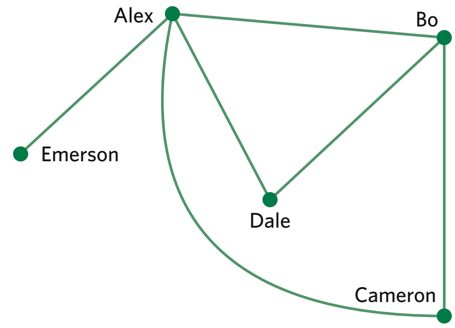
Part **a**: **93%** of students answered this question correctly.

Part **b**: **88%** of students answered this question correctly.

**19.** The Sunny Coast Cricket Club has five new players join its team: Alex, Bo, Cameron, Dale and Emerson. The following graph shows the players who have played cricket together before joining the team. For example, the edge between Alex and Bo shows that they have previously played cricket together.

Who had played cricket with both Alex and Bo before joining the team? (1 MARK)

*VCAA 2020 Exam 2 Networks and decision mathematics Q1b*




**58%** of students answered this question correctly.

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**20.** Consider the following graph.

The number of faces is

**A.** 2

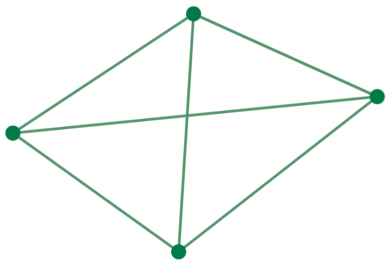
**B.** 3

**C.** 4

**D.** 5

**E.** 6

*VCAA 2021 Exam 1 Networks and decision mathematics Q3*



**34%** of students answered this question correctly.

## Questions from multiple lessons

### Data analysis

**21.** The following scatterplot shows *income*, in thousands of dollars, plotted against *happiness*, on a scale from one to ten. A least squares line has been fitted to the data and the resulting residual plot is also shown.

The equation of this least squares line is

*happiness* $= 1.00 + 0.139 \times$ *income*

The coefficient of determination is $r^2 = 0.768$.

Which of the following statements is **not** true?

**A.** Using the least squares line to predict the *happiness* of a person with an *income* of \$50 000 is an example of interpolation.

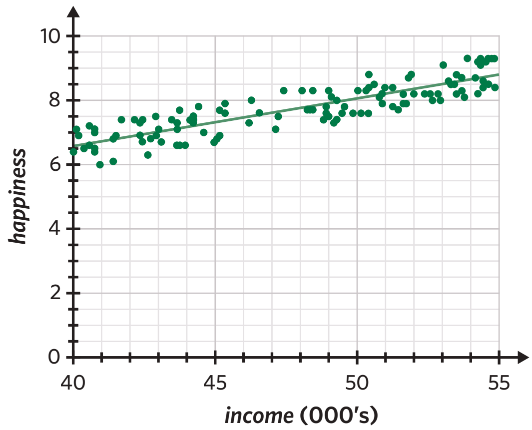
**B.** The value of the correlation coefficient is close to 0.88.

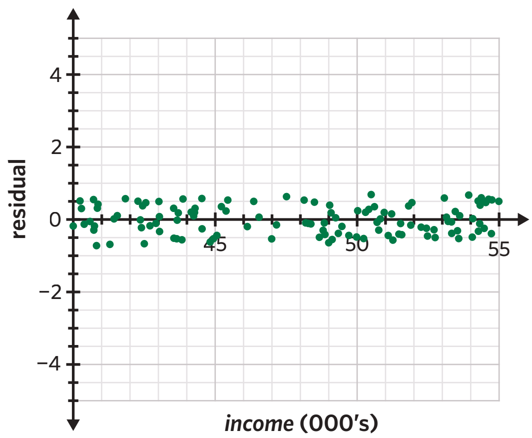
**C.** 23.2% of the variation in *happiness* is not explained by variation in *income*.

**D.** On average, *happiness* increases by 1.00 for every \$10 000 increase in *income*.

**E.** Ignoring any outliers, the association between *happiness* and *income* can be described as strong, linear and positive.

*Adapted from VCAA 2016 Exam 1 Data analysis Q9*



## Recursion and financial modelling

**22.** Five lines of an amortisation table for an annuity investment are shown.

The interest rate for this investment remains constant, but the payment value may vary.

| payment number | payment | interest | principal addition | balance of investment |
|---|---|---|---|---|
| 8 | 75.00 | 9.46 | 84.46 | 1661.77 |
| 9 | 75.00 | 9.97 | 84.97 | 1746.74 |
| 10 | 75.00 | 10.48 | 85.48 | 1832.22 |
| 11 | 75.00 | 10.99 | 85.99 | 1918.22 |
| 12 | | | | 2000.00 |

The balance of the investment after payment number 12 is $2000.

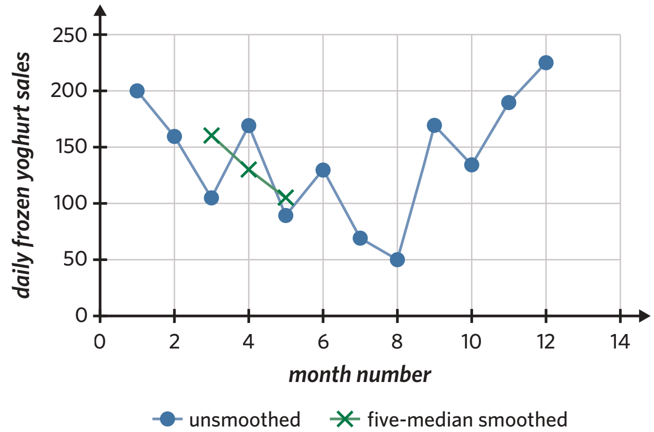
The value of payment number 12 is closest to

**A.** $70 **B.** $75 **C.** $82 **D.** $87 **E.** $90

*Adapted from VCAA 2017 Exam 1 Recursion and financial modelling Q23*

## Data analysis

**23.** A local frozen yoghurt establishment decided to take note of their average daily sales every month of the year. The *daily frozen yoghurt sales* were recorded each month and were plotted against the *month number* (1 = January, 2 = February, and so on) in the following graph.




The data was collected over a period of one year.

**a.** Five-median smoothing has been used to smooth the time series plot.

The first three smoothed points are shown as crosses (X). Complete the five-median smoothing by marking smoothed values with crosses. (2 MARKS)

The *daily ice cream* sales each month of the store next door were also recorded in the following table.

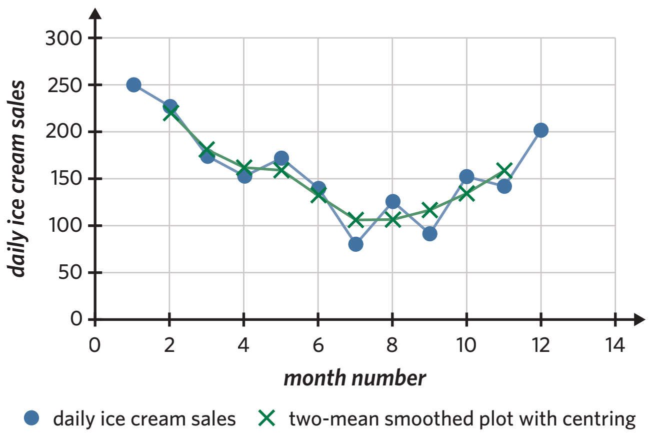
| | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ***month number*** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| ***daily ice cream sales*** | 251 | 227 | 174 | 153 | 172 | 139 | 81 | 126 | 93 | 152 | 143 | 201 |

The data in the table has been used to plot *daily ice cream sales* against *month number* in a time series plot.

**b.** Two-mean smoothing with centring has been used to smooth the time series plot.

The smoothed values are marked with crosses (X). Using the data given in the table, show that the two-mean smoothed ice cream sales centred on July is 106.75. (2 MARKS)

*Adapted from VCAA 2016 Exam 2 Data analysis Q4*



# 8B Graphs, networks and matrices

**STUDY DESIGN DOT POINT**

- use of matrices to represent graphs, digraphs and networks and their application

8A 8B 8C 8D 8E 8F 8G 8H 8I 8J

**KEY SKILLS**

During this lesson, you will be:

- constructing an adjacency matrix from a graph
- constructing a graph from an adjacency matrix
- representing directed graphs.

**KEY TERMS**

- Adjacency matrix
- Directed graph (digraph)

Matrices can be used to represent the links between the vertices in a graph, by displaying the number of edges connecting each vertex. These matrices can be used to construct both undirected and directed graphs. Conversely, these matrices can be constructed if a graph is provided.

## Constructing an adjacency matrix from a graph

An **adjacency matrix** represents the number of connections between vertices. It is a square matrix that has a row and column for each vertex and contains the number of edges between each vertex.

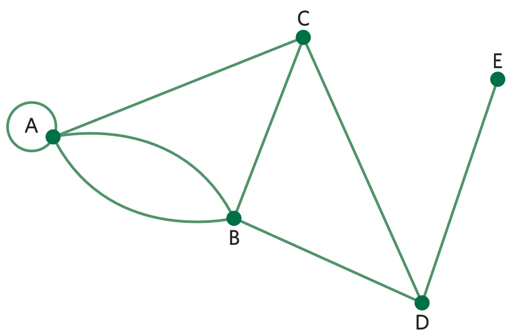
The order of the matrix will represent the number of vertices in the graph since it has a row and column for each vertex. The number of edges between vertices is represented as an element within the matrix. For example, an element of 3 indicates there are three edges connecting two different vertices, or a vertex with itself. A loop counts as one edge.

Columns and rows should add up to the degree of that particular vertex since the degree shows the total number of connections at a vertex. However, if a loop is involved, the columns and rows will add up to one less than the degree of the vertex for each loop involved, as a loop is counted as two degrees but only one edge.

Undirected graphs produce adjacency matrices that are symmetric about the leading diagonal. This is because the number of edges between vertices A and B is the same as the number of edges between vertices B and A.

**Worked example 1**

Construct an adjacency matrix from the following graph.




Continues →

## Explanation

**Step 1:** Determine the order of the matrix.

All adjacency matrices are square.

There are five vertices.

The order will be $5 \times 5$.

**Step 2:** Label the matrix.

Each vertex will have its own row and column.

$$\begin{matrix} & A & B & C & D & E & \\ \end{matrix}$$
$$\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}\begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

**Step 3:** Fill in row A of the matrix.

There is one edge from vertex A back to vertex A. Element $m_{AA}$ is 1.

There are two edges between vertex A and vertex B. Element $m_{AB}$ is 2.

There is one edge between vertex A and vertex C. Element $m_{AC}$ is 1.

There are no edges between vertex A and vertex D. Element $m_{AD}$ is 0.

There are no edges between vertex A and vertex E. Element $m_{AE}$ is 0.

$$\begin{matrix} A & B & C & D & E \end{matrix}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}\begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

**Step 4:** Fill in row B of the matrix.

There are two edges between vertex B and vertex A. Element $m_{BA}$ is 2.

There are no loops at vertex B. Element $m_{BB}$ is 0.

There is one edge between vertex B and vertex C. Element $m_{BC}$ is 1.

There is one edge between vertex B and vertex D. Element $m_{BD}$ is 1.

There are no edges between vertex B and vertex E. Element $m_{BE}$ is 0.

$$\begin{matrix} A & B & C & D & E \end{matrix}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}\begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

**Step 5:** Repeat for the remaining rows.

## Answer

$$\begin{matrix} A & B & C & D & E \end{matrix}$$
$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

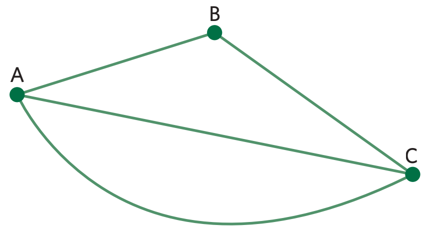
# Constructing a graph from an adjacency matrix

Adjacency matrices contain the information needed to construct a graph.

Each row and column in the matrix must be represented as a vertex. The value of each element can then be used to determine the number of edges connecting the two corresponding vertices. If the elements in the leading diagonal are all zeros, then the graph will not contain any loops.

For example, element $m_{AC}$ in the following matrix is 2. This indicates that there are two edges between vertex A and vertex C.

$$\begin{matrix} A & B & C & \\ \end{matrix}$$
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$




## Worked example 2

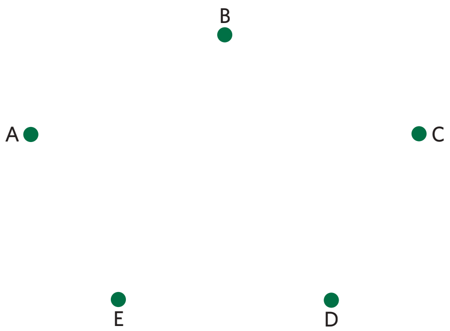
Construct a graph from the following adjacency matrix.

$$\begin{matrix} A & B & C & D & E \end{matrix}$$
$$\begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

### Explanation

**Step 1:** Draw and label the vertices.

The order of the matrix is 5 × 5. There will be five vertices.




**Step 2:** Draw the edges connecting to vertex A.

Note that these can be straight or curved lines.

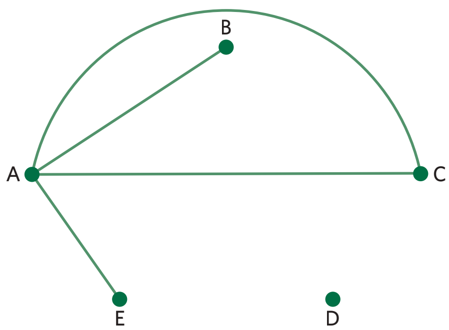
$m_{AA} = 0$, so there is no loop at vertex A.

$m_{AB} = 1$, so there is one edge between vertices A and B.

$m_{AC} = 2$, so there are two edges between vertices A and C.

$m_{AD} = 0$, so there are no edges between vertices A and D.

$m_{AE} = 1$, so there is one edge between vertices A and E.




Continues →

**Step 3:** Draw the edges connecting to vertex B.

$m_{BA} = 1$, so there is one edge between vertices B and A.

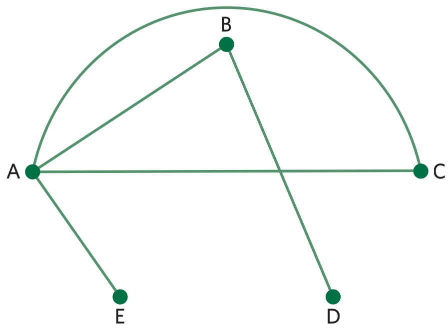
$m_{BB} = 0$, so there is no loop at vertex B.

$m_{BC} = 0$, so there are no edges between vertices B and C.

$m_{BD} = 1$, so there is one edge between vertices B and D.

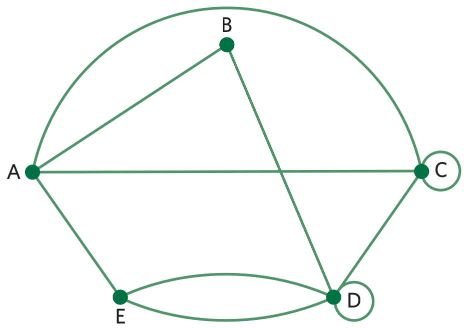
$m_{BE} = 0$, so there are no edges between vertices B and E.

Note that the edge between vertices A and B is already drawn so only the edge from B to D needs to be added.




**Step 4:** Repeat for the remaining vertices.

**Answer**

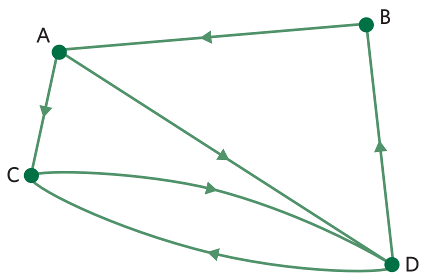



Note: Graphs may look different. A correct graph must simply include all vertices and edges leading to and from them.

## Representing directed graphs

A **directed graph** is a network containing arrows on each edge that show the way in which one can travel between two vertices.

For example, the following directed graph shows the roads connecting four towns.




The edge from A to D shows a one-way road from town A to town D.

The edges between C and D show that traffic is allowed in both directions between towns C and D.

A directed graph can be represented in an adjacency matrix by labelling the rows as ‘from’ and the columns as ‘to’ in order to show the directional flow of each edge. Each element in the matrix represents the number of edges going from the row vertex to the column vertex. Adjacency matrices for directed graphs do not have to be symmetric.

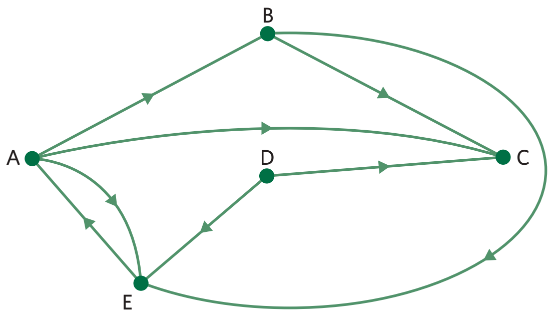
See worked example 3

Directional information contained in an adjacency matrix can be used to construct a digraph.

See worked example 4

## Worked example 3

Represent the following directed graph in an adjacency matrix.




### Explanation

**Step 1:** Determine the order of the matrix.

All adjacency matrices are square.

There are five vertices.

The order will be $5 \times 5$.

**Step 2:** Label the matrix.

Each vertex will have its own row and column.

Label the rows with ‘from’ and the columns with ‘to’.

$$\begin{array}{cl} \begin{array}{ccccc} & & \text{to} & & \\ A & B & C & D & E \end{array} & \\ \left[\begin{array}{ccccc} \phantom{0}&\phantom{0}&\phantom{0}&\phantom{0}&\phantom{0} \\ &&&& \\ &&&& \\ &&&& \\ &&&& \end{array}\right] & \begin{array}{l} A \\ B \\ C \quad \text{from} \\ D \\ E \end{array} \end{array}$$

**Step 3:** Fill in row A of the matrix.

There are no loops at vertex A. Element $m_{AA}$ is 0.

There is one edge from vertex A to vertex B. Element $m_{AB}$ is 1.

There is one edge from vertex A to vertex C. Element $m_{AC}$ is 1.

There are no edges from vertex A to vertex D. Element $m_{AD}$ is 0.

There is one edge from vertex A to vertex E. Element $m_{AE}$ is 1.

$$\begin{array}{cl} \begin{array}{ccccc} & & \text{to} & & \\ A & B & C & D & E \end{array} & \\ \left[\begin{array}{ccccc} 0&1&1&0&1 \\ &&&& \\ &&&& \\ &&&& \\ &&&& \end{array}\right] & \begin{array}{l} A \\ B \\ C \quad \text{from} \\ D \\ E \end{array} \end{array}$$

**Step 4:** Fill in row B of the matrix.

There are no edges from vertex B to vertex A. Element $m_{BA}$ is 0.

There are no loops at vertex B. Element $m_{BB}$ is 0.

There is one edge from vertex B to vertex C. Element $m_{BC}$ is 1.

There are no edges from vertex B to vertex D. Element $m_{BD}$ is 0.

There is one edge from vertex B to vertex E. Element $m_{BE}$ is 1.

$$\begin{array}{cl} \begin{array}{ccccc} & & \text{to} & & \\ A & B & C & D & E \end{array} & \\ \left[\begin{array}{ccccc} 0&1&1&0&1 \\ 0&0&1&0&1 \\ &&&& \\ &&&& \\ &&&& \end{array}\right] & \begin{array}{l} A \\ B \\ C \quad \text{from} \\ D \\ E \end{array} \end{array}$$

**Step 5:** Repeat for the remaining rows.

Continues →

## Answer

$$
\begin{array}{c}
\text{to} \\
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{l} \\ \\ A \\ B \\ C \ \ \text{from} \\ D \\ E \end{array}
$$

## Worked example 4

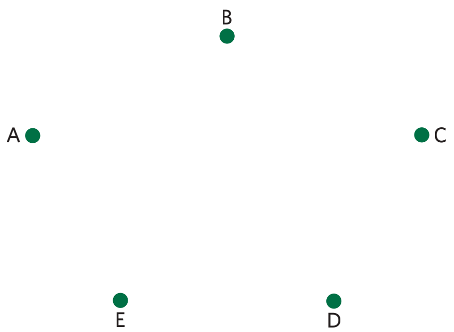
Construct a digraph from the adjacency matrix.

$$
\begin{array}{c}
\text{to} \\
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix}
0 & 2 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{l} \\ \\ A \\ B \\ C \ \ \text{from} \\ D \\ E \end{array}
$$

### Explanation

**Step 1:** Draw and label the vertices.

The order of the matrix is $5 \times 5$. There will be five vertices.




**Step 2:** Draw the edges connecting from vertex A.

Note that these can be straight or curved lines.

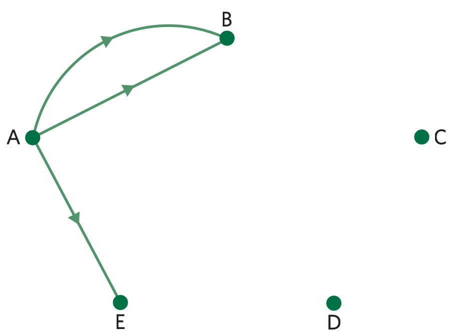
$m_{AA} = 0$. There is no loop at vertex A.

$m_{AB} = 2$. There are two edges from vertex A to B.

$m_{AC} = 0$. There are no edges from vertex A to C.

$m_{AD} = 0$. There are no edges from vertex A to D.

$m_{AE} = 1$. There is one edge from vertex A to E.




**Step 3:** Draw the edges connecting from vertex B.

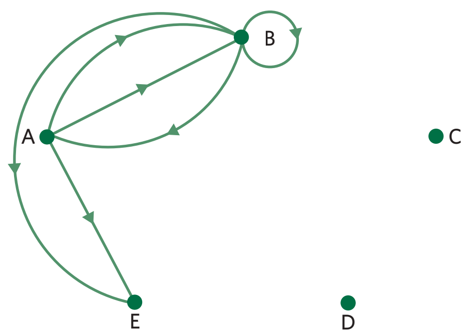
$m_{BA} = 1$. There is one edge from vertex B to A.

$m_{BB} = 1$. There is a loop at vertex B.

$m_{BC} = 0$. There are no edges from vertex B to C.

$m_{BD} = 0$. There are no edges from vertex B to D.

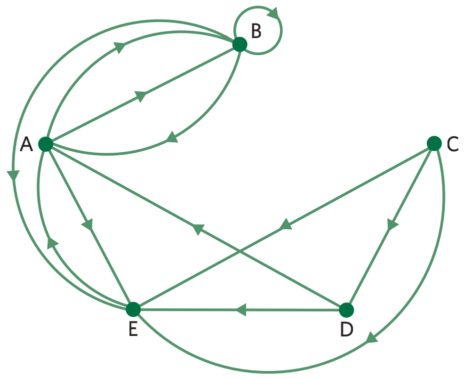
$m_{BE} = 1$. There are no edges from vertex B to E.




**Step 4:** Repeat for the remaining vertices.

Continues →

**Answer**

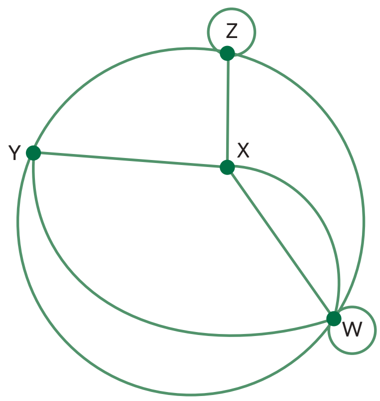



Note: Graphs may look different. A correct graph must simply include all vertices and edges leading to and from them, with the correct directional information.

## Exam question breakdown

*VCAA 2017 Exam 1 Networks and decision mathematics Q3*

Consider the following graph.




The adjacency matrix for this graph, with some elements missing, is shown.

$$\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \left[\begin{array}{cccc} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{array}\right] \end{array} \begin{array}{c} \\ W \\ X \\ Y \\ Z \end{array}$$

This adjacency matrix contains 16 elements when complete.

Of the 12 missing elements

**A.** eight are ‘1’ and four are ‘2’.

**B.** four are ‘1’ and eight are ‘2’.

**C.** six are ‘1’ and six are ‘2’.

**D.** two are ‘0’, six are ‘1’ and four are ‘2’.

**E.** four are ‘0’, four are ‘1’ and four are ‘2’.

### Explanation

**Step 1:** Determine the missing elements in row W of the matrix.

There are two edges between vertex W and vertex X. Element $m_{WX}$ is 2.

There are two edges between vertex W and vertex Y. Element $m_{WY}$ is 2.

There is one edge between vertex W and vertex Z. Element $m_{WZ}$ is 1.

**Step 2:** Determine the missing elements in row X of the matrix.

There are two edges between vertex X and vertex W. Element $m_{XW}$ is 2.

There is one edge between vertex X and vertex Y. Element $m_{XY}$ is 1.

There is one edge between vertex X and vertex Z. Element $m_{XZ}$ is 1.

**Step 3:** Determine the missing elements in row Y of the matrix.

There are two edges between vertex Y and vertex W. Element $m_{YW}$ is 2.

There is one edge between vertex Y and vertex X. Element $m_{YX}$ is 1.

There is one edge between vertex Y and vertex Z. Element $m_{YZ}$ is 1.

**Step 4:** Determine the missing elements in row Z of the matrix.

There is one edge between vertex Z and vertex W. Element $m_{ZW}$ is 1.

There is one edge between vertex Z and vertex X. Element $m_{ZX}$ is 1.

There is one edge between vertex Z and vertex Y. Element $m_{ZY}$ is 1.

Continues →

**Step 5:** Determine the total number of each missing element.

Eight of the missing elements are ‘1’.

Four of the missing elements are ‘2’.

**Answer**

A

92% of students answered this question correctly.

**5%** of students incorrectly chose either D or E. It was important to be aware that the graph was a complete graph which means each vertex is directly connected to every other vertex by at least one edge, so none of the missing elements should be ‘0’.



# 8B Questions

## Constructing an adjacency matrix from a graph

**1.** Which of the following is not an adjacency matrix for an undirected graph?

**A.** $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

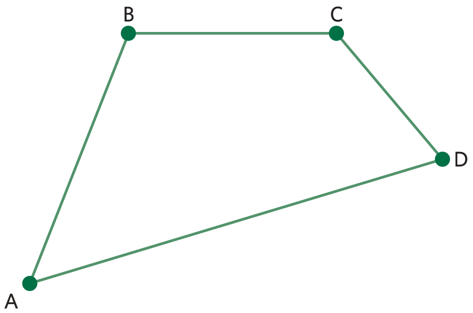
**B.** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

**C.** $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

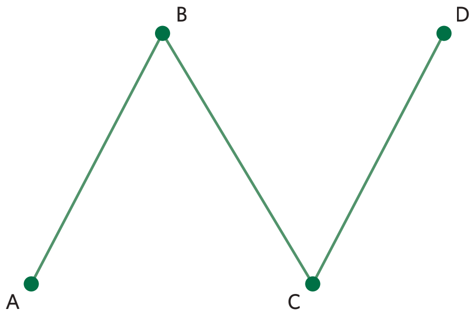
**D.** $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

**2.** Construct an adjacency matrix from the following graphs.

**a.**

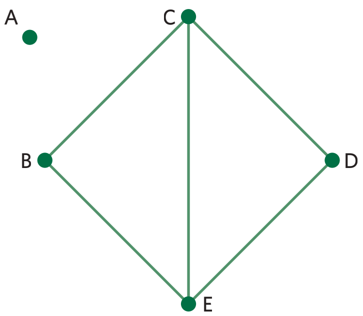



**b.**

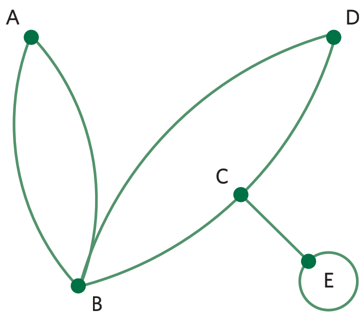



**3.** Construct an adjacency matrix from the following graphs.

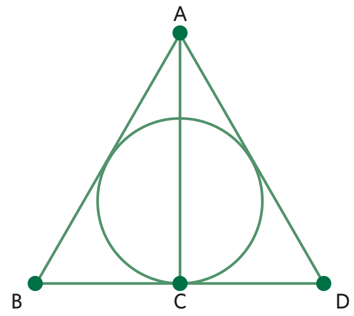
**a.**




**b.**



4. Harry Potter can't seem to remember how to draw the symbol of the Deathly Hallows. Create an adjacency matrix that will help Harry remember which vertices are connected.

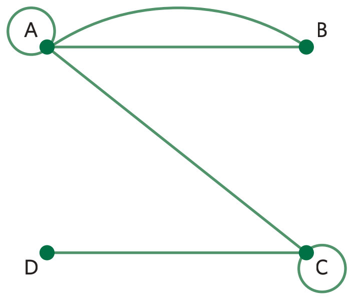



## Constructing a graph from an adjacency matrix

5. Consider the following adjacency matrix.

| A | B | C | D | |
|---|---|---|---|---|
| 1 | 2 | 1 | 0 | A |
| 2 | 0 | 0 | 1 | B |
| 1 | 0 | 1 | 1 | C |
| 0 | 1 | 1 | 0 | D |

The matrix is used to construct the following incorrect graph.




To make the graph correct,

**A.** an extra edge needs to be added between vertices A and B.

**B.** an edge needs to be added between vertices B and D.

**C.** an edge needs to be removed between vertices A and B.

**D.** a loop needs to be added on vertex D.

6. Construct a graph from the following adjacency matrices.

**a.**

| A | B | C | D | |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | A |
| 1 | 0 | 0 | 0 | B |
| 1 | 0 | 0 | 1 | C |
| 0 | 0 | 1 | 0 | D |

**b.**

| A | B | C | D | |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | A |
| 1 | 1 | 0 | 1 | B |
| 1 | 0 | 1 | 0 | C |
| 0 | 1 | 0 | 0 | D |

7. Construct a graph from the following adjacency matrices.

**a.**

| A | B | C | D | E | |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | A |
| 1 | 1 | 1 | 0 | 2 | B |
| 0 | 1 | 0 | 0 | 0 | C |
| 0 | 0 | 0 | 1 | 1 | D |
| 0 | 2 | 0 | 1 | 0 | E |

**b.**

| A | B | C | D | E | F | |
|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 1 | 0 | A |
| 2 | 0 | 0 | 0 | 1 | 0 | B |
| 1 | 0 | 1 | 2 | 0 | 0 | C |
| 2 | 0 | 2 | 0 | 1 | 0 | D |
| 1 | 1 | 0 | 1 | 1 | 0 | E |
| 0 | 0 | 0 | 0 | 0 | 0 | F |

**8.** The following adjacency matrix shows the railway connections between Melbourne CBD and seven regional towns in Victoria.

Construct a graph to show Victoria's regional train services.

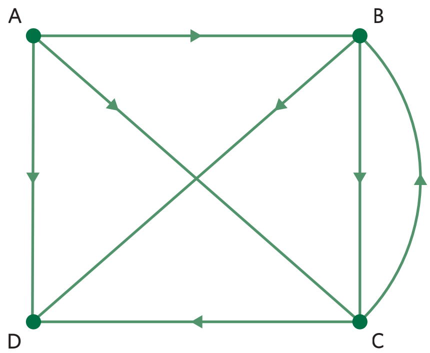
| | Ararat | Ballarat | Bendigo | Echuca | Geelong | Maryborough | Melbourne | Warrnambool |
|---|---|---|---|---|---|---|---|---|
| Ararat | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| Ballarat | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| Bendigo | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| Echuca | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Geelong | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| Maryborough | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Melbourne | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| Warrnambool | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

## Representing directed graphs

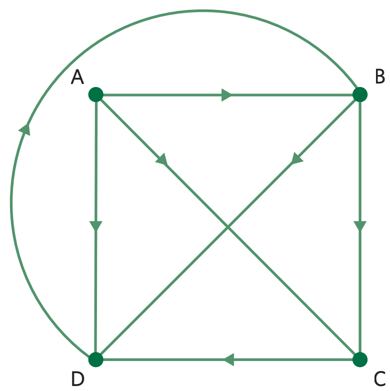
**9.** Which is the correct graph to represent the following adjacency matrix?

| from \ to | A | B | C | D |
|---|---|---|---|---|
| A | 0 | 1 | 1 | 1 |
| B | 0 | 0 | 1 | 1 |
| C | 0 | 1 | 0 | 1 |
| D | 0 | 1 | 0 | 0 |

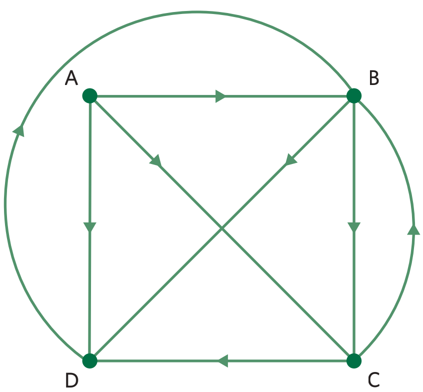
**A.**



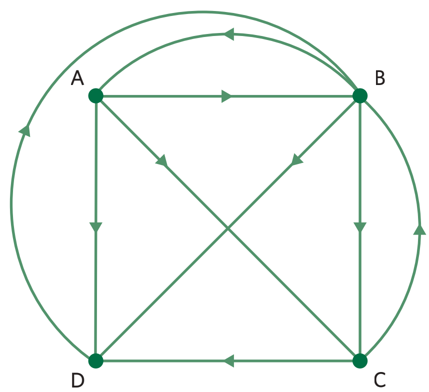

**B.**




**C.**

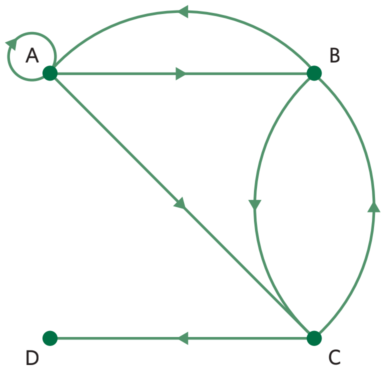



**D.**

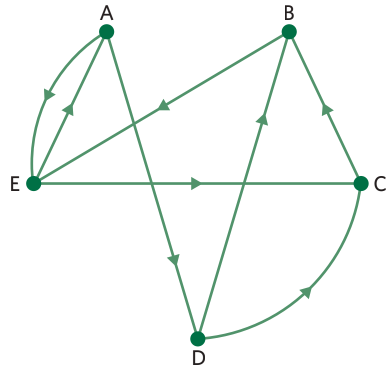


**10.** Construct an adjacency matrix to represent each of the following directed graphs.

**a.**




**b.**




**11.** Use the following adjacency matrices to construct directed graphs.

**a.**

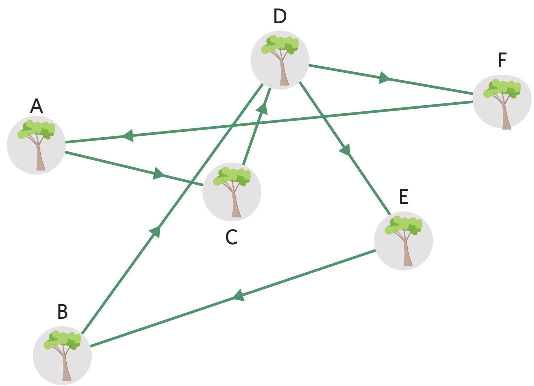
$$\text{from}\ \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \overset{\text{to}}{\begin{array}{cccc} A & B & C & D \\ \hline 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}}$$

**b.**

$$\text{from}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \overset{\text{to}}{\begin{array}{ccccc} A & B & C & D & E \\ \hline 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{array}}$$

**12.** Kayla decided to take her dogs for a walk in the local park. She starts and ends at tree A. The following graph shows the route she took through the park.

Kayla's dogs really enjoyed the walk so she wants to record the route in an adjacency matrix to remember it for the future. Construct an adjacency matrix to show Kayla's walk.




## Joining it all together

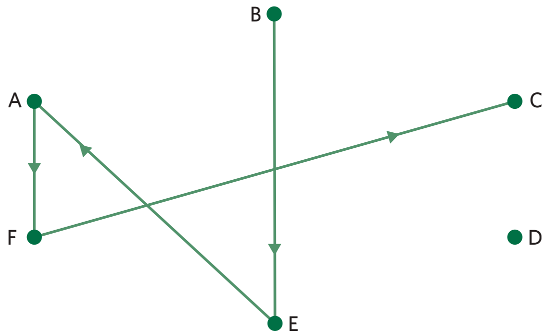
**13.** A museum has several exhibits on display. The exhibits are connected by various different pathways. These pathways are summarised in the following adjacency matrix.

$$\begin{array}{c|cccccc|} & A & B & C & D & E & F \\ \hline A & 0 & 1 & 1 & 0 & 0 & 1 \\ B & 1 & 0 & 0 & 1 & 1 & 0 \\ C & 1 & 0 & 0 & 0 & 1 & 1 \\ D & 0 & 1 & 0 & 0 & 0 & 1 \\ E & 0 & 1 & 1 & 0 & 0 & 0 \\ F & 1 & 0 & 1 & 1 & 0 & 0 \end{array}$$

**a.** Construct a graph showing the pathways between the exhibits.

**b.** The museum is undergoing renovations and, as a result, the pathway between exhibit A and exhibit B is no longer accessible. They decided to establish a new pathway between exhibits A and E. Construct a new adjacency matrix showing the museum's pathways.

**c.** Sarah is visiting the museum. She only has time to see five of the exhibits. The graph shows the exhibits she decides to see and the route she takes.




Show this information in an adjacency matrix.

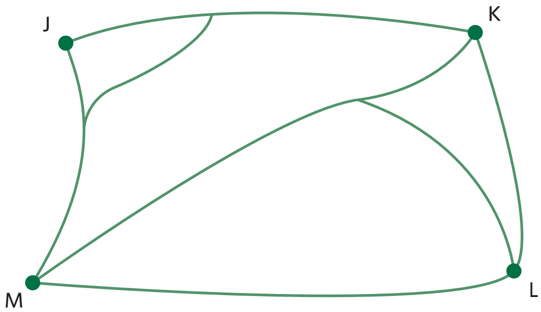
## Exam practice

**14.** The adjacency matrix shows the number of pathway connections between four landmarks: J, K, L and M.

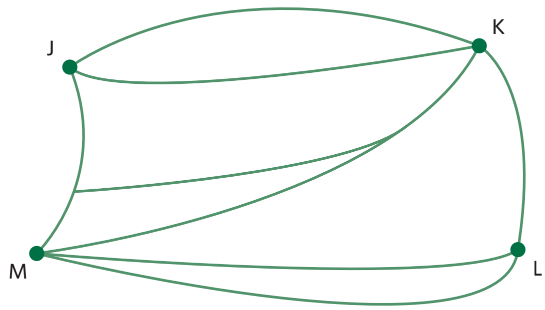
$$\begin{array}{c} \begin{matrix} J & K & L & M \end{matrix} \\ \begin{bmatrix} 1 & 3 & 0 & 2 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix} \end{array} \begin{matrix} J \\ K \\ L \\ M \end{matrix}$$

A network of pathways that could be represented by the adjacency matrix is

**A.**

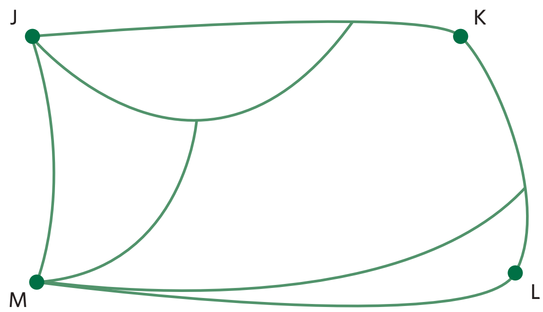



**B.**

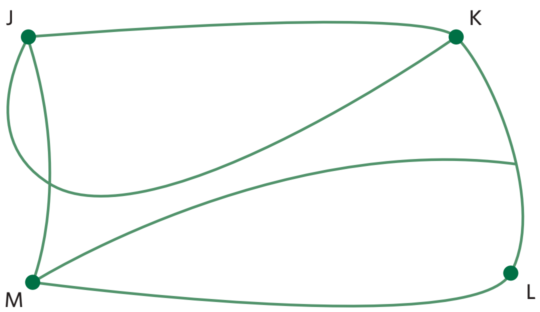



**C.**


**D.**



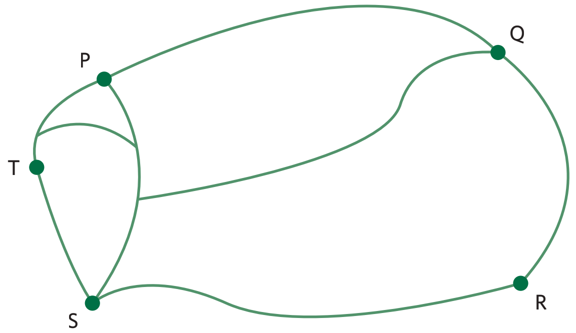

**E.**




*VCAA 2020 Exam 1 Networks and decision mathematics Q8*

**75%** of students answered this question correctly.

**15.** The map shows all the road connections between five towns, P, Q, R, S and T.




The road connections could be represented by the adjacency matrix

**A.**
$$\begin{array}{c} \begin{matrix} P & Q & R & S & T \end{matrix} \\ \begin{bmatrix} 1 & 3 & 0 & 2 & 2 \\ 3 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} P \\ Q \\ R \\ S \\ T \end{matrix} \end{array}$$

**B.**
$$\begin{array}{c} \begin{matrix} P & Q & R & S & T \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} P \\ Q \\ R \\ S \\ T \end{matrix} \end{array}$$

**C.**
$$\begin{array}{c} \begin{matrix} P & Q & R & S & T \end{matrix} \\ \begin{bmatrix} 0 & 3 & 0 & 2 & 2 \\ 3 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} P \\ Q \\ R \\ S \\ T \end{matrix} \end{array}$$

**D.**
$$\begin{array}{c} \begin{matrix} P & Q & R & S & T \end{matrix} \\ \begin{bmatrix} 0 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} P \\ Q \\ R \\ S \\ T \end{matrix} \end{array}$$

**E.**
$$\begin{array}{c} \begin{matrix} P & Q & R & S & T \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} P \\ Q \\ R \\ S \\ T \end{matrix} \end{array}$$

*VCAA 2019 Exam 1 Networks and decision mathematics Q6*

**39%** of students answered this question correctly.

**16.** The network shows the pathways between five buildings: J, K, L, M and N.

An adjacency matrix for this network is formed.

The number of zeros in this matrix is

**A.** 8

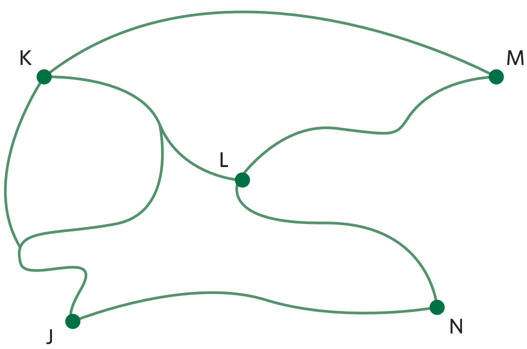
**B.** 9

**C.** 10

**D.** 11

**E.** 12

*VCAA 2021 Exam 1 Networks and decision mathematics Q7*




**33%** of students answered this question correctly.

## Questions from multiple lessons

### Data analysis

**17.** The seasonal index for a shop's sales of umbrellas in March is 0.8.

To correct the March sales of umbrellas for seasonality, March's actual number of sales should be

**A.** increased by 25%

**B.** increased by 80%

**C.** reduced by 20%

**D.** reduced by 25%

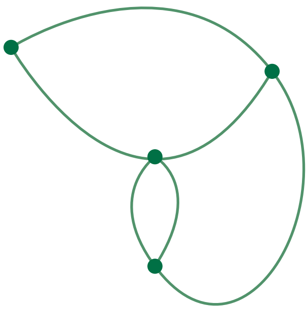
**E.** reduced by 80%

*Adapted from VCAA 2017 Exam 1 Data analysis Q16*

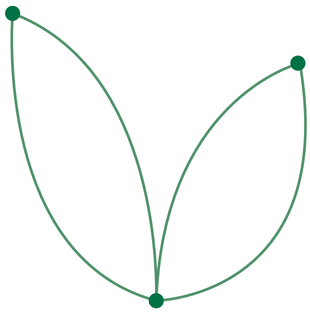
## Networks and decision mathematics

**18.** Which of the following graphs contains a loop?

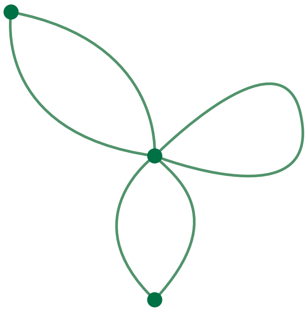
**A.**



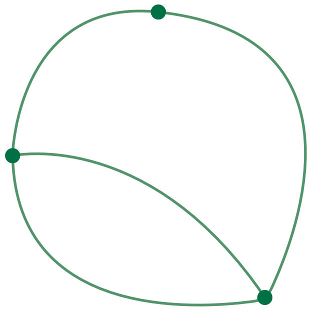
**B.**



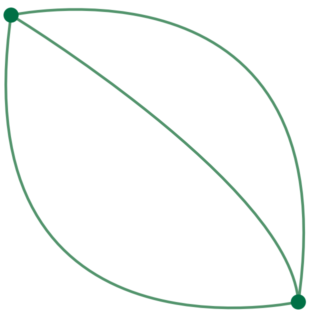
**C.**



**D.**



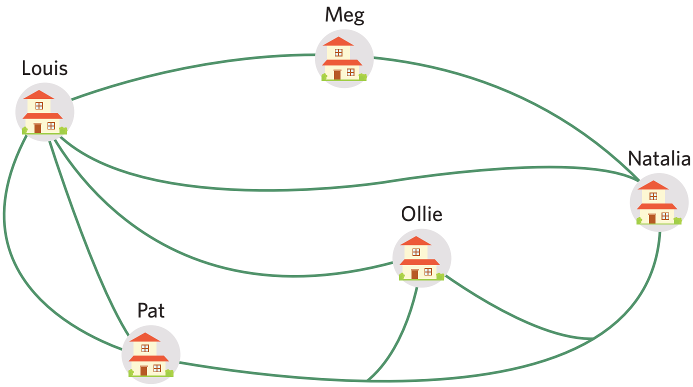
**E.**



*Adapted from VCAA 2017 Exam 1 Network and decision mathematics Q1*

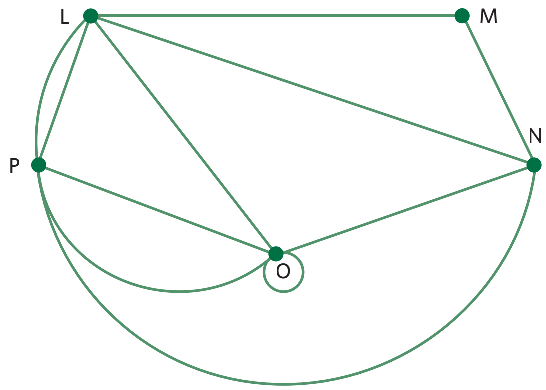
## Networks and decision mathematics *Year 11 content*

**19.** Shown is a map of the roads between five friends' houses; Louis (L), Meg (M), Natalia (N), Ollie (O), and Pat (P).




**a.** Meg is currently at her house. Which two friends can she visit, travelling along one road only? (1 MARK)

**b.** The map has been converted into the following graph. However, the graph is missing an edge.




**i.** Which two vertices is the missing edge between? (1 MARK)

**ii.** What does the loop at O represent in the context of travel from Ollie's house? (1 MARK)

*Adapted from VCAA 2016 Exam 2 Networks and decision mathematics Q1*

# 8C Exploring and travelling problems

## STUDY DESIGN DOT POINTS

- the concepts, conventions and notations of walks, trails, paths, cycles and circuits
- Eulerian trails and Eulerian circuits: the conditions for a graph to have an Eulerian trail or an Eulerian circuit, properties and applications
- Hamiltonian paths and cycles: properties and applications



## KEY SKILLS

During this lesson, you will be:

- identifying types of walks
- identifying Eulerian trails and circuits
- identifying Hamiltonian paths and cycles.

## KEY TERMS

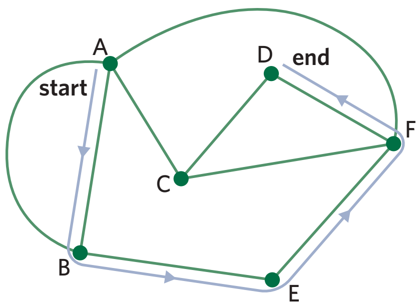
- Route
- Walk
- Trail
- Path
- Circuit
- Cycle
- Eulerian trail
- Eulerian circuit
- Hamiltonian path
- Hamiltonian cycle

Graphs can be used to represent physical locations similar to maps, with locations shown as vertices and roads/paths shown as edges. These can be used to model exploring and travelling problems, with different ways of travelling through graphs described with different terms.

## Identifying types of walks

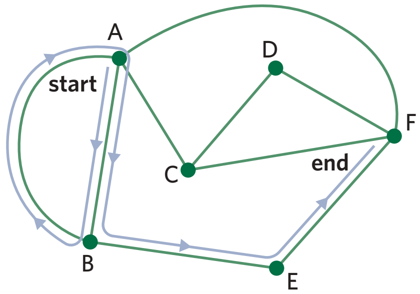
A **route** is a list of the vertices travelled through, in order, when moving from one vertex to another. It shows the pathway travelled in a graph.

For example, the route in the following graph can be written as A–B–E–F–D.




A **walk** is a continuous sequence of edges that passes through any number of vertices, in any order, starting and finishing at any vertex.

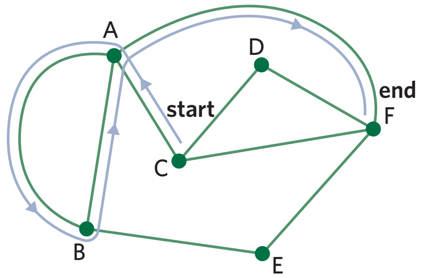
For example, the walk A–B–A–B–E–F is shown in the following graph.



There are different types of walks that exhibit varying properties. These include trails, paths, circuits and cycles.

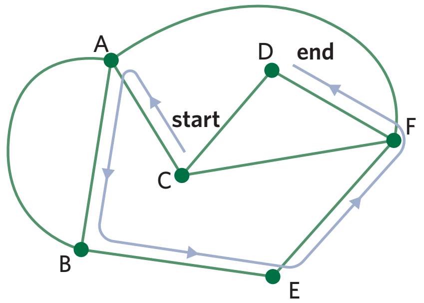
A **trail** is a walk in which no edges are repeated. However, they may pass through the same vertex multiple times.

For example, the walk C–A–B–A–F in the following graph is a trail since there are no repeated edges.



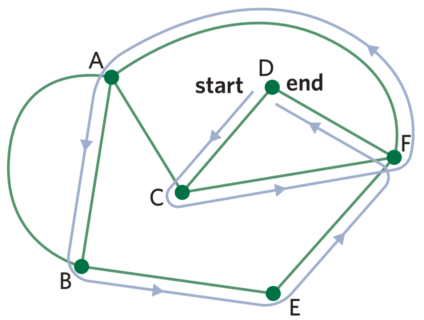

A **path** is a walk in which no edges or vertices are repeated.

For example, the walk C–A–B–E–F–D in the following graph is a path since there are no repeated edges or vertices.



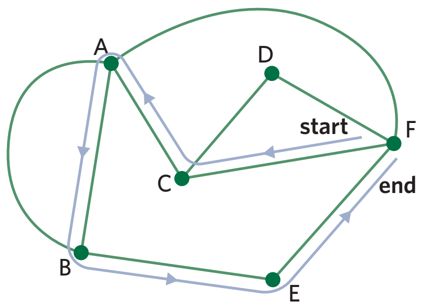

A **circuit** is a trail beginning and ending at the same vertex. Since it is a type of trail, no edges are repeated. Vertices may be repeated.

For example, the walk D–C–F–A–B–E–F–D in the following graph is a circuit because it has no repeated edges, and starts and ends at the same vertex.




A **cycle** is a path beginning and ending at the same vertex. Since it is a type of path, no edges or vertices are repeated, except the start and ending vertex.

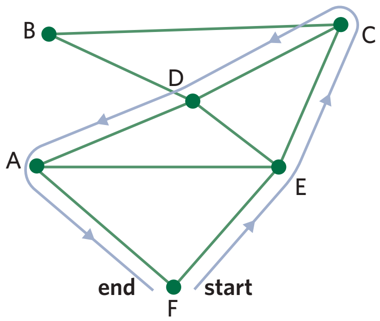
For example, the walk F–C–A–B–E–F in the following graph is a cycle because it has no repeated edges or vertices, and starts and ends at the same vertex.



## Worked example 1

Determine if the walk in each graph is a trail, path, circuit, cycle, or just a walk.

a.




### Explanation

**Step 1:** Determine whether the walk starts and ends at the same vertex.

The walk starts and ends at vertex F.

Since it starts and ends at the same vertex, consider whether it is a circuit or a cycle.

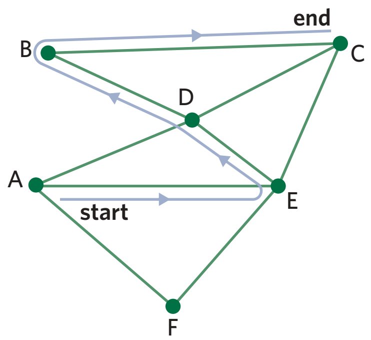
**Step 2:** Determine whether any edges or vertices are repeated.

There are no repeated edges and no repeated vertices, except the start and ending vertex.

### Answer

Cycle

b.




### Explanation

**Step 1:** Determine whether the walk starts and ends at the same vertex.

The walk starts at vertex A and ends at vertex C.

Since it starts and ends at different vertices, consider whether it is a trail or a path.

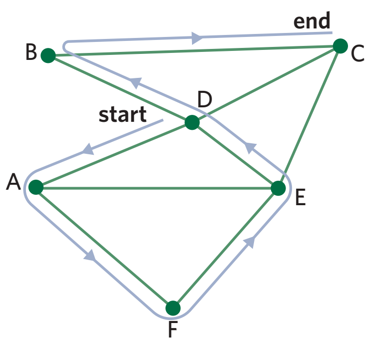
**Step 2:** Determine whether any edges or vertices are repeated.

There are no repeated edges or vertices.

### Answer

Path

c.




Continues →

### Explanation

**Step 1:** Determine whether the walk starts and ends at the same vertex.

The walk starts at vertex D and ends at vertex C.

Since it starts and ends at different vertices, consider whether it is a trail or a path.

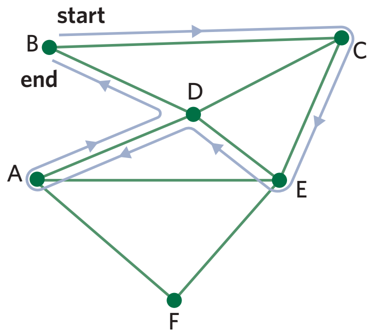
**Step 2:** Determine whether any edges or vertices are repeated.

There are no repeated edges. Vertex D is repeated.

### Answer

Trail

**d.**




### Explanation

**Step 1:** Determine whether the walk starts and ends at the same vertex.

The walk starts and ends at vertex B.

Since it starts and ends at the same vertex, consider whether it is a circuit or a cycle.

**Step 2:** Determine whether any edges or vertices are repeated.

The edge between vertex A and vertex D is repeated.

Vertex D is repeated.

### Answer

Walk

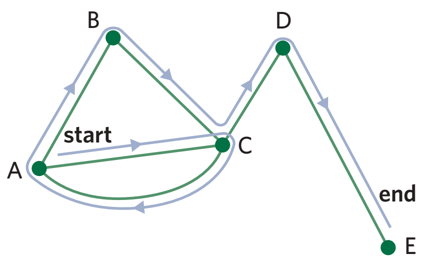
# Identifying Eulerian trails and circuits

An **Eulerian trail** is a walk that includes every edge in a graph exactly once. For an Eulerian trail to exist, the graph must:

- be connected
- have exactly two vertices of an odd degree, with all other vertices of an even degree

The Eulerian trail will always start and finish at vertices with an odd degree.

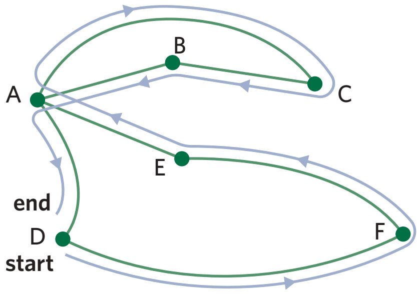
For example, an Eulerian trail A–C–A–B–C–D–E is possible in the following connected graph because vertices A and E have an odd degree and all other vertices have an even degree. The Eulerian trail will start and end at these two vertices since they have an odd degree.




An **Eulerian circuit** is an Eulerian trail that starts and ends at the same vertex. For an Eulerian circuit to exist, the graph must:

- be connected
- have all vertices of even degree.

For example, an Eulerian circuit D–F–E–A–C–B–A–D is possible in the following connected graph because all vertices have an even degree.

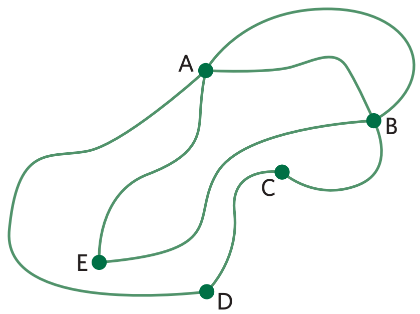



It is possible for multiple different Eulerian trails and circuits to exist in a graph.

## Worked example 2

The following graph represents a map of a walking track in the Dandenong Ranges.

The vertices represent lookout points.




**a.** Is an Eulerian trail possible in this graph?

### Explanation

**Step 1:** Check if the graph is connected.

All vertices in the graph are connected.

**Step 2:** Count the degrees of all of the vertices.

In order for an Eulerian trail to be possible, exactly two vertices must be of an odd degree.

Vertices A and B are of degree 4.

Vertices C, D and E are of degree 2.

All vertices are of an even degree.

### Answer

Not possible

**b.** Is an Eulerian circuit possible in this graph?

### Explanation

**Step 1:** Check if the graph is connected.

All vertices in the graph are connected.

**Step 2:** Count the degrees of all of the vertices.

In order for an Eulerian circuit to be possible, all vertices must be of an even degree.

Vertices A and B are of degree 4.

Vertices C, D and E are of degree 2.

All vertices are of an even degree.

### Answer

Possible

Continues →

**c.** Write down an Eulerian circuit starting from lookout B.

**Explanation**

**Step 1:** Find the starting vertex.

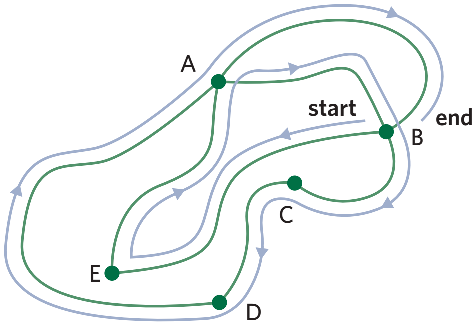
The starting vertex is lookout B.

**Step 2:** Follow the edges to create an Eulerian circuit.

Starting from lookout B, follow the edges until all have been included exactly once.

Vertices can be repeated.

The walk must end at lookout B.




**Answer**

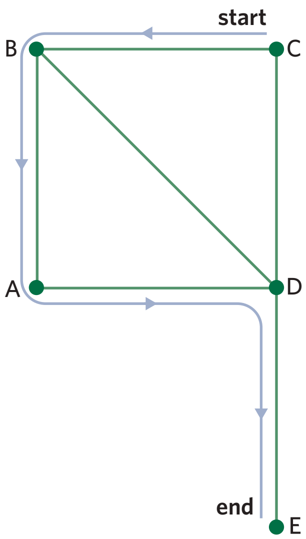
B–E–A–B–C–D–A–B

Note: There are other possible Eulerian circuits.

## Identifying Hamiltonian paths and cycles

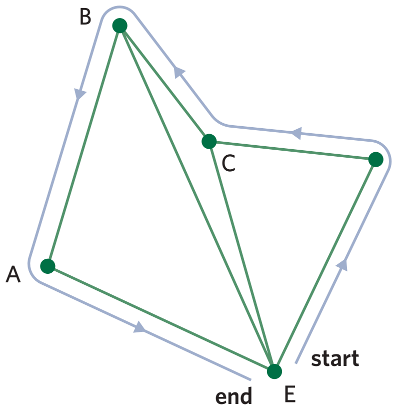
A **Hamiltonian path** is a walk that includes every vertex in a graph exactly once, with no repeated edges. Although Hamiltonian paths do not repeat edges, they do not necessarily have to include every edge.

For example, the walk C–B–A–D–E in the following graph is a Hamiltonian path since it includes all vertices and does not repeat any edges.




A **Hamiltonian cycle** is a Hamiltonian path that starts and ends at the same vertex. As with a Hamiltonian path, a Hamiltonian cycle does not repeat edges but does not have to include every edge.

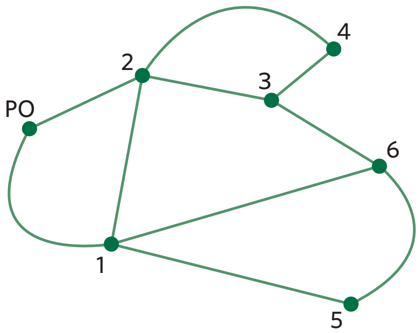
For example, the walk E–D–C–B–A–E in the following graph is a Hamiltonian cycle since it includes all vertices, without repeating edges, and starts and ends at vertex E.




It is possible for multiple Hamiltonian paths and cycles to exist in a graph.

## Worked example 3

Max the mailman is doing his morning delivery run and has to deliver packages to six houses. These houses and the post office where the packages are stored are shown in the graph.




**a.** Write down a Hamiltonian path that allows Max to deliver all of his packages.

### Explanation

**Step 1:** Determine the starting vertex.

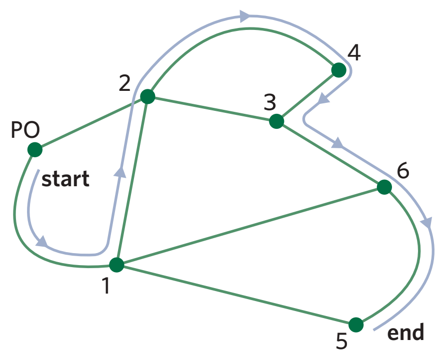
Max will start at the post office because that is where he must pick up his packages.

**Step 2:** Follow the edges from the starting vertex to create a Hamiltonian path.

Max can travel from the post office to either house 1 or house 2.

He must then travel to every other house exactly once.

He can end at any house.




### Answer

PO–1–2–4–3–6–5

Note: There are other possible Hamiltonian paths.

Continues →

**b.** Write down a Hamiltonian cycle that allows Max to deliver all of his packages before returning to the post office.

## Explanation

**Step 1:** Determine the starting vertex.

Max must start at the post office because that is where he must pick up his packages.

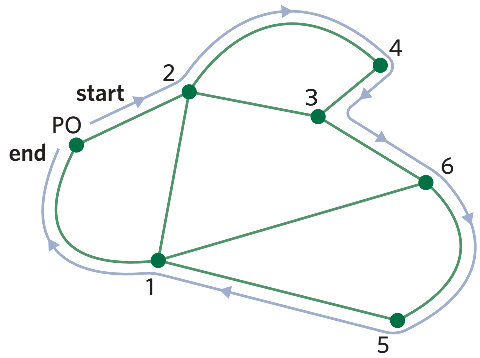
As this is a Hamiltonian cycle, the post office is also the ending vertex.

**Step 2:** Follow the edges from the starting vertex to create a Hamiltonian cycle.

Max can travel from the post office to either house 1 or house 2.

He must then travel to every other house exactly once.

He must end at the post office.




## Answer

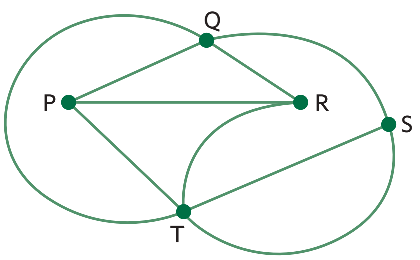
PO–2–4–3–6–5–1–PO

Note: There are other possible Hamiltonian cycles.

## Exam question breakdown

*VCAA 2018 Exam 1 Networks and decision mathematics Q4*

Consider the graph.




Which one of the following is **not** a path for this graph?

**A.** P–R–Q–T–S

**B.** P–Q–R–T–S

**C.** P–R–T–S–Q

**D.** P–T–Q–S–R

**E.** P–T–R–Q–S

## Explanation

Determine whether any edges or vertices are repeated in each option.

A: P–R–Q–T–S has no repeated edges or vertices. It is a path. ✗

B: P–Q–R–T–S has no repeated edges or vertices. It is a path. ✗

C: P–R–T–S–Q has no repeated edges or vertices. It is a path. ✗

D: P–T–Q–S–R is not possible unless a vertex is repeated since S and R are not connected. It is not a path. ✓

E: P–T–R–Q–S has no repeated edges or vertices. It is a path. ✗

88% of students answered this question correctly.

7% of students incorrectly chose option A. Students needed to check each route and ensure it had no repeated edges or vertices in order for it to meet the definition of a path.

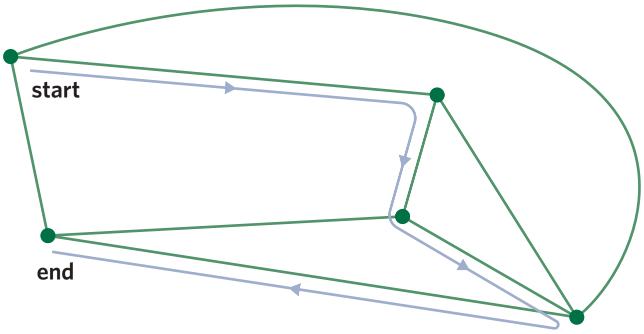
## Answer

C

# 8C Questions

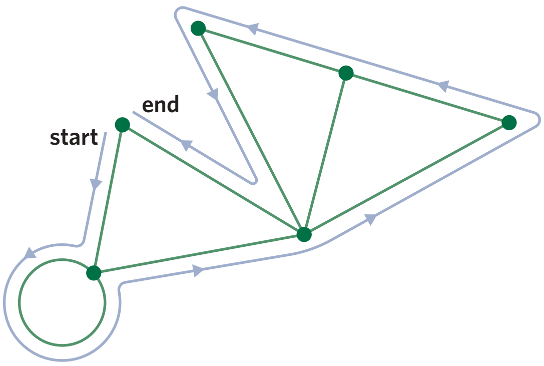
## Identifying types of walks

1. What type of walk is shown in the following graph?
   - **A.** Trail
   - **B.** Path
   - **C.** Circuit
   - **D.** Cycle

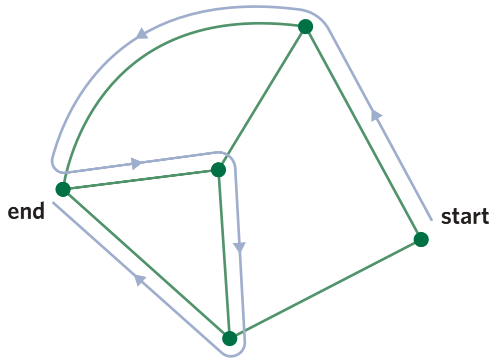



2. Determine if the walk in each of the following graphs is a trail, path, circuit, cycle, or just a walk.

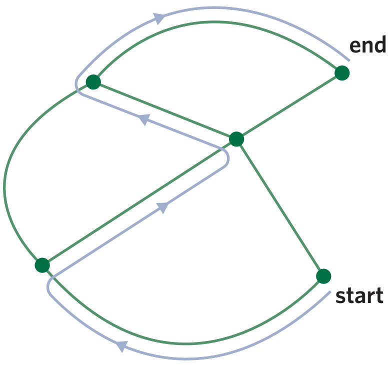
**a.**



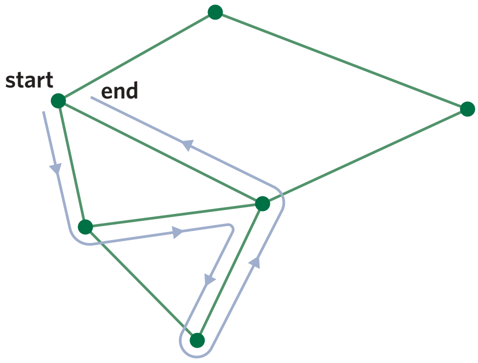

**b.**



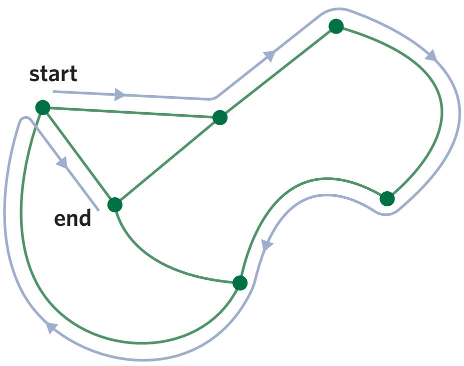

**c.**



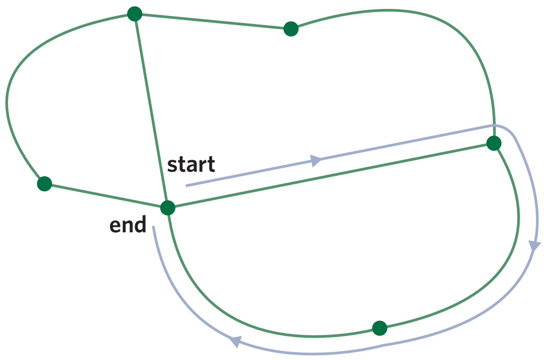

**d.**




**e.**



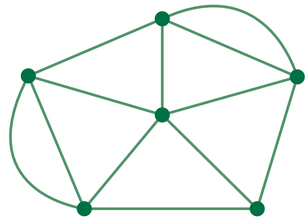

**f.**




3. Chelsea is going for a walk around her neighbourhood. She wants to make sure she starts and ends at her house and does not walk along any of the roads more than once. She likes her walks to be interesting, so she also does not want to visit any of the intersections more than once. What type of walk is Chelsea hoping to take?

## Identifying Eulerian trails and circuits

**4.** How many edges need to be removed from the following graph for an Eulerian circuit to be possible?



**A.** 0

**B.** 1

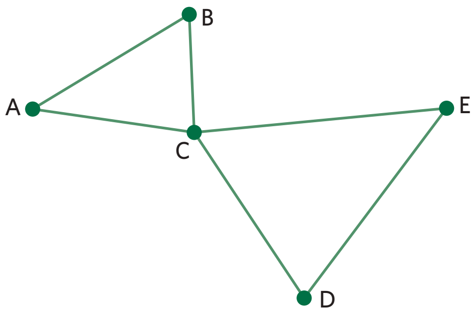
**C.** 2

**D.** 3

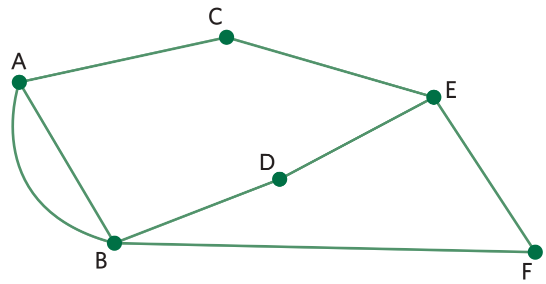
**5.** For each of the following graphs:

- Identify whether an Eulerian trail or Eulerian circuit is possible.
- If possible, write down an Eulerian trail or Eulerian circuit starting from vertex A.

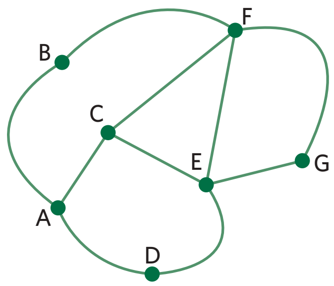
**a.**



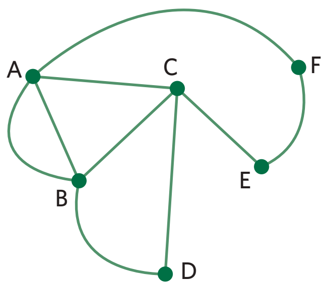

**b.**




**c.**



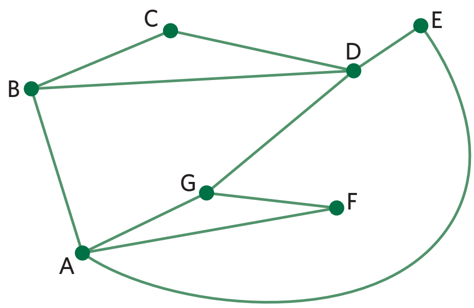

**d.**




**6.** Consider the following graph.

**a.** Add one edge to the graph to make an Eulerian circuit possible.

**b.** Write down a possible Eulerian circuit.

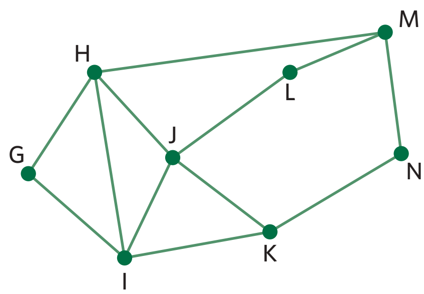



**7.** Jesse is skating at the local skatepark. The skatepark has eight ramps which are represented as vertices on the following graph.

**a.** Jesse is currently at ramp K and decides to follow an Eulerian trail. At which ramp will Jesse finish?

**b.** Write down a possible Eulerian trail that Jesse could take, starting from ramp K.

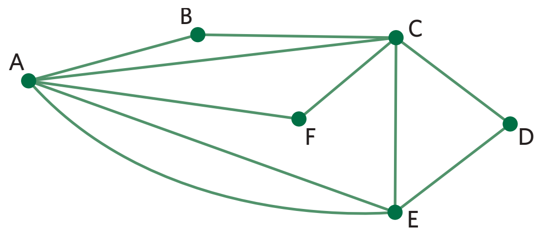
**c.** If Jesse wanted to start and end at the same ramp, but still skate along every path, which two ramps would need to be connected?



## Identifying Hamiltonian paths and cycles



**8.** Consider the following graph.




Which of the following is **not** a Hamiltonian path?

**A.** F–A–B–C–E–D

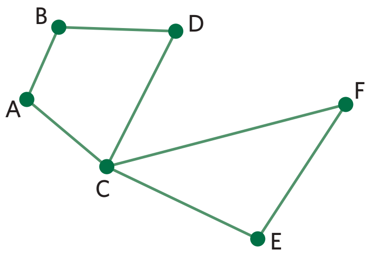
**B.** F–C–B–A–E–D

**C.** D–C–E–A–F–B

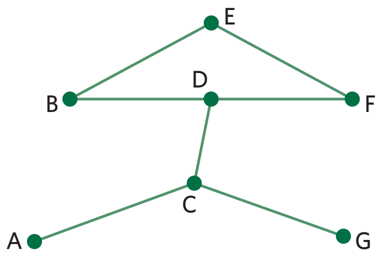
**D.** B–C–F–A–E–D

**9.** Determine if a Hamiltonian path is possible in each of the following graphs. If so, write one down.

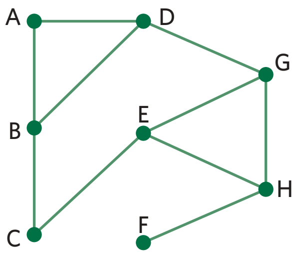
**a.**




**b.**

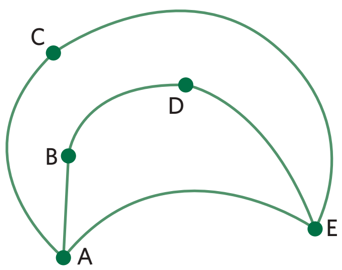



**c.**

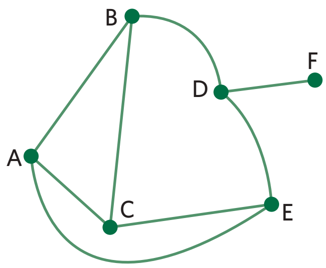



**10.** Determine if a Hamiltonian cycle is possible in each of the following graphs. If so, write one down.

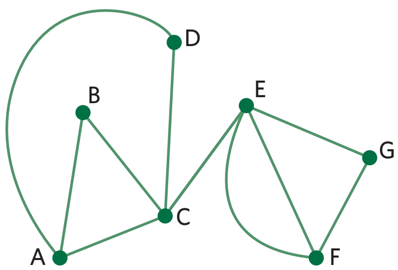
**a.**



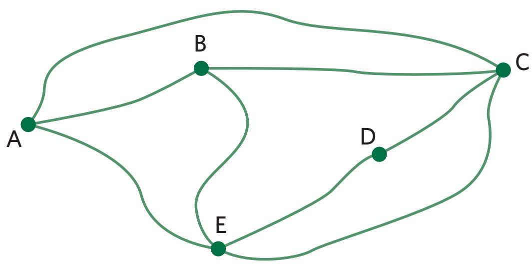

**b.**




**c.**




**11.** Five friends each live in different towns. The following graph shows the roads that link the five towns.



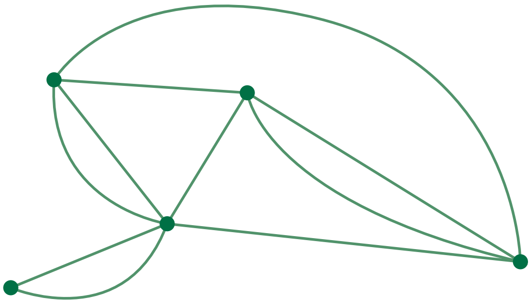

**a.** Natasha lives in town A and wants to visit each of her friends without travelling along the same road more than once. If she returns home after visiting her friends, what is the name of the walk she would have taken?

**b.** Write down a possible Hamiltonian path that Natasha could take in order to visit each of her friends.

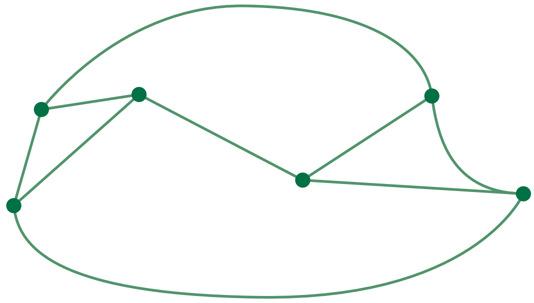
## Joining it all together

**12.** Consider the following five graphs.

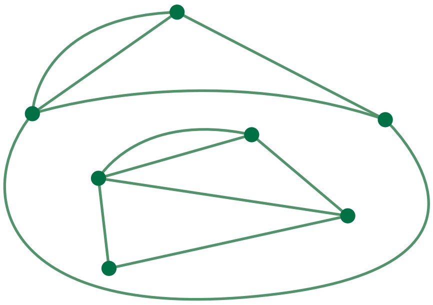
**I.**



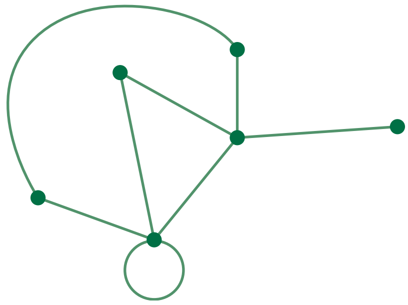
**II.**



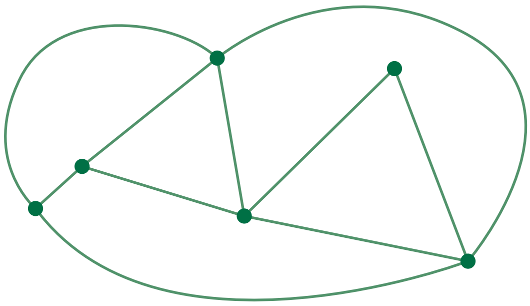
**III.**



**IV.**



**V.**



Which of the graphs

**a.** does not contain a Hamiltonian path?

**b.** contain a Hamiltonian cycle?

**c.** contain an Eulerian trail?

**d.** contain an Eulerian circuit?

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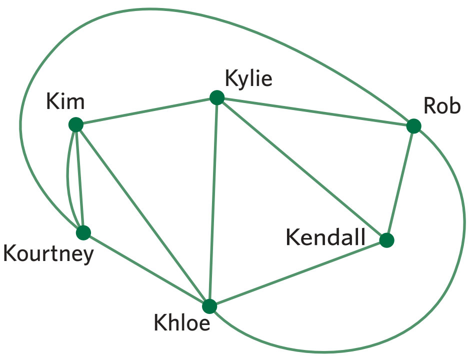
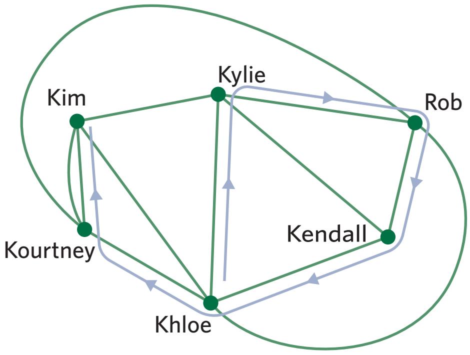
**13.** The following graph displays the location of the Los Angeles houses of the six Kardashian siblings. The edges represent roads and the vertices represent houses where each of the siblings live.



**a.** Khloe is bored on a Sunday afternoon and decides to visit each of her siblings at their houses. She takes the route shown.




What type of walk did she take?

**b.** Kendall is back in Los Angeles after a long time in Europe and wants to drive around the neighbourhood. She decides to drive along every road once, starting from her house.

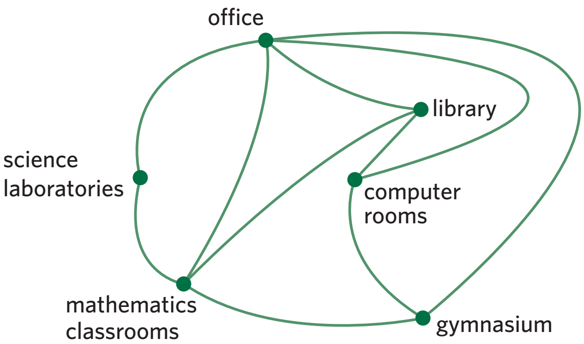
Which sibling's house will she end up at?

**c.** A new road is being built which will add an extra edge connecting Rob and Khloe's houses. Starting from her house, Kendall still wants to drive along every road once. Based on the new graph, which house will she now finish her trip at?

**d.** Kim decides she also wants to visit everyone. However, she decides to travel more efficiently than Kendall and does not want to travel along every road. She plans to visit each person once, starting and ending at her house. What type of walk is Kim planning on taking? Write down a possible route she could take.

## Exam practice

**14.** Fencedale High School has six buildings. The network shows these buildings represented by vertices. The edges represent the paths between the buildings.




A school tour is to start and finish at the office, visiting each building only once.

What is the mathematical term for this route? (1 MARK)

*VCAA 2019 Exam 2 Networks and decision mathematics Q1bi*

**70%** of students answered this question correctly.

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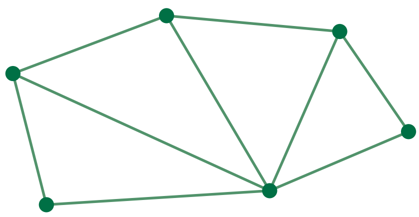
**15.** Consider the following graph.



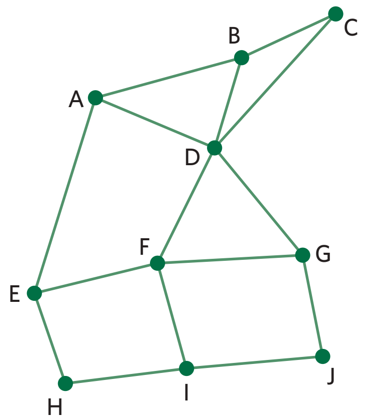
The minimum number of extra edges that are required so that an Eulerian circuit is possible in this graph is

**A.** 0
**B.** 1
**C.** 2
**D.** 3
**E.** 4

*VCAA 2019 Exam 1 Networks and decision mathematics Q2*

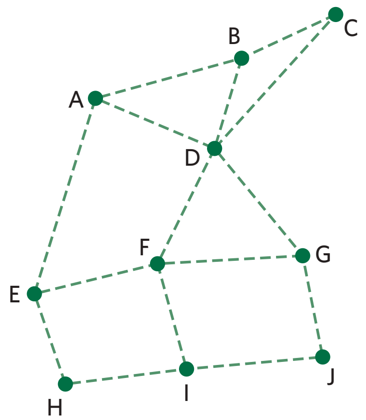
**64%** of students answered this question correctly.

**16.** In one area of the town of Zenith, a postal worker delivers mail to 10 houses labelled as vertices A to J on the following graph.




The postal worker has delivered the mail at F and will continue her deliveries by following a Hamiltonian path from F.

Draw in a possible Hamiltonian path for the postal worker on the following diagram. (1 MARK)




*VCAA 2018 Exam 2 Networks and decision mathematics Q2c*

**62%** of students answered this question correctly.

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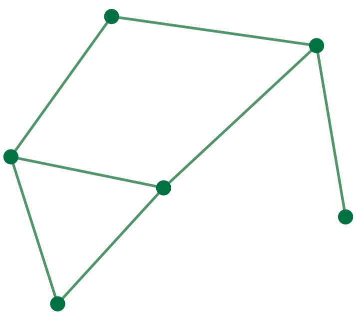
**17.**



Consider the following five statements about the graph:

- The graph is planar.
- The graph contains a cycle.
- The graph contains a bridge.
- The graph contains an Eulerian trail.
- The graph contains a Hamiltonian path.

How many of these statements are true?

**A.** 1 **B.** 2 **C.** 3
**D.** 4 **E.** 5

*VCAA 2021 Exam 1 Networks and decision mathematics Q5*

**17%** of students answered this question correctly.

## Questions from multiple lessons

### Recursion and financial modelling

**18.** Which of the following recurrence relations could model the value of a perpetuity investment, $P_n$, after $n$ months?

**A.** $P_0 = 340\,000, \quad P_{n+1} = 1.0031 \times P_n - 1024$

**B.** $P_0 = 245\,000, \quad P_{n+1} = 1.0056 \times P_n - 1382$

**C.** $P_0 = 375\,000, \quad P_{n+1} = 1.0018 \times P_n - 670$

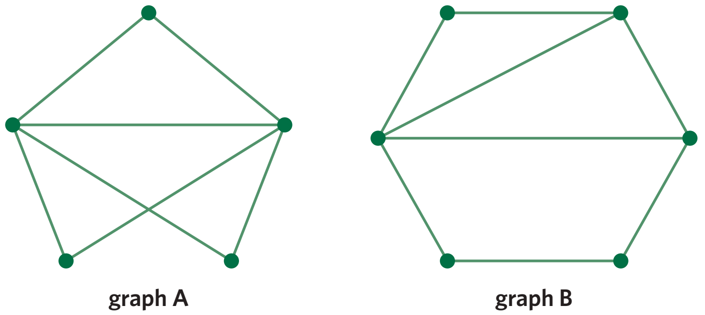
**D.** $P_0 = 310\,000, \quad P_{n+1} = 1.0046 \times P_n - 1426$

**E.** $P_0 = 285\,000, \quad P_{n+1} = 1.0064 \times P_n - 1844$

*Adapted from VCAA 2018 Exam 1 Recursion and financial modelling Q21*

### Network and decision mathematics

**19.** Consider graphs A and B shown.




The sum of the degrees of the vertices of graph B is

**A.** two less than the sum of the degrees of the vertices of graph A.

**B.** one less than the sum of the degrees of the vertices of graph A.

**C.** equal to the sum of the degrees of the vertices of graph A.

**D.** one more than the sum of the degrees of the vertices of graph A.

**E.** two more than the sum of the degrees of the vertices of graph A.

*Adapted from VCAA 2017 Exam 1 Networks and decision mathematics Q2*

### Recursion and financial modelling

**20.** Chloe has sold her investment property and has received \$865 200 after taxes.

She can invest this money in two ways.

**Option 1**

Chloe could put the \$865 200 into a perpetuity investment. She would then receive \$3287.76 per month for the rest of her life.

**Option 2**

Chloe could invest the \$865 200 in an annuity, instead of a perpetuity.

The annuity earns interest at 4.55% per annum, compounded fortnightly.

The balance of Chloe's annuity at the end of the first year would be \$850 160.19.

**a.** What is the interest rate, per annum, of the **Option 1** investment? (1 MARK)

**b.** For the **Option 2** investment,

**i.** what fortnightly payment, to the nearest dollar, would Chloe receive? (1 MARK)

**ii.** how much interest would Chloe's annuity earn in the second year of the investment? Round to the nearest cent. (2 MARKS)

*Adapted from VCAA 2018 Exam 2 Recursion and financial modelling Q6*

# 8D Minimum connector problems

STUDY DESIGN DOT POINTS

- trees and spanning trees
- minimum spanning trees in a weighted connected graph and their determination by inspection or by Prim's algorithm
- use of minimal spanning trees to solve minimal connector problems

8A 8B 8C **8D** 8E 8F 8G 8H 8I 8J

KEY SKILLS

During this lesson, you will be:

- identifying and finding the weight of a spanning tree
- finding the minimum spanning tree.

KEY TERMS

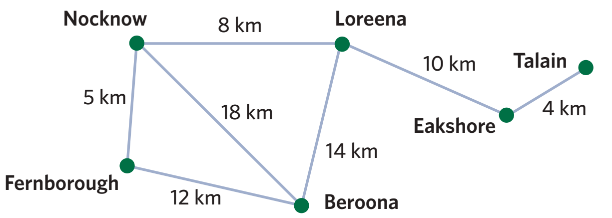
- Weighted graph
- Tree
- Spanning tree
- Minimum spanning tree
- Prim's algorithm

Graphs can provide additional information by including numeric values, known as weights, at each edge. A useful application of these weights is to minimise the total weight of a graph, while still keeping all vertices connected. For example, determining the minimum length of power lines required to connect different towns. This is known as a connector problem and can be solved using minimum spanning trees.

## Identifying and finding the weight of a spanning tree

A **weighted graph** is a graph that has numeric values attached to each edge, which provides further information about the connections between the vertices. These weights represent physical quantities such as time, distance or cost.

For example, in the following graph the weights represent the lengths of train tracks between different stations.

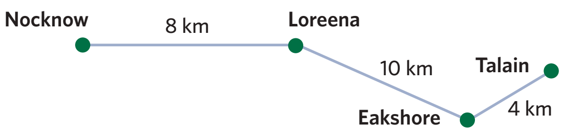



A **tree** is a type of graph that has no loops, duplicate edges or cycles. That is, it uses the least number of edges to connect the vertices.

The number of edges in a tree is always one less than the number of vertices, $e = v - 1$.

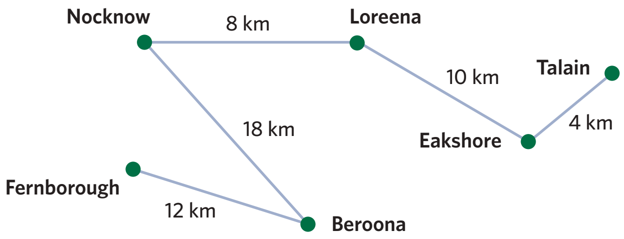
A tree can be a subgraph of a larger graph, so not all vertices in the larger graph need to be included.

For example, the following graph is a tree of the previous graph.



A **spanning tree** is a tree which connects all vertices in the original graph. There may be multiple spanning trees in a single graph.

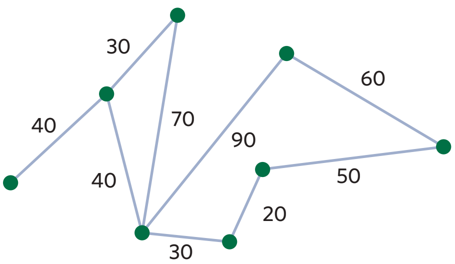
For example, the following graph is a spanning tree of the original graph.




The weight of a graph or tree is the sum of the weights of all its edges. For example, this spanning tree has a weight of 52 km.

## Worked example 1

Consider the following graph.




**a.** Construct a spanning tree for the graph

### Explanation

**Step 1:** Determine the number of vertices and edges required.

A spanning tree connects all vertices. Therefore the spanning tree will use all 8 vertices.

$e = v - 1$

$= 8 - 1$

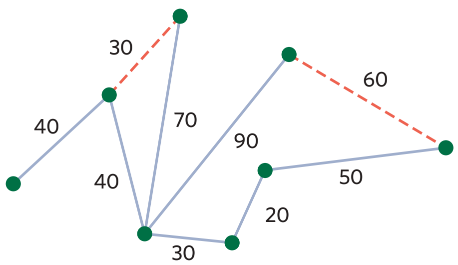
$= 7$

The spanning tree will have 7 edges.

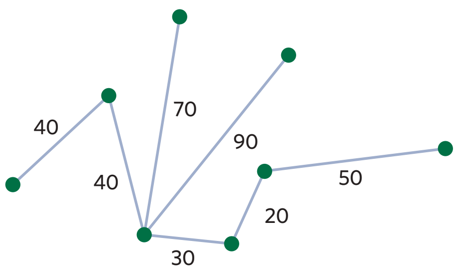
**Step 2:** Remove excess edges.

The original graph has 9 edges so 2 edges can be removed.

Ensure all vertices remain connected.




### Answer




Note: There are other possible spanning trees.

Continues →

**b.** Calculate the weight of the spanning tree.

**Explanation**

Sum the weights of the edges of the spanning tree constructed in part **a**.

*weight* = 40 + 40 + 70 + 90 + 30 + 20 + 50

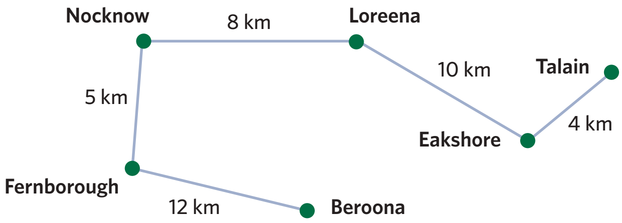
**Answer**

340

# Finding the minimum spanning tree

The **minimum spanning tree** of a graph is the spanning tree with the lowest total weight. A key application of minimum spanning trees involves using them to solve connector problems. These are problems that involve finding the minimum distance, cost, weight or time to connect all vertices in a graph.

For example, the following graph is the minimum spanning tree of the original graph.




The minimum spanning tree has a weight of 39 km.

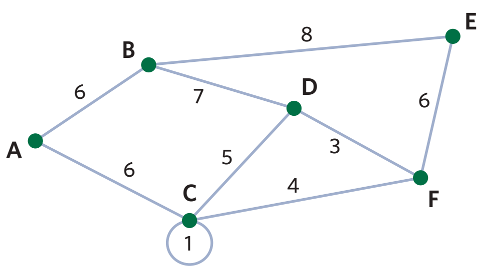
The minimum spanning tree of a graph can be found using a method known as **Prim's algorithm**, which has the following steps.

1. Select any vertex.
2. Select the edge with the lowest weight connected to that vertex.
3. There are now two connected vertices. Inspect all edges connected to either vertex and select the edge with the lowest weight that will connect a new vertex to the tree.
4. Ignore any edges which would create duplicate edges, cycles or loops.
5. Continue the process until all vertices are connected.

Note: If two edges have the same weight, it does not matter which is selected.

**Worked example 2**

For the following graph:




Continues →

**a.** Use Prim's algorithm to find the minimum spanning tree.

### Explanation

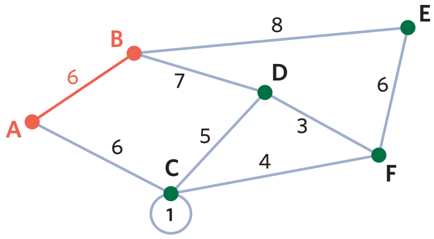
**Step 1:** Select any vertex.

For this example, start with vertex A.

**Step 2:** Select the edge with the lowest weight connected to that vertex.

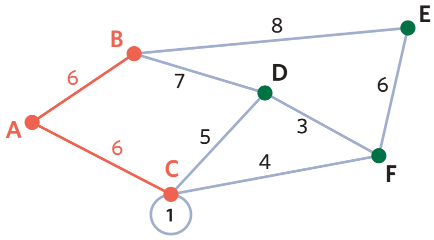
Both edges connected to vertex A are weighted 6 so it does not matter which is chosen.

For this example, choose the edge connected to vertex B.




**Step 3:** Inspect all edges connected to either of the two connected vertices. Select the edge with the lowest weight that will connect a new vertex to the tree.

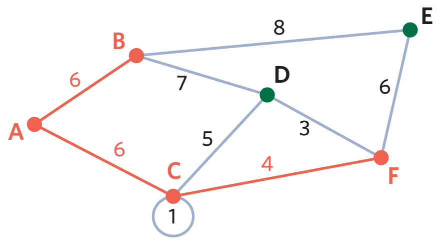
The edge connecting vertex A to C has the lowest weight.



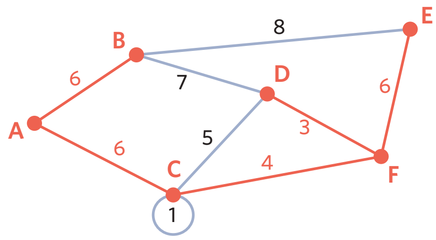

**Step 4:** Inspect all edges connected to the three connected vertices. Select the edge with the lowest weight that will connect a new vertex to the tree.

The loop at vertex C is the lowest weighted edge, however it should not be used as it does not connect a new vertex to the tree.

The edge connecting vertex C to F has the lowest weight.

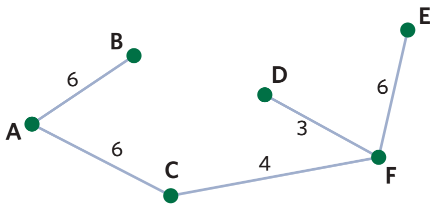



**Step 5:** Continue this process until all vertices are connected.




**Step 6:** Redraw the minimum spanning tree on its own.

### Answer




**b.** Calculate the weight of the minimum spanning tree.

### Explanation

Sum the weights of the edges of the minimum spanning tree.

$weight = 6 + 6 + 4 + 3 + 6$

### Answer

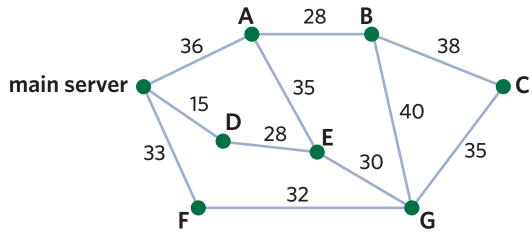
25

## Exam question breakdown

*VCAA 2019 Exam 1 Networks and decision mathematics Q5*

The following diagram shows the distances, in metres, along a series of cables connecting a main server to seven points, A to G, in a computer network.

The minimum length of cable, in metres, required to ensure that each of the seven points is connected to the main server directly or via another point is




**A.** 175 **B.** 203 **C.** 208 **D.** 221 **E.** 236

### Explanation

**Step 1:** Identify the connector problem.

The question requires the minimum length of cable needed to connect all points to be determined. This is the minimum spanning tree of the graph.

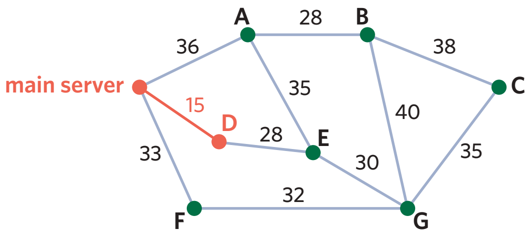
Prim's algorithm can be used to find the minimum spanning tree.

**Step 2:** Select any vertex.

Start with the main server.

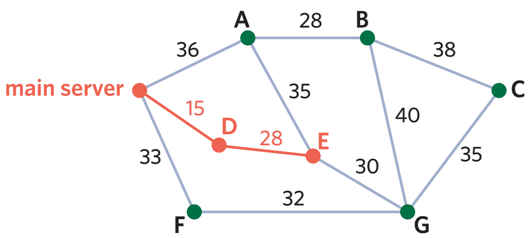
**Step 3:** Select the edge with the lowest weight connected to that vertex.

The edge connecting the main server to point D has the lowest weight.




**Step 4:** Inspect all edges connected to either of the two connected vertices. Select the edge with the lowest weight that will connect a new vertex to the tree.

The edge connecting vertex D to E has the lowest weight.

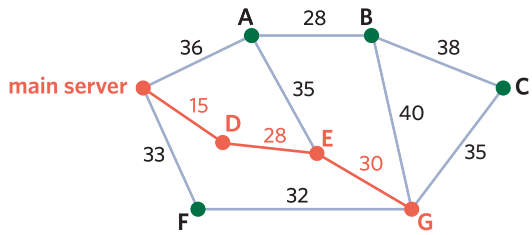



### Answer

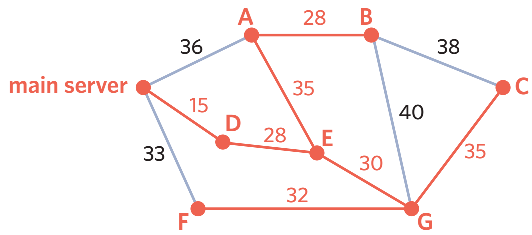
B

**Step 5:** Inspect all edges connected to the three connected vertices. Select the edge with the lowest weight that will connect a new vertex to the tree.

The edge connecting vertex E to G has the lowest weight.




**Step 6:** Continue this process until all vertices are connected.




**Step 7:** Calculate the weight of the minimum spanning tree.

Sum the weights of the edges.

$15 + 28 + 35 + 28 + 30 + 32 + 35 = 203$

**75%** of students answered this question correctly.

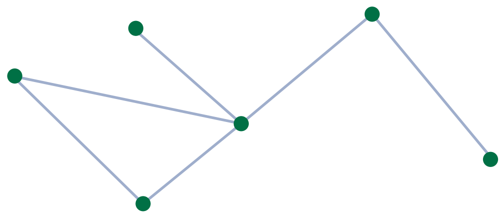
Students needed to be careful not to select any edges which would create duplicate edges, cycles, or loops. Edges needed to be selected that connected new vertices to the tree.

# 8D Questions

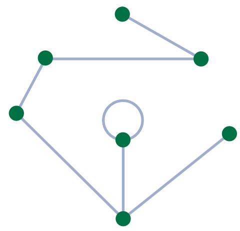
## Identifying and finding the weight of a spanning tree

**1.** Which of the following graphs is a tree?

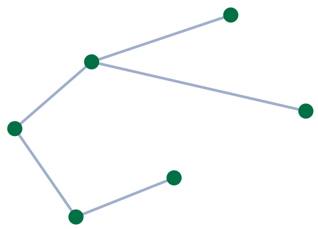
**A.**



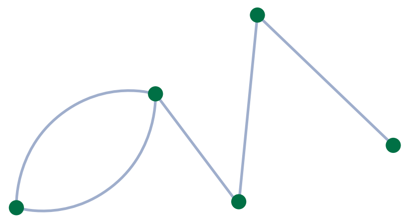
**B.**



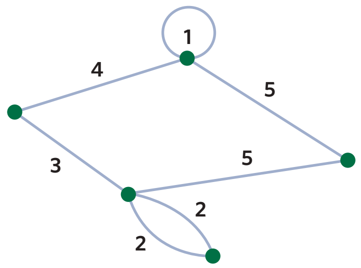
**C.**



**D.**



**2.** Consider the following weighted graph.

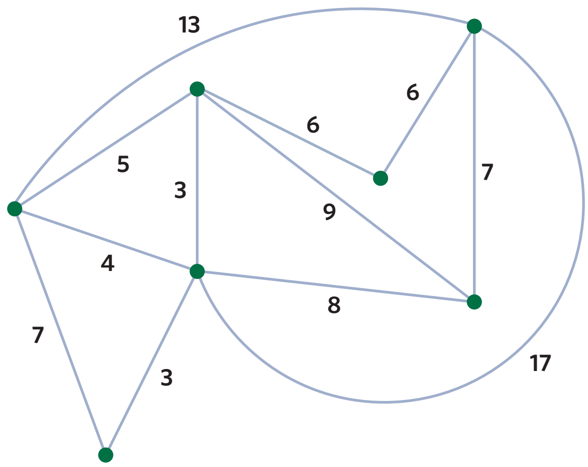



**a.** How many vertices and edges will a spanning tree of the weighted graph contain?

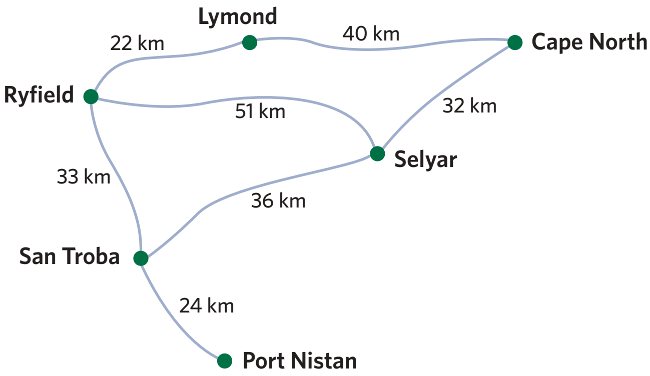
**b.** Find a possible spanning tree within the weighted graph.

**c.** Calculate the weight of the spanning tree found in part **a**.

**3.** How many edges need to be removed from the following graph in order to create a spanning tree?



**4.** The following graph shows the roads connecting six towns.

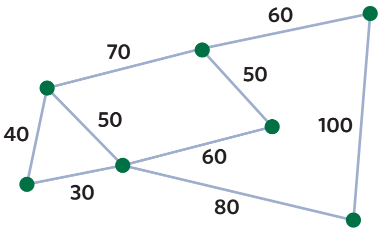



The roads require urgent maintenance and repairs. However, while roadworks are underway, all towns need to remain accessible.

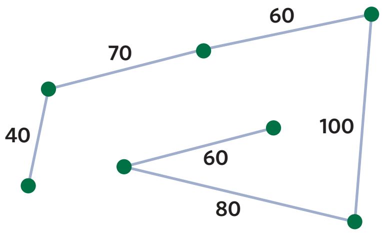
Find three possible spanning trees that ensure each town can still be accessed.

## Finding the minimum spanning tree

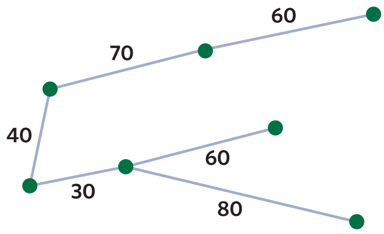
**5.** Which of the following graphs is the minimum spanning tree of the graph provided?



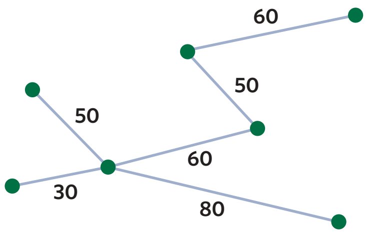

**A.**



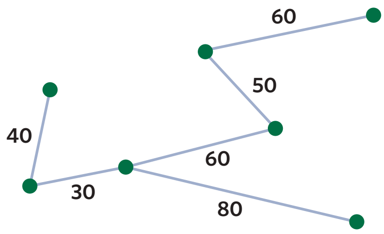

**B.**



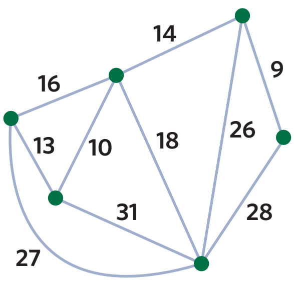

**C.**




**D.**

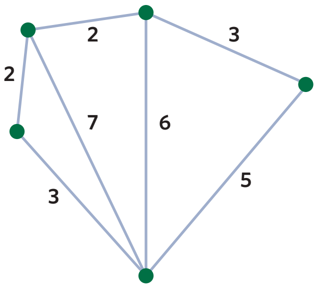



**6.** Draw the minimum spanning tree of the following weighted graph.

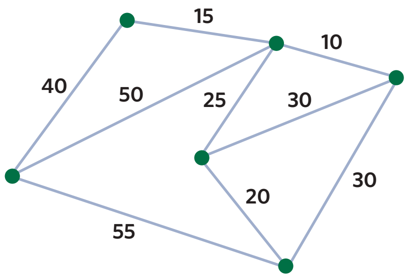


7. Calculate the weight of the minimum spanning tree for each of the following graphs.

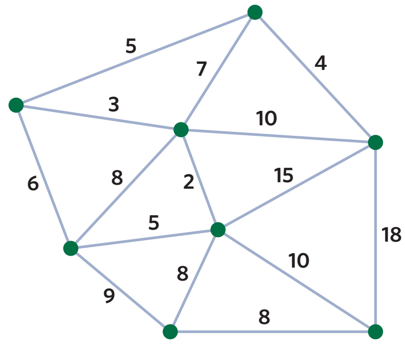
a.



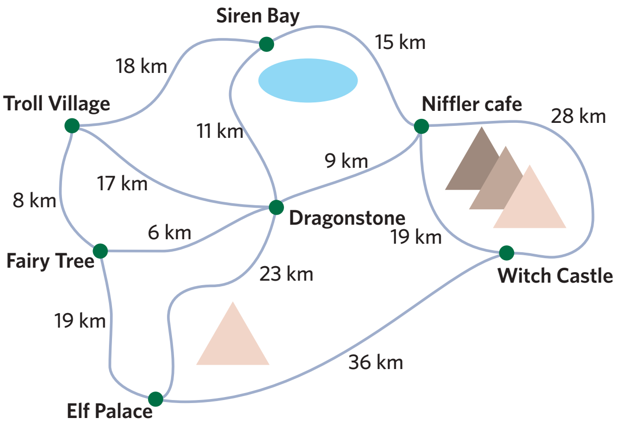

b.




c.




8. Since transportation dust has been discovered, Dragonride Services have been going out of business. In an effort to save some gold, they have decided to cut down on some of their flight paths.




a. Draw the flight map with the shortest possible flight distance that still accesses all of the landmarks.

b. What is the total distance of the flight map?

## Joining it all together

9. The total weight of the minimum spanning tree of the following graph is 150.

a. How many edges will be in the minimum spanning tree?

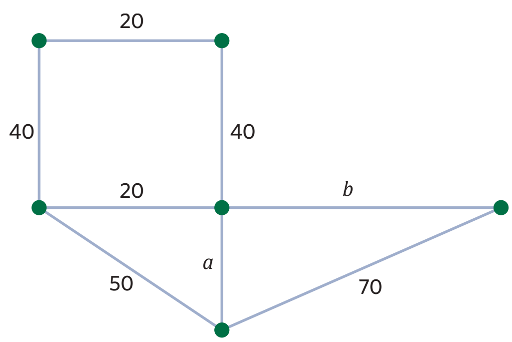
b. The values of $a$ and $b$ could be:

A. $a = 10, \quad b = 30$

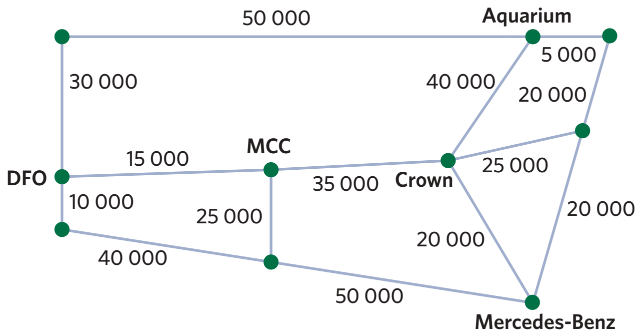
B. $a = 20, \quad b = 60$

C. $a = 10, \quad b = 60$

D. $a = 30, \quad b = 50$



**10.** Global leaders from every country are scheduled to meet at the Melbourne Convention Centre (MCC) for a conference on climate change. The government is worried about the security of the event and decides to block off as many roads surrounding the event as possible.




Only major landmarks (DFO, Mercedes-Benz, Crown, Aquarium, and the MCC) still need to be accessible.

The numbers represent the operational cost in dollars of keeping each road open.

**a.** Create an appropriate street map which will minimise the cost for the government.

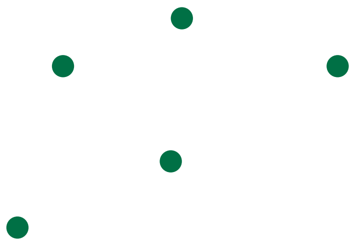
Due to security purposes, the roads connecting DFO to Mercedes-Benz ($10 000, $40 000 and $50 000) must stay open.

**b.** Draw a new appropriate street map which is still as cheap as possible, while including these streets.

**c.** Suppose the government decides that traffic congestion will increase by too much if only the landmarks are accessible. They decide that all points on the original map need to be accessible. What is the minimum cost required to keep the necessary roads open?

## Exam practice

**11.** Consider the following graph with five isolated vertices.



To form a tree, the minimum number of edges that must be added to the graph is

**A.** 1

**B.** 4

**C.** 5

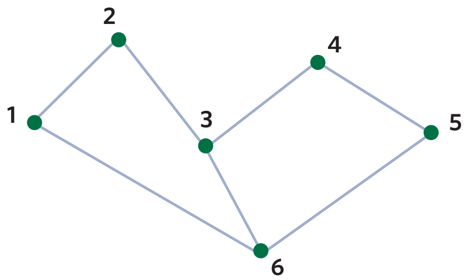
**D.** 6

**E.** 10

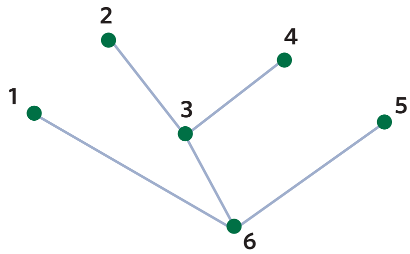
*VCAA 2018 Exam 1 Networks and decision mathematics Q1*

**92%** of students answered this question correctly.

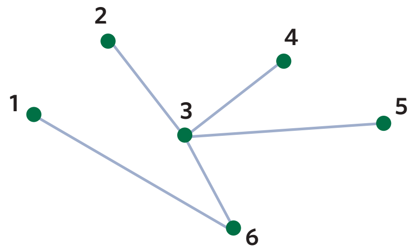
**12.** Which graph is **not** a spanning tree for the following network?



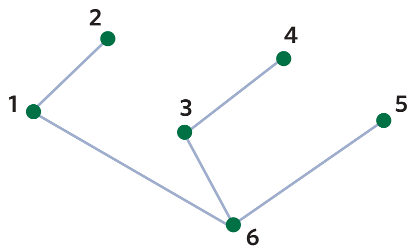

**A.**



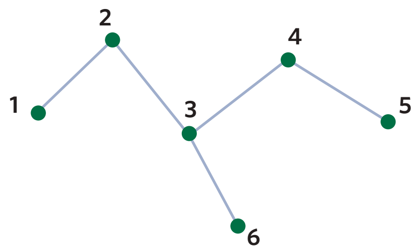

**B.**



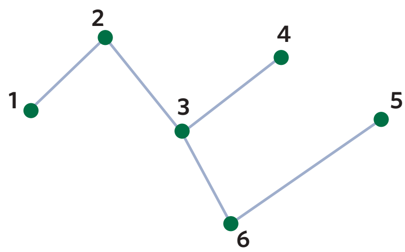

**C.**




**D.**




**E.**




*VCAA 2020 Exam 1 Networks and decision mathematics Q3*

**81%** of students answered this question correctly.

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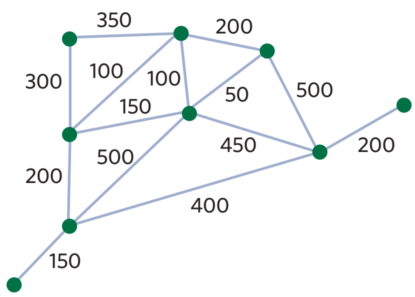
**13.** While on holiday, four friends visit a theme park where there are nine rides. On the following graph, the positions of the rides are indicated by the vertices.

The number on the edges represent the distances, in metres, between rides.

Electrical cables are required to power the rides.

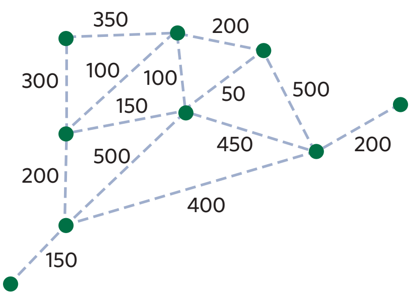
These cables will form a connected graph.

The shortest total length of cable will be used.




**a.** Give a mathematical term to describe a graph that represents these cables. (1 MARK)

**b.** Draw the graph that represents these cables using the following diagram (1 MARK)




*VCAA 2017 Exam 2 Networks and decision mathematics Q3ai,ii*

Part **a**: **63%** of students answered this question correctly.

Part **b**: **53%** of students answered this question correctly.

# Questions from multiple lessons

## Data analysis *Year 11 content*

**14.** The following table shows the average maximum daily *temperature* (°C) in Melbourne for each month in the first half of the year.

| | Jan | Feb | Mar | Apr | May | Jun |
|---|---|---|---|---|---|---|
| ***temperature* (°C)** | 26 | 27 | 24 | 20 | 17 | 15 |

Using four-mean smoothing with centring, what is the smoothed value for April?

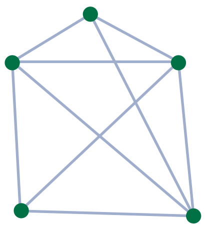
**A.** 19 °C **B.** 20.5 °C **C.** 22 °C **D.** 23.5 °C **E.** 24.25 °C

*Adapted from VCAA 2018 Exam 1 Data analysis Q15*

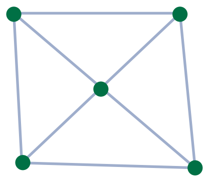
## Networks and decision mathematics *Year 11 content*

**15.** Which one of the following graphs is **not** a planar graph?

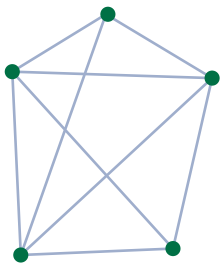
**A.**



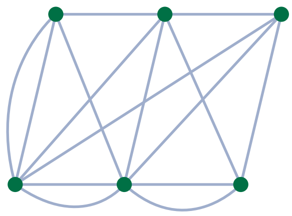
**B.**



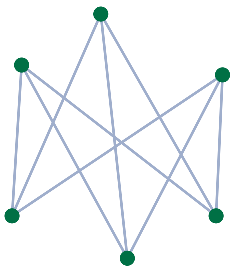
**C.**



**D.**



**E.**



*Adapted from VCAA 2018 Exam 1 Networks and decision mathematics Q6*

## Data analysis *Year 11 content*

**16.** The equation of the least squares regression line relating *pond size* ($m^2$) and *number of fish* is:

*number of fish* = 2 + 0.85 × *pond size*

**a.** Interpret the slope of the regression line in terms of the variables *pond size* and *number of fish*. (1 MARK)

**b.** There are 23 fish in the pond when the pond has an area of 20 $m^2$. What is the residual value when the *pond size* is 20 $m^2$? (1 MARK)

**c.** The correlation coefficient is 0.73. What percentage of the variation in *number of fish* can be explained by the variation in *pond size*? Round your answer to two decimal places if required. (1 MARK)

*Adapted from VCAA 2017 Exam 2 Data analysis Q3*

# 8E Flow problems

## STUDY DESIGN DOT POINTS

- use of networks to model flow problems: capacity, sinks and sources
- solution of small-scale network flow problems by inspection and the use of the 'maximum-flow minimum-cut' theorem to aid the solution of larger scale problems

8A 8B 8C 8D **8E** 8F 8G 8H 8I 8J

## KEY SKILLS

During this lesson, you will be:

- navigating directed graphs
- calculating cut capacities
- determining the maximum flow.

## KEY TERMS

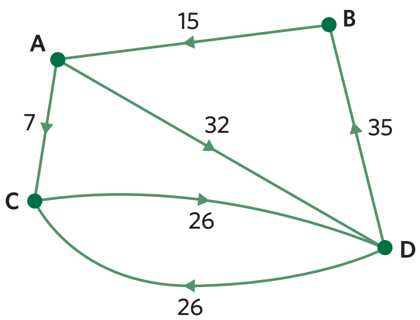
- Flow
- Source
- Sink
- Cut
- Cut capacity
- Maximum flow
- Minimum cut

Directed graphs can be used to solve flow problems, where the maximum capacity of a network, when moving from one side to another, needs to be calculated. A practical application of a flow problem is the movement of water through a system of pipes. The amount of water that can flow through the system is dictated by the capacity of the smallest pipe.

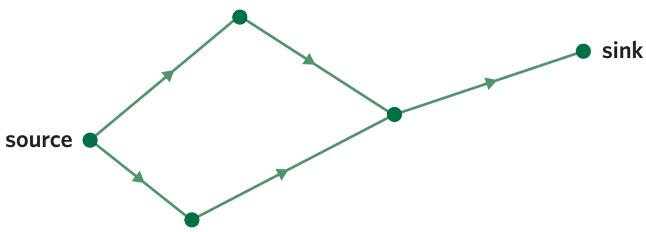
## Navigating directed graphs

A directed graph is a network containing arrows on each edge that show the way in which one can travel between two vertices. Weights can be added to the edges to represent physical quantities such as time, distance or cost.

For example, the following directed graph shows the roads connecting four towns, where the weights represent the length, in km, of each road.



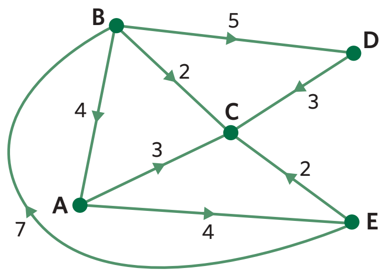

**Flow** occurs when networks only allow for movement in one direction, starting at one vertex (the **source**) and ending at another vertex (the **sink**).



## Worked example 1

For the following directed graphs, determine if it is possible to travel from vertex A to vertex D. If so, identify a route and calculate the weight of the route.

**a.**




### Explanation

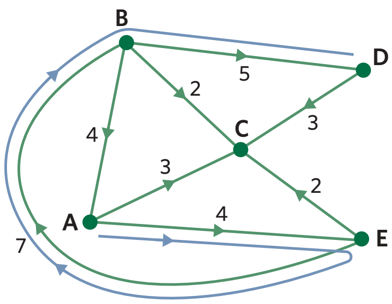
**Step 1:** Follow the edges starting from A.

From A, travel is possible either to C or E.

Travelling to C results in a dead end. Travelling to E is the only option.

**Step 2:** Continue this pattern until D is reached.

The route is A–E–B–D




**Step 3:** Calculate the weight of the route.

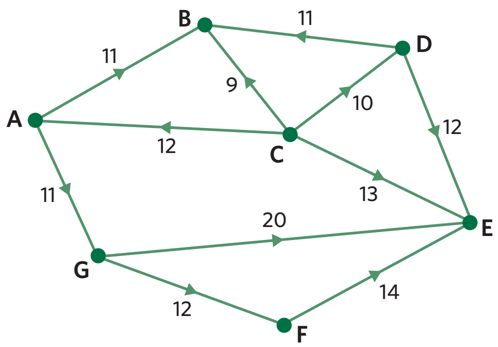
Sum the weights of the edges (A to E, E to B and B to D) in the route.

$weight = 4 + 7 + 5$

### Answer

Yes, A–E–B–D, 16

**b.**




### Explanation

From A, you can travel either to B or G.

Travelling to B results in a dead end. Travelling to G is the only option.

After travelling to G, you will end up at E, which is a dead end.

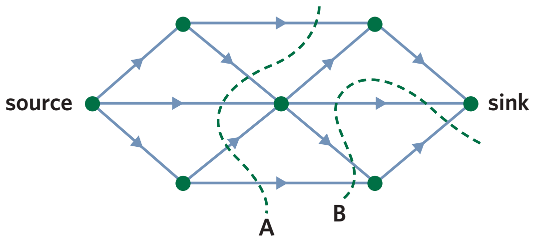
### Answer

No

## Calculating cut capacities

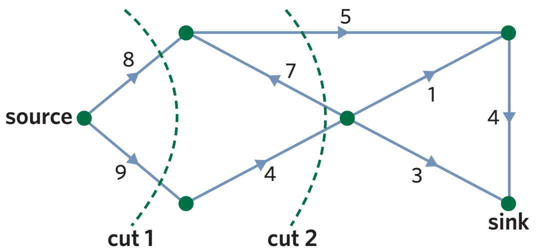
A **cut** is a line passing through a network that cuts the network in two. A valid cut must completely stop any network flow from the source to the sink.

For example, line A is a cut because it completely prevents any flow from the source to the sink. Line B is not a cut because it does not prevent all flow from source to sink.




The **cut capacity** is the total capacity, or weight, of all edges that a cut passes through. It is calculated by summing the weights of all edges that flow from source to sink included in the cut. If an edge flows from the sink side of the cut to the source side of the cut, it is not included in the cut capacity.

For example, consider the following network.



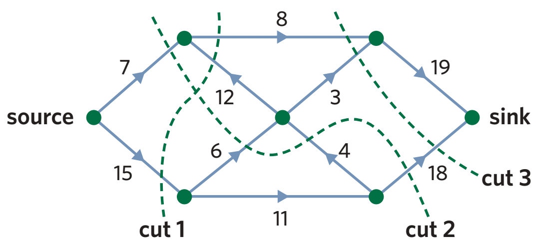

The capacity of cut 1 is $8 + 9 = 17$.

The capacity of cut 2 is $5 + 4 = 9$.

The edge with a weight of 7 is not included in the calculation for cut 2 as it flows from the sink side of the cut to the source side of the cut.

### Worked example 2

Calculate the capacity of each cut in the following network.




#### Explanation

**Step 1:** Determine if any edges will not be counted.

If an edge flows from the sink side of the cut to the source side of the cut, it is not counted in the cut capacity. Hence, the edge with a weight of 12 in cut 1 will not be counted.

All edges in cuts 2 and 3 will be counted.

**Step 2:** Calculate the sum of the capacities of the edges.

Cut 1: $8 + 15 = 23$

Cut 2: $7 + 6 + 4 + 18 = 35$

Cut 3: $8 + 3 + 18 = 29$

#### Answer

Cut 1: 23

Cut 2: 35

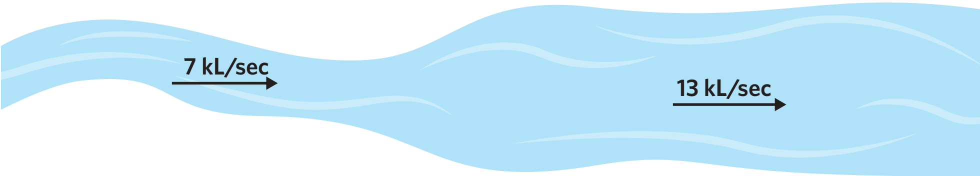
Cut 3: 29

# Determining the maximum flow

The **maximum flow** is equal to the minimum capacity through a network.

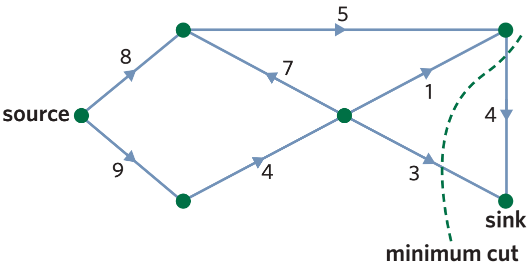
The maximum flow can best be explained using an example.

The start of a river system has the capacity to allow 7 kilolitres (kL) of water to flow through it each second. The river system then widens and the capacity of the next section is 13 kL per second. However, since the first section of river is smaller, only 7 kL will be allowed into the river system each second. Therefore, the full capacity of the second section will not be used. The maximum flow of the river system is 7 kL/sec.



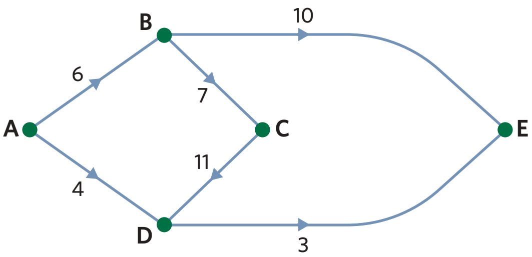

The **minimum cut** of a network is the cut with the smallest cut capacity. The capacity of the minimum cut is the maximum flow through the network.

For example, the minimum cut capacity of the following network is 7. Therefore, the maximum flow of the network is also 7.




## Worked example 3

The following network shows the number of cars that can travel between five different towns each second.



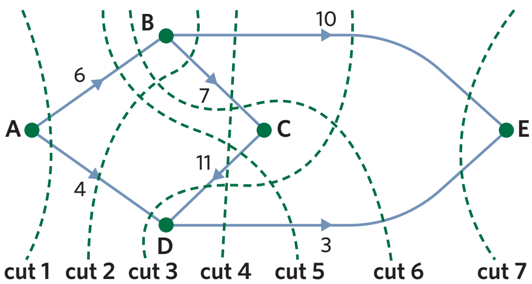

What is the maximum number of cars that can travel from town A to town E each second?

### Explanation

**Step 1:** Identify all possible cuts through the network.

A cut must separate the source from the sink and completely stop the network flow.

There are seven possible cuts for this network.




Continues →

**Step 2:** Calculate the capacity of each cut.

The cut capacity is the sum of the capacities of all edges (from source to sink) that the cut passes through.

Cut 1: $6 + 4 = 10$

Cut 2: $10 + 7 + 4 = 21$

Cut 3: $10 + 11 + 4 = 25$

Cut 4: $10 + 7 + 3 = 20$

Note: In cut 4, the edge from vertex C to vertex D is not counted.

Cut 5: $6 + 3 = 9$

Note: In cut 5, the edge from vertex C to vertex D is not counted.

Cut 6: $6 + 3 = 9$

Note: In cut 6, the edge from vertex B to vertex C is not counted.

Cut 7: $10 + 3 = 13$

**Step 3:** Identify the minimum cut(s).

The minimum cuts are cut 4 and cut 5, with a capacity of 9 cars per second.

The maximum flow is equal to the capacity of the minimum cuts.

### Answer

9 cars

## Exam question breakdown

*VCAA 2020 Exam 1 Networks and decision mathematics Q9*

The flow of liquid through a series of pipelines, in litres per minute, is shown in the following directed network.

Five cuts labelled A to E are shown on the network. The number of these cuts with a capacity equal to the maximum flow of liquid from the source to the sink, in litres per minute, is

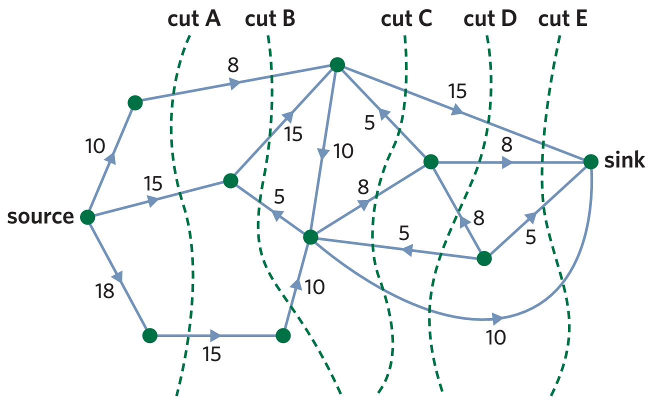
**A.** 1

**B.** 2

**C.** 3

**D.** 4

**E.** 5




### Explanation

**Step 1:** Calculate the capacity of each cut.

The cut capacity is the sum of the capacities of all edges (from source to sink) that the cut passes through.

Cut A: $8 + 15 + 15 = 38$

Cut B: $8 + 15 + 10 = 33$

Cut C: $15 + 8 + 10 = 33$

Cut D: $15 + 8 + 10 = 33$

Cut E: $15 + 8 + 5 + 10 = 38$

**Step 2:** Determine the maximum flow.

The maximum flow is equal to the capacity of the minimum cut.

The capacity of the minimum cut is 33 L/min.

**Step 3:** Count the number of cuts with the minimum capacity.

Cuts B, C, and D all have a capacity of 33 L/min.

**60%** of students answered this question correctly.

**18%** of students incorrectly chose option B. This is likely because they included edges in their cut capacity calculations that were flowing in the wrong direction. It is important that only edges flowing from the source side of the cut to the sink side of the cut are included in the cut capacity.

### Answer

C

# 8E Questions

## Navigating directed graphs

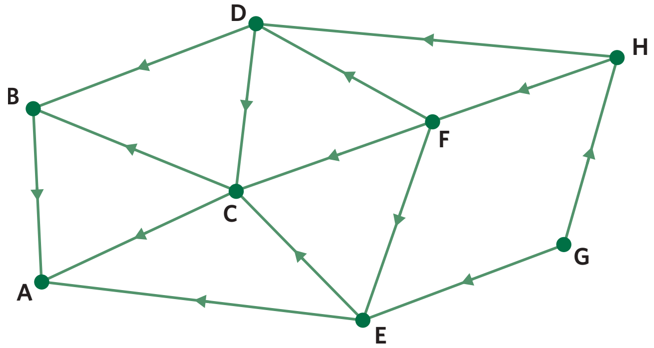
**1.** For the following directed graph, the source and sink, respectively, are

**A.** vertex A and vertex H

**B.** vertex G and vertex A

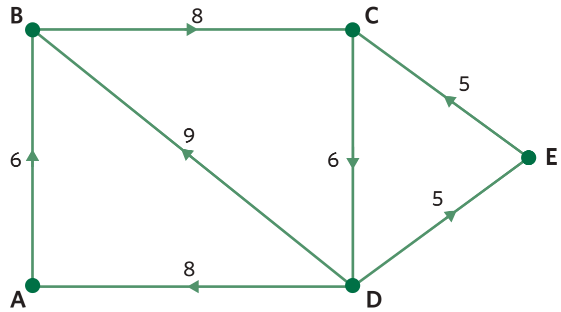
**C.** vertex H and vertex B

**D.** vertex H and vertex C

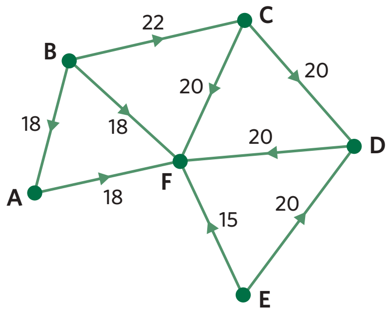



**2.** For each of the following directed graphs, is it possible to travel from vertex B to vertex E? If so, write down a route and calculate the weight of the route.

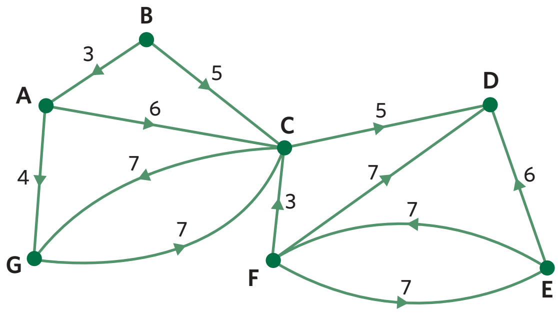
**a.**



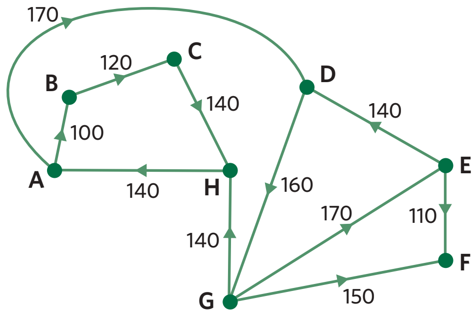

**b.**




**c.**




**d.**




**3.** The following network represents a number of one-way roads, with intersections represented as vertices. The numerical values above each edge represent the lengths of the roads, in km.

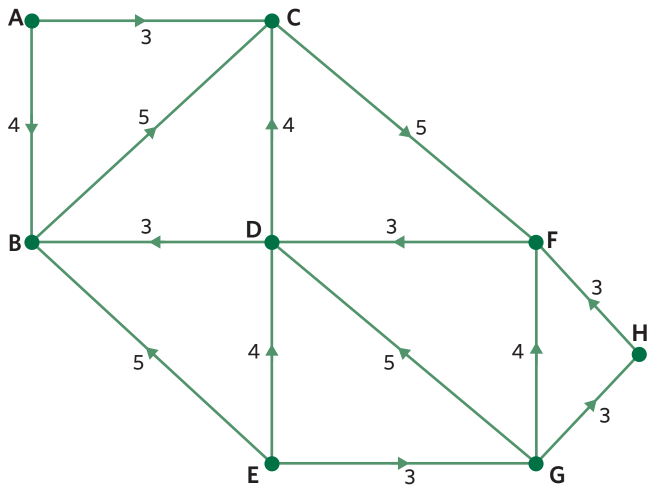
Which route is possible and has a weight of 13?

**A.** intersection A to intersection F.

**B.** intersection C to intersection G.

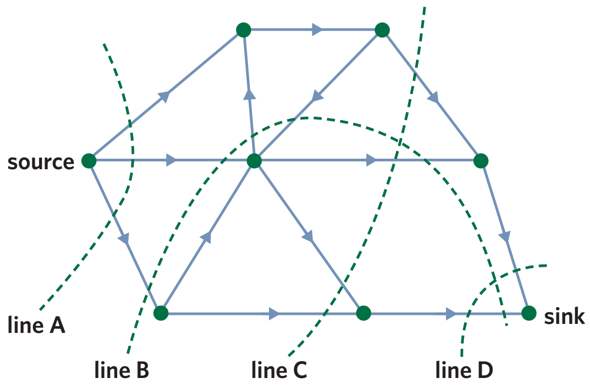
**C.** intersection G to intersection A.

**D.** intersection B to intersection D.



## Calculating cut capacities

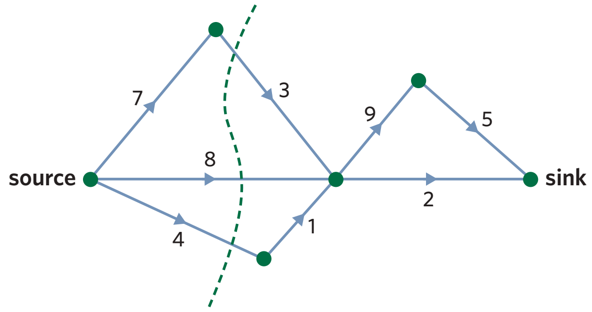
**4.** On the following network, which of the lines is **not** a cut?



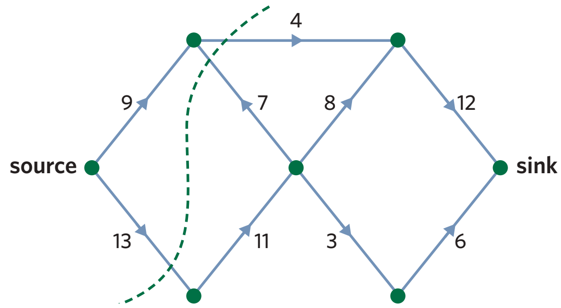

**A.** line A **B.** line B **C.** line C **D.** line D

**5.** Calculate the capacity of each of the following cuts.

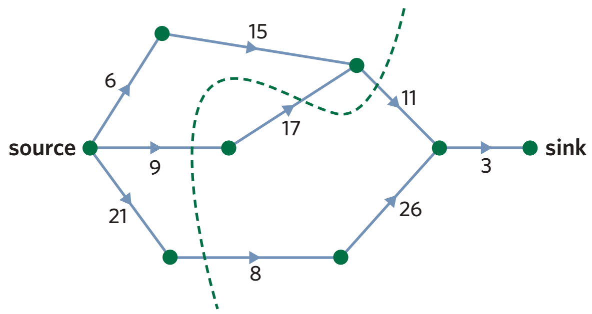
**a.**



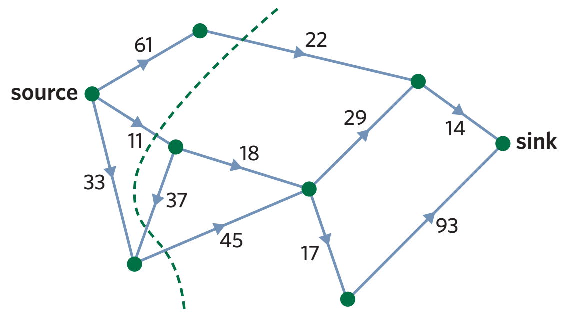

**b.**



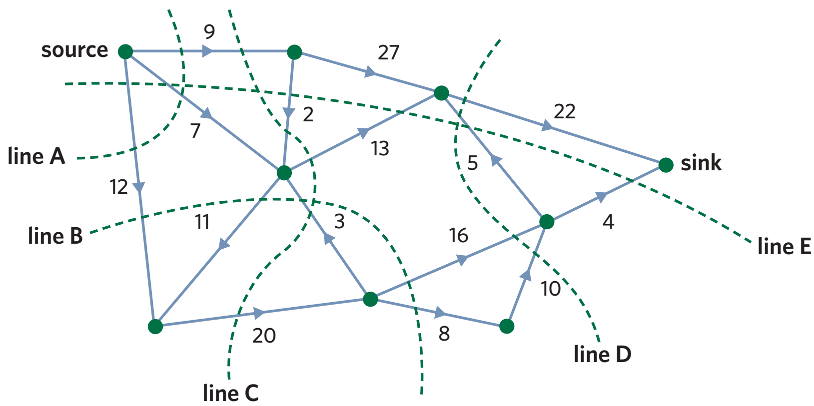

**c.**




**d.**




**6.** Consider the following network.

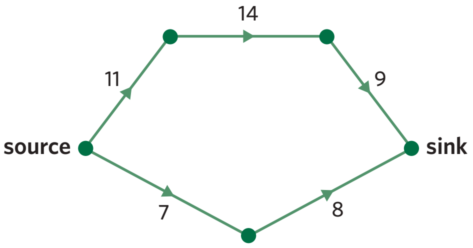



**a.** Which of the lines are valid cuts that stop flow from the source to the sink?

**b.** Determine the capacity of each cut identified in part **a**.

## Determining the maximum flow

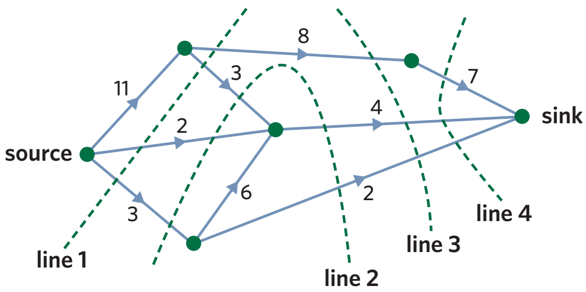
**7.** The maximum flow from source to sink through the following network is



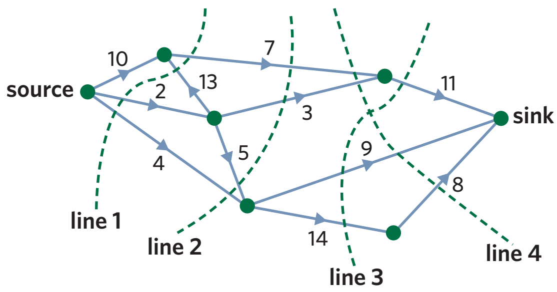

**A.** 16 **B.** 17 **C.** 18 **D.** 22

**8.** Four lines have been drawn on each of the following networks. One of these four lines is the minimum cut. For each network, determine which line is the minimum cut.

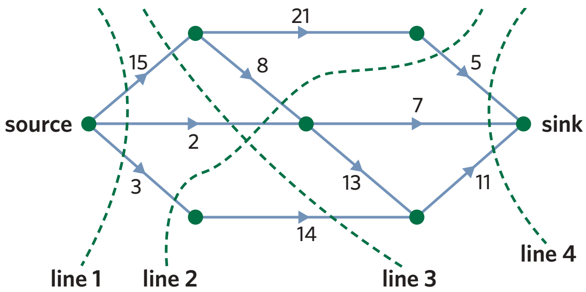
**a.**



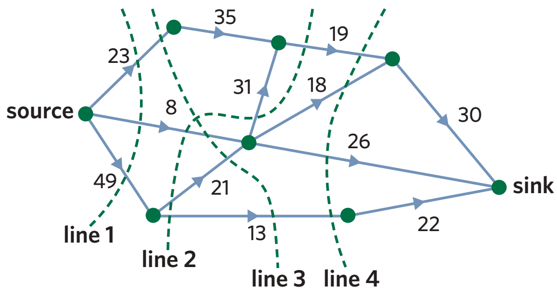

**b.**




**c.**

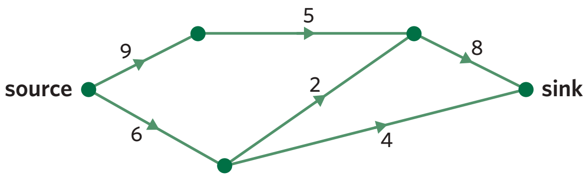



**d.**

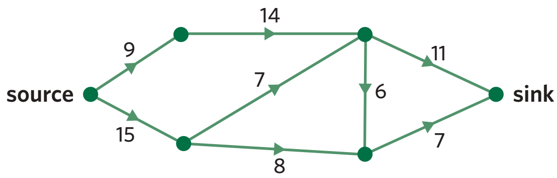



**9.** Find the maximum flow from source to sink for each of the following networks.

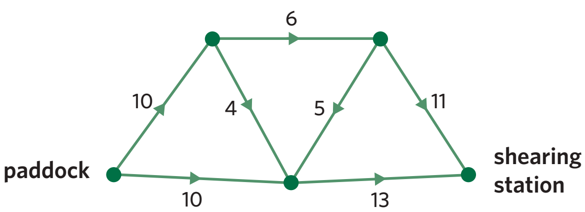
**a.**




**b.**




**10.** The following network displays the route from a paddock in which sheep are currently herded to a shearing station. The number of sheep per second that can travel between different gates on the route is shown.

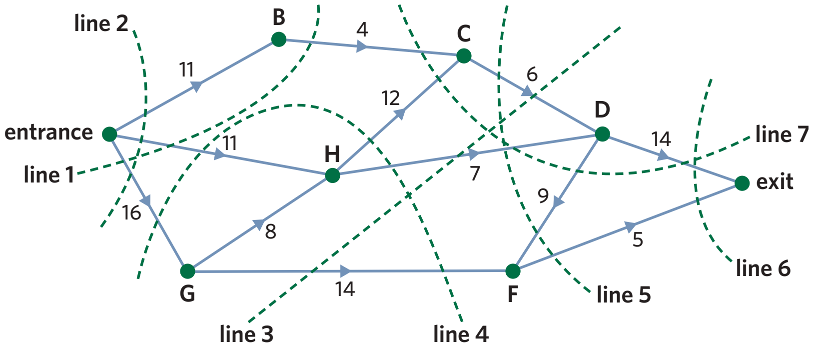



Find the maximum number of sheep that can travel each second from the paddock to the shearing station.

## Joining it all together

**11.** The following network is a map of the paths between stalls and rides at the community fair. It shows the number of people that can travel along each of the paths every minute, as well as which direction they are able to walk.

There are seven lines on the map. One of these lines is the minimum cut.




**a.** It is possible to travel from

**A.** attraction D to attraction G.

**B.** attraction C to attraction G.

**C.** attraction F to attraction C.

**D.** attraction B to attraction F.

**b.** Which of the lines are not valid cuts that stop flow from the entrance to the exit?

**c.** What is the maximum number of people that can move from the entrance to the exit each minute?

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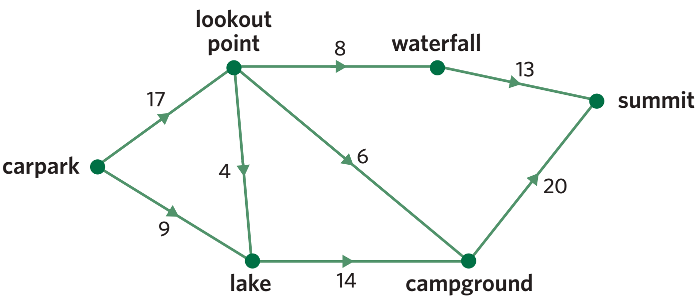
**12.** The following network shows the paths that connect landmarks on a hiking trail, and the number of people per hour that can travel along each path.



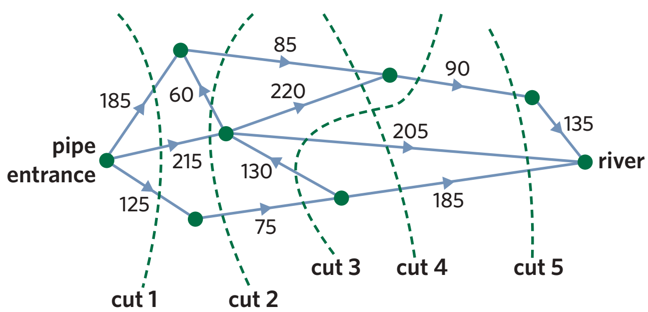

**a.** What is the maximum number of people that can travel from the carpark to the campground each hour?

**b.** What is the maximum number of people that can travel from the carpark to the summit each hour?

**c.** This week, park rangers are doing some work on the path from the waterfall to the summit, so now only 5 people can travel along this path each hour. How many less people can now travel from the carpark to the summit each hour?

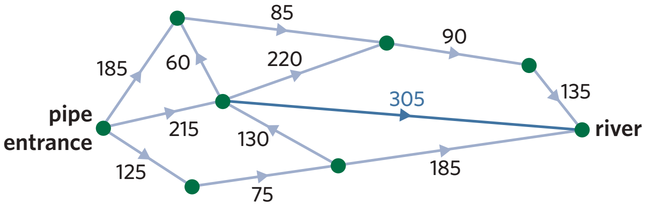
---

**13.** The following network shows the layout of a pipe system that transports water into a river. The edges represent pipes while the vertices are intersections of pipes. All values are in L/sec. There are five cuts on the network, one of them being the minimum cut.




**a.** What is the maximum amount of water that can be transported from the pipe entrance to the river each second?

In an effort to maximise transportation rate of the pipe system, the council increased the size of one pipe. They increased the capacity of the highlighted pipe from 205 L/sec of water to 305 L/sec.




**b.** What is now the maximum amount of water that can be transported from the pipe entrance to the river each second?

## Exam practice

**14.** The following directed graph shows the flow of water, in litres per minute, in a system of pipes connecting the source to the sink.

The maximum flow, in litres per minute, from the source to the sink is

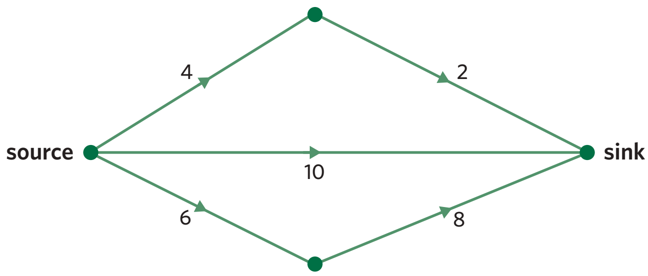
**A.** 10

**B.** 14

**C.** 18

**D.** 20

**E.** 22



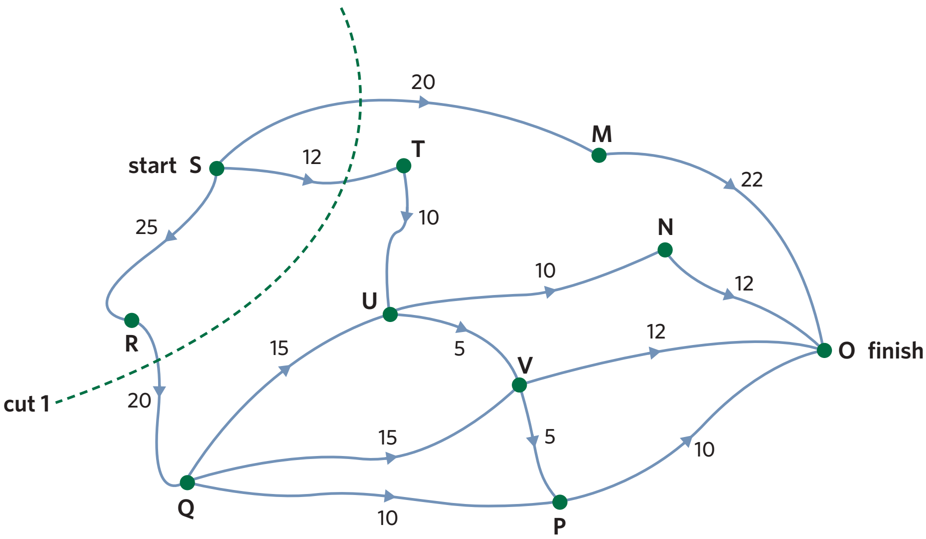

*VCAA 2016 Exam 1 Networks and decision mathematics Q2*

**54%** of students answered this question correctly.

**15.** A training program has a cricket team starting from exercise station S and running to exercise station O.

For safety reasons, the cricket coach has placed a restriction on the maximum number of people who can use the tracks in the fitness park.

The directed graph shows the capacity of the tracks, in number of people per minute.




When considering the possible flow of people through this network, many different cuts can be made.

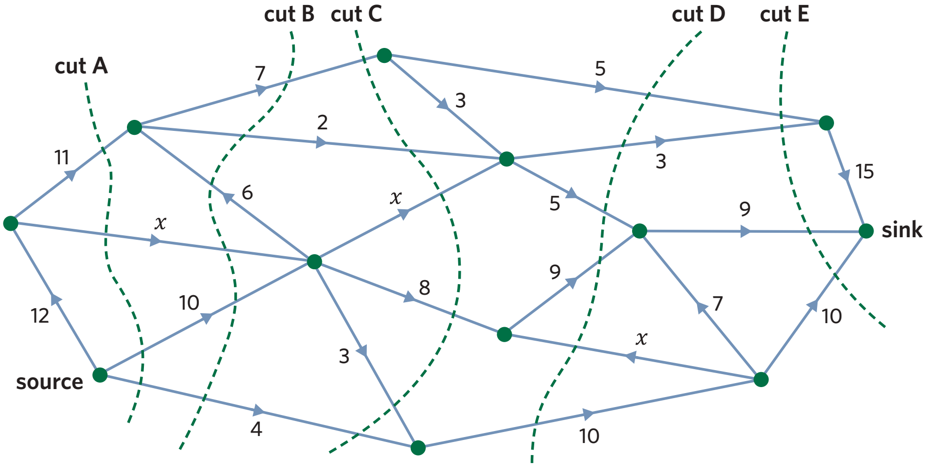
**a.** Determine the capacity of cut 1. (1 MARK)

**b.** What is the maximum flow from S to O, in number of people per minute? (1 MARK)

*VCAA 2020 Exam 2 Networks and decision mathematics Q4b,c*

Part **a**: **69%** of students answered this question correctly.
Part **b**: **32%** of students answered this question correctly.

**16.** The flow of oil through a series of pipelines, in litres per minute, is shown in the following network.




The weightings of three of the edges are labelled $x$.

Five cuts labelled A–E are shown on the network.

The maximum flow of oil from the source to the sink, in litres per minute, is given by the capacity of

**A.** cut A if $x = 1$

**B.** cut B if $x = 2$

**C.** cut C if $x = 2$

**D.** cut D if $x = 3$

**E.** cut E if $x = 3$

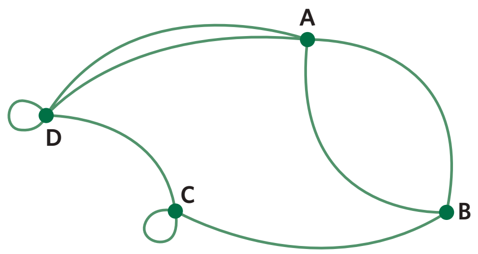
*VCAA 2017 Exam 1 Networks and decision mathematics Q8*

**44%** of students answered this question correctly.

## Questions from multiple lessons

### Networks and decision mathematics

**17.** Consider the following graph.




The adjacency matrix for this graph, with some elements missing, is shown.

$$\begin{array}{c} \quad A \quad B \quad C \quad D \\ \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$

The adjacency matrix contains 16 elements when complete.

Of the 12 missing elements

**A.** four are ‘1’ and eight are ‘2’.

**B.** eight are ‘1’ and four are ‘2’.

**C.** two are ‘0’, six are ‘1’ and four are ‘2’.

**D.** six are ‘1’ and six are ‘2’.

**E.** four are ‘0’, four are ‘1’ and four are ‘2’.

*Adapted from VCAA 2017 Exam 1 Networks and decision mathematics Q3*

## Recursion and financial modelling

**18.** Five lines of an amortisation table for a reducing balance loan with fortnightly repayments are shown.

| repayment number | repayment | interest | principal reduction | balance of loan |
|---|---|---|---|---|
| 17 | \$1050 | \$348.68 | \$701.32 | \$173 639.60 |
| 18 | \$1050 | \$347.28 | \$702.72 | \$172 936.88 |
| 19 | \$1050 | \$345.87 | \$704.13 | \$172 232.75 |
| 20 | \$1050 | \$310.02 | \$739.98 | \$171 492.77 |
| 21 | \$1050 | \$308.69 | \$741.31 | \$170 751.46 |

The interest rate for this loan changed at the start of the period that applies to repayment 20.

This change in interest rate is best described as

**A.** an increase of 0.52% per annum.

**B.** a decrease of 0.0002% per annum.

**C.** an increase of 0.02% per annum.

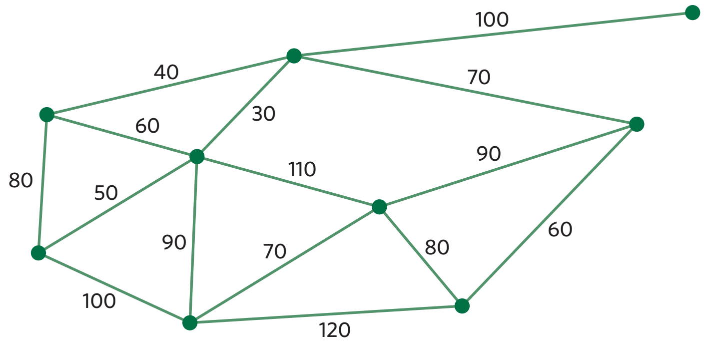
**D.** a decrease of 0.52% per annum.

**E.** a decrease of 0.02% per annum.

*Adapted from VCAA 2018 Exam 1 Recursion and financial modelling Q23*

## Networks and decision mathematics *Year 11 content*

**19.** A village has nine homes. They are represented as vertices in the following graph. The edges represent possible power lines that could be built between the houses, with the numbers representing the distances, in metres, between them.




Power lines are required so that the houses can use electricity.

These power lines will create a connected graph.

The shortest length of power line will be used.

**a.** Give a mathematical term to describe a graph that represents these power lines. (1 MARK)

**b.** Draw the graph that represents these power lines. (1 MARK)

*Adapted from VCAA 2017 Exam 2 Networks and decision mathematics Q3*

# 8F Shortest path problems

**STUDY DESIGN DOT POINTS**

- determination of the shortest path between two specified vertices in a graph, digraph or network by inspection
- Dijkstra's algorithm and its use to determine the shortest path between a given vertex and each of the other vertices in a weighted graph or network

8A 8B 8C 8D 8E **8F** 8G 8H 8I 8J

**KEY SKILLS**

During this lesson, you will be:

- determining the shortest path by inspection
- applying Dijkstra's algorithm.

**KEY TERMS**

- Shortest path
- Dijkstra's algorithm

When looking at weighted graphs, it is important to be able to find the shortest path to maximise efficiency when travelling between vertices. For practical examples of weighted graphs, the shortest path can represent the cheapest option to get from point A to point B, or the shortest distance between points.

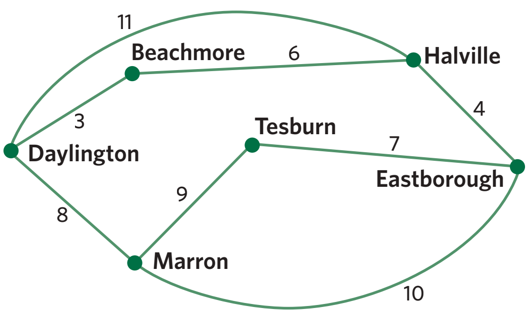
## Determining the shortest path by inspection

The **shortest path** between two vertices on a weighted graph is the path with the minimum sum of weights. For simple weighted graphs, the shortest path can often be found by inspection.

### Worked example 1

The weighted graph shown displays the distance, in kilometres, between six train stations.

Find the length and route of the shortest path from Daylington to Eastborough by eye.




**Explanation**

**Step 1:** List all the possible paths to get from Daylington (D) to Eastborough (E).

D–H–E, D–M–E, D–M–T–E, D–B–H–E

**Step 2:** Find the sum of the weights for each of these paths, and determine the minimum value.

D–H–E: $11 + 4 = 15$

D–M–E: $8 + 10 = 18$

D–M–T–E: $8 + 9 + 7 = 24$

D–B–H–E: $3 + 6 + 4 = 13$

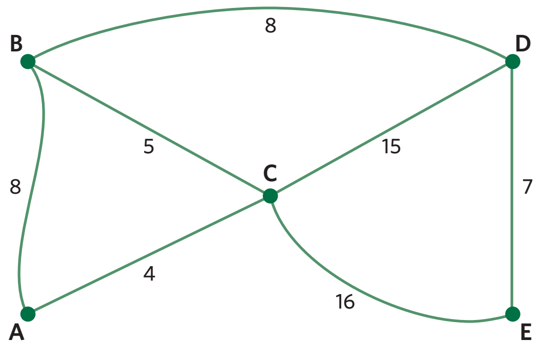
**Answer**

13 km, Daylington–Beachmore–Halville–Eastborough

## Applying Dijkstra's algorithm

For more complex weighted graphs, determining the shortest path by inspection can be tedious. **Dijkstra's algorithm** can be used to determine the length and route of the shortest path between vertices on any weighted network.

The following steps detail the use of Dijkstra's algorithm to find the shortest path between vertex A and D in the given weighted network.




1. Construct a table.
   – The starting vertex should be the first row vertex.
   – Each of the other vertices should be listed as column vertices in any order.

| | B | C | D | E |
|---|---|---|---|---|
| A | | | | |

2. Complete the first row.
   Write down the distance of direct connections from the starting vertex to each of the other vertices.
   – If there is no edge between the starting vertex and the column vertex, mark it with a cross.

| | B | C | D | E |
|---|---|---|---|---|
| A | 8 | 4 | X | X |

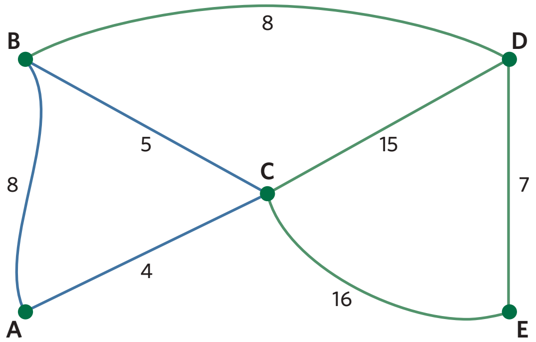
3. Identify the smallest number in this row.
   – If there are multiple equally small numbers, choose any one of them.
   – Put a square around the chosen number.
   – Copy the number into the next row, and put a square around it.
   – The column vertex of this number will become the new row vertex.

| | B | C | D | E |
|---|---|---|---|---|
| A | 8 | [4] | X | X |
| C | | [4] | | |

4. Complete the next row.
   For the rest of the columns, add the number that recently had a square put around it to the distance from the row vertex to the column vertex.
   – If this number is less than or equal to the number above it in the column, write it down.
   – If the number is greater than the number above it in the column, ignore it and copy the value from above.
   – If there is no connection from the starting vertex to the column vertex, via the row vertex, copy the number from above. If there is no number above, mark it with a cross.

| | B | C | D | E |
|---|---|---|---|---|
| A | 8 | 4 | X | X |
| C | 8 | 4 | 19 | 20 |

Although there is a connection from the starting vertex (A) to vertex B, via the row vertex (C), the direct connection from vertex A to B has a shorter length of 8.




5. Identify the smallest number in this row that is not already in a square.
   - If there are multiple equally small numbers, choose any one of them.
   - Put a square around the chosen number.
   - The column vertex of this number will become the new row vertex.
   - Copy down all numbers in squares into the next row.

| | B | C | D | E |
|---|---|---|---|---|
| A | 8 | 4 | X | X |
| C | 8 | 4 | 19 | 20 |
| B | 8 | 4 | | |

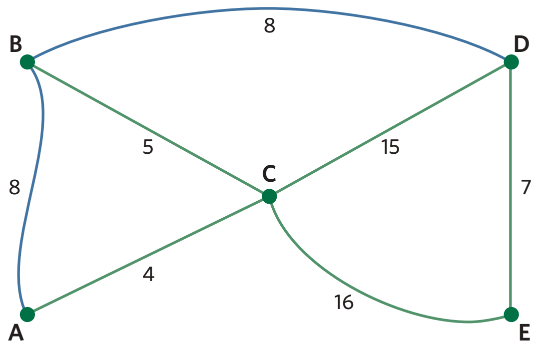
6. Complete the table by repeating steps 4 and 5.
   - Continue until the value in the destination vertex column has a square around it.

| | B | C | D | E |
|---|---|---|---|---|
| A | 8 | 4 | X | X |
| C | 8 | 4 | 19 | 20 |
| B | 8 | 4 | 16 | 20 |

7. Identify the length of the shortest path.
   The number in a square in the column of the destination vertex is the length of the shortest path.
   The length of the shortest path is 16.
8. Identify the shortest path.
   - Start from the number in the square in the destination vertex column.
   - Draw a line up the column from this final value until you reach the highest row with this same value. This value does not need to be in a square.
   - Locate the row vertex for the row you are now at and draw a horizontal line to this vertex's column.
   - Repeat until you reach the row of the starting vertex.
   - The horizontal lines show the route of the shortest path.

| | B | C | D | E |
|---|---|---|---|---|
| A | 8 | 4 | X | X |
| C | 8 | 4 | 19 | 20 |
| B | 8 | 4 | 16 | 20 |

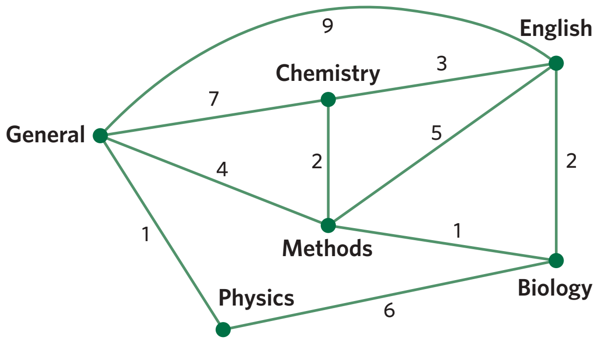
The shortest path is A–B–D.




## Worked example 2

Reuben is struggling to find his way around his new school and is constantly running late. Luckily his friend in General Maths drew him a weighted graph of the school, representing the time it takes, in minutes, to walk to each class.

Find the time and route of the shortest path from General (G) to English (E).




### Explanation

**Step 1:** Construct the table.

For efficiency purposes, label each vertex as their initial.

The first row is the starting vertex (G).

| | P | B | M | C | E |
|---|---|---|---|---|---|
| G | | | | | |

**Step 2:** Complete the first row.

There are edges directly connecting vertex G to vertices P, M, C and E. Write the weights of these edges in the table.

There is no edge directly connecting vertex G to vertex B. Mark column B with a cross.

| | P | B | M | C | E |
|---|---|---|---|---|---|
| G | 1 | X | 4 | 7 | 9 |

**Step 3:** Identify the smallest number in this row.

1 is the smallest number in the row.

Put a square around the '1' and copy it to the next row.

The '1' is located in column P, so vertex P will become the next row vertex.

| | P | B | M | C | E |
|---|---|---|---|---|---|
| G | [1] | X | 4 | 7 | 9 |
| P | [1] | | | | |

**Step 4:** Complete the second row.

For the rest of the columns, add the number that recently had a square put around it (1) to the distance from the row vertex (P) to each column vertex.

Vertex B: $1 + 6 = 7$

There is no number above, so write it down.

Vertex P is not directly connected to vertices M, C and E. As there is a number above each, copy them down.

| | P | B | M | C | E |
|---|---|---|---|---|---|
| G | [1] | X | 4 | 7 | 9 |
| P | [1] | 7 | 4 | 7 | 9 |

Continues →

**Step 5:** Identify the smallest number in this row that is not already in a square.

4 is the smallest number in the row that is not already in a square.

Put a square around the '4' and copy it to the next row.

The '4' is located in column M, so vertex M will become the next row vertex.

| | P | B | M | C | E |
|---|---|---|---|---|---|
| **G** | 1 | X | 4 | 7 | 9 |
| **P** | 1 | 7 | 4 | 7 | 9 |
| **M** | 1 | | 4 | | |

**Step 6:** Complete the table by repeating steps 4 and 5.

Continue until the value in the destination vertex column has a square around it.

| | P | B | M | C | E |
|---|---|---|---|---|---|
| **G** | 1 | X | 4 | 7 | 9 |
| **P** | 1 | 7 | 4 | 7 | 9 |
| **M** | 1 | 5 | 4 | 6 | 9 |
| **B** | 1 | 5 | 4 | 6 | 7 |
| **C** | 1 | 5 | 4 | 6 | 7 |

**Step 7:** Identify the length of the shortest path.

The number in a square in column E is the length of the shortest path.

The length of the shortest path is 7.

**Step 8:** Identify the route of the shortest path.

Start from the square in column E.

Draw a line up the column from this final value until you reach the highest row with this same value. This is row B.

Draw a horizontal line to this vertex's column (B).

Repeat until you reach the row of the starting vertex.

The horizontal lines show the route of the shortest path.

| | P | B | M | C | E |
|---|---|---|---|---|---|
| **G** | 1 | X | 4 | 7 | 9 |
| **P** | 1 | 7 | 4 | 7 | 9 |
| **M** | 1 | 5 | 4 | 6 | 9 |
| **B** | 1 | 5 | 4 | 6 | 7 |
| **C** | 1 | 5 | 4 | 6 | 7 |

The shortest path is G–M–B–E.

### Answer

7 minutes, General–Methods–Biology–English

## Exam question breakdown

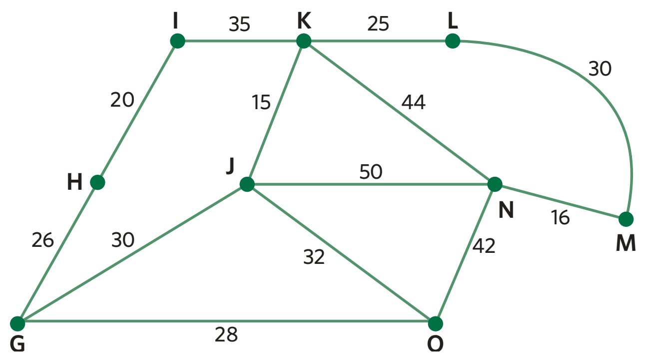
*VCAA 2021 Exam 2 Networks and decision mathematics Q2a*

George lives in town G and Maggie lives in town M.

The diagram shows the network of main roads between town G and town M.

The vertices G, H, I, J, K, L, M, N and O represent towns.

The edges represent the main roads. The numbers on the edges indicate the distances, in kilometres, between adjacent towns.




What is the shortest distance, in kilometres, between town G and town M? (1 MARK)

Continues →

## Explanation

**Step 1:** Apply Dijkstra's algorithm to the weighted network.

Town G is the starting vertex and town M is the destination vertex.

| | H | I | J | K | L | M | N | O |
|---|---|---|---|---|---|---|---|---|
| G | 26 | X | 30 | X | X | X | X | 28 |
| H | 26 | 46 | 30 | X | X | X | X | 28 |
| O | 26 | 46 | 30 | X | X | X | 70 | 28 |
| J | 26 | 46 | 30 | 45 | X | X | 70 | 28 |
| K | 26 | 46 | 30 | 45 | 70 | X | 70 | 28 |
| I | 26 | 46 | 30 | 45 | 70 | X | 70 | 28 |
| N | 26 | 46 | 30 | 45 | 70 | 86 | 70 | 28 |
| L | 26 | 46 | 30 | 45 | 70 | 86 | 70 | 28 |

**Step 2:** Identify the length of the shortest path.

The number in a square in column M is the length of the shortest path.

The shortest path from town G to town M is 86 km.

### Answer

86 km

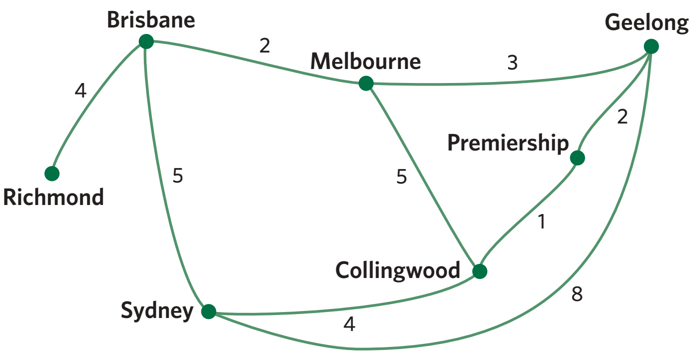
**85%** of students answered this question correctly.

Some students correctly found the route but did not find the distance of the path, as asked for in the question.

# 8F Questions

## Determining the shortest path by inspection

**1.** Seven fictitious towns, connected by roads, are represented in the following weighted network. The number on each edge is the distance, in kilometres, of the road.




Which option outlines the shortest path between the towns Richmond and Premiership?

**A.** Richmond–Brisbane–Melbourne–Geelong–Premiership

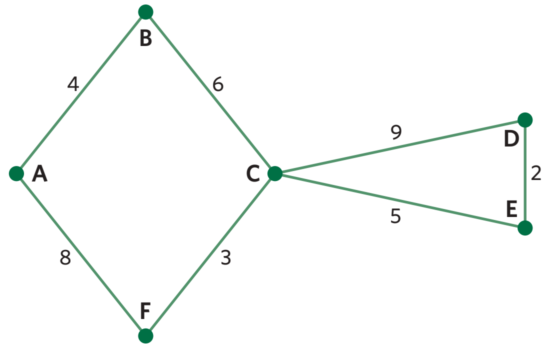
**B.** Richmond–Brisbane–Sydney–Collingwood–Premiership

**C.** Richmond–Brisbane–Melbourne–Collingwood–Premiership

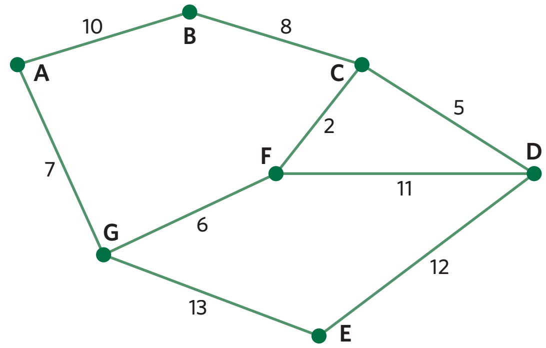
**D.** Richmond–Brisbane–Sydney–Geelong–Premiership

2. Find, by inspection, the length and route of the shortest path from vertex A to vertex D for each of the following weighted networks.

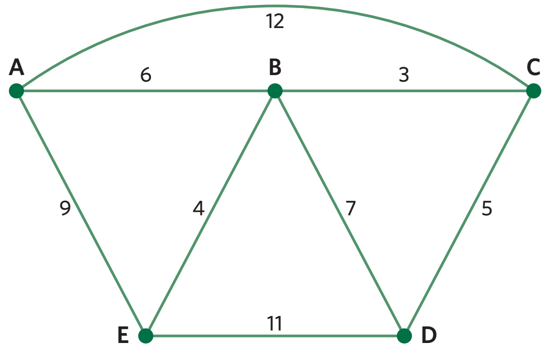
a.



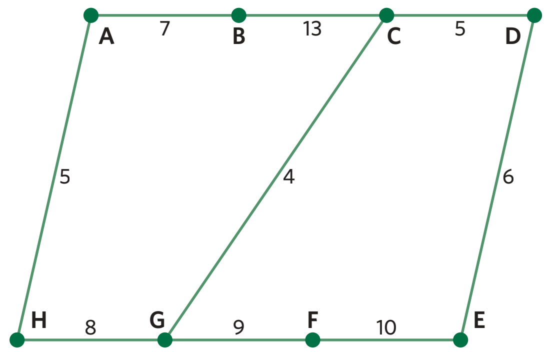

b.



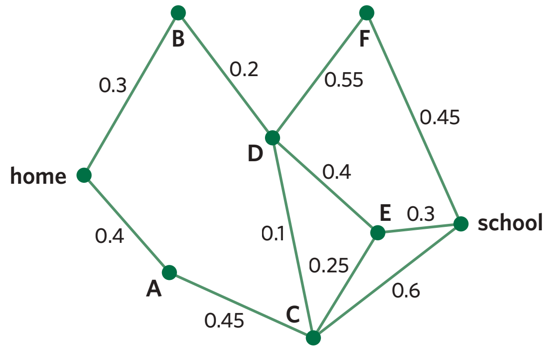

c.




d.




3. Fiona is preparing for her first day at her new school, and is planning on walking to school. The given weighted network shows the distance, in kilometres, of the roads between her house and her school. Intersections are labelled A–F.




a. Fiona decides to walk to school via the route home–A–C–D–E–school. What distance will she walk?

b. Find, by inspection, the shortest distance that Fiona can walk to school.

## Applying Dijkstra's algorithm

4. The following table is a complete Dijkstra's algorithm solution to find the shortest path through a particular weighted network.

| | B | C | D | E | F |
|---|---|---|---|---|---|
| A | 5 | 4 | X | X | X |
| C | 5 | 4 | 7 | 10 | 12 |
| B | 5 | 4 | 7 | 10 | 12 |
| D | 5 | 4 | 7 | 10 | 11 |
| E | 5 | 4 | 7 | 10 | 11 |

a. What is the length of the shortest path from A to D?

A. 4    B. 7    C. 11    D. 14

b. What is the length of the shortest path from A to F?

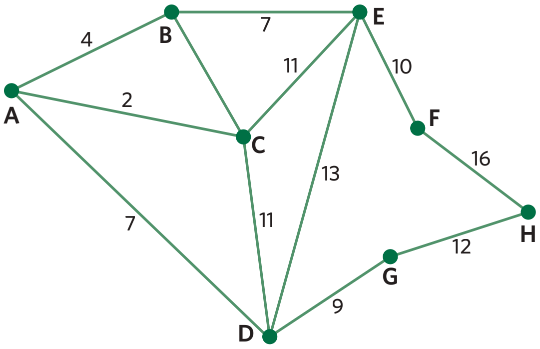
c. What is the route of the shortest path from A to F?

**5.** Consider the table containing a complete Dijkstra's algorithm solution.

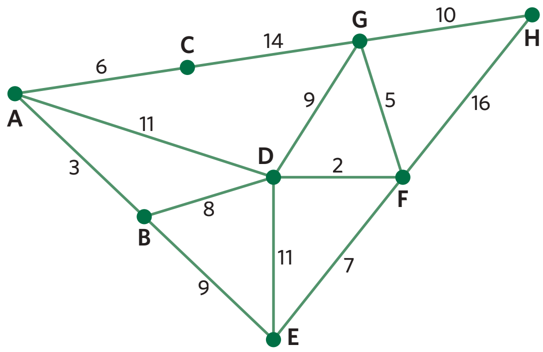
| | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| **A** | 4 | 6 | 11 | X | X | X | X |
| **B** | 4 | 6 | 11 | 13 | X | X | X |
| **C** | 4 | 6 | 11 | 13 | X | 20 | X |
| **D** | 4 | 6 | 11 | 13 | 13 | 20 | X |
| **E** | 4 | 6 | 11 | 13 | 13 | 20 | X |
| **F** | 4 | 6 | 11 | 13 | 13 | 18 | 29 |
| **G** | 4 | 6 | 11 | 13 | 13 | 18 | 28 |

**a.** Which of the following weighted networks could this Dijkstra's algorithm be used for?

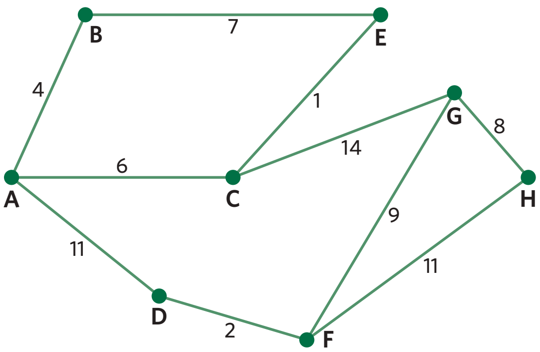
**A.**



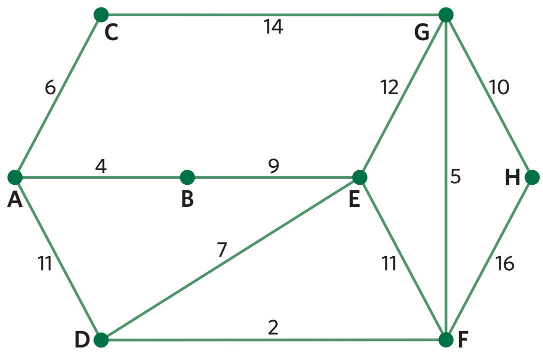

**B.**




**C.**




**D.**




**b.** What is the length of the shortest path from A to F?

**c.** What is the route of the shortest path from A to G?

---

**6.** The first three rows of a Dijkstra's algorithm solution have been filled out for a particular weighted network. Which of the following statements is not true?

| | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| **A** | 8 | X | 7 | X | X | 13 | X |
| **D** | 8 | 16 | 7 | X | 9 | 12 | X |
| **B** | 8 | 14 | 7 | X | 9 | 11 | 15 |

**A.** The shortest path between vertices A and F has a length of 9.

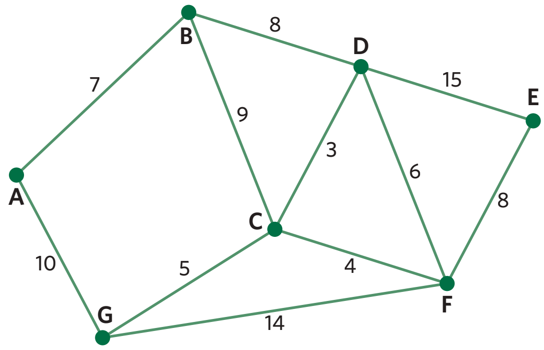
**B.** The edge connecting vertices D and G has a length of 5.

**C.** There must be an edge connecting vertices B and F.

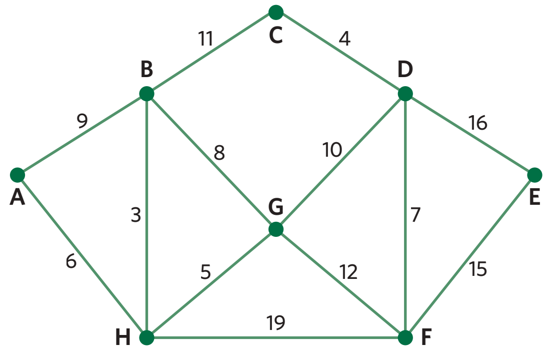
**D.** Vertex E is not connected directly to vertices A, B or D.

7. Use Dijkstra's algorithm to find, for each of the following weighted networks:
- the length of the shortest path from vertex A to vertex E.
- the route of the shortest path from vertex A to vertex E.

a.



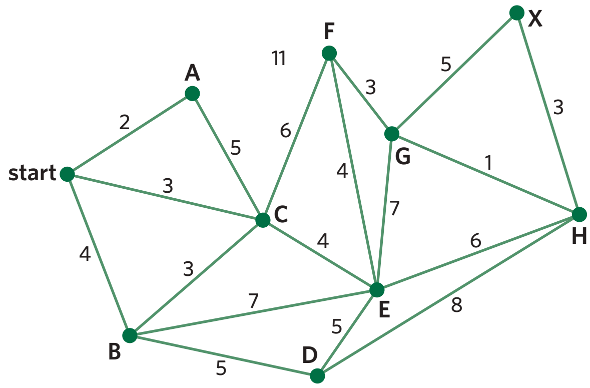

b.




8. During an orienteering task at a school camp, students are given a map and are tasked with getting to point X in the shortest time possible. One group of students converted the map into the following weighted network diagram. Each weight represents the time, in minutes, it would take to travel down the path.

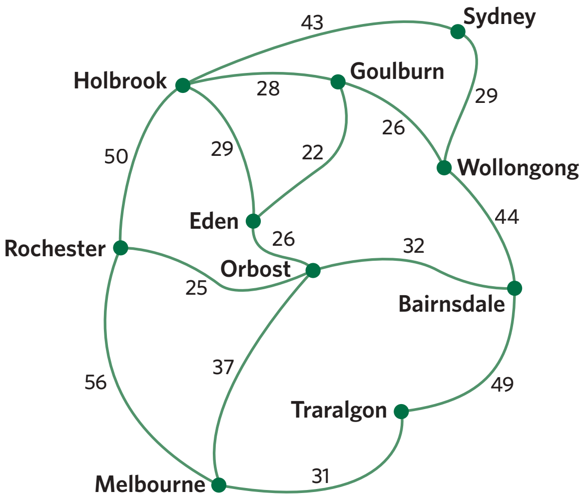
   a. What is the shortest possible time it would take the students to get from the start to point X?

   b. What is the route of the shortest path?




## Joining it all together

9. Tom is a truck driver who frequently travels between Melbourne and Sydney. He has calculated his petrol costs for different routes passing through small towns on the way. All values are correct to the nearest dollar.




   a. By inspection, what is the minimum cost of petrol from Melbourne to Bairnsdale?

   **A.** \$63 **B.** \$69 **C.** \$75 **D.** \$80

   b. When driving from Melbourne to Sydney, what is the minimum amount that Tom has to spend on petrol?

c. While following this route that allows for the minimum cost, which one of the following towns will Tom pass through?

A. Rochester B. Wollongong C. Goulburn D. Eden

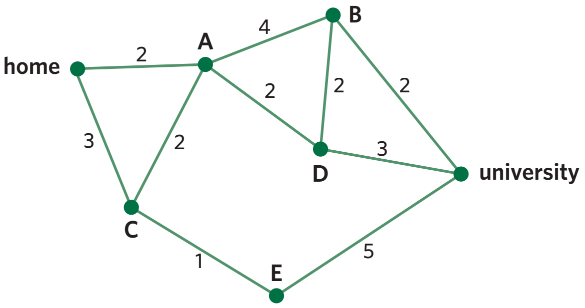
d. Tom has remembered that his favourite bakery is in Goulburn, and decides that he is willing to pay the extra money in order to visit it on his way to Sydney. What is his new minimum cost for petrol?

## Exam practice

**10.** Niko drives from his home to university.

The following network shows the distances, in kilometres, along a series of streets connecting Niko's home to the university.

The vertices A, B, C, D and E represent the intersection of these streets.




The shortest path for Niko from his home to the university could be found using

A. a minimum cut.

B. Prim's algorithm.

C. Dijkstra's algorithm.

D. a minimum spanning tree.

E. an Eulerian trail.

*Adapted from VCAA 2018 Exam 1 Networks and decision mathematics Q2*

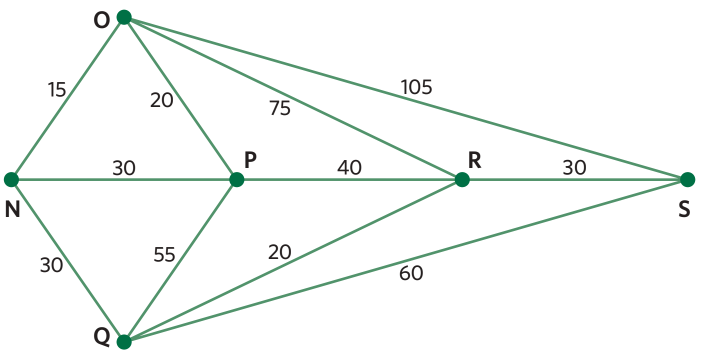
**68%** of students answered this type of question correctly.

**11.** Bus routes connect six towns.

The towns are Northend (N), Opera (O), Palmer (P), Quigley (Q), Rosebush (R) and Seatown (S).

The following graph gives the cost, in dollars, of bus travel along these routes.

Bai lives in Northend (N) and he will travel by bus to take a holiday in Seatown (S).




If Bai takes the cheapest route from Northend (N) to Seatown (S), which other town(s) will he pass through? (1 MARK)

*VCAA 2017 Exam 2 Networks and decision mathematics Q1b*

**79%** of students answered this question correctly.

# Questions from multiple lessons

## Data analysis

**12.** You are completing some research on how students at your school use public transport. You have decided to focus on collecting information on two variables:

- *trips per week* (less than 5, 5–10, more than 10)
- *public transport type* (bus, tram, train)

These variables are

**A.** numerical and categorical respectively.

**B.** ordinal and nominal respectively.

**C.** nominal and ordinal respectively.

**D.** both ordinal.

**E.** both nominal.

*Adapted from VCAA 2017 Exam 1 Data analysis Q7*

## Networks and decision mathematics

**13.** A connected planar graph has 7 faces.

This graph could have

**A.** eight vertices and nine edges.

**B.** seven vertices and twelve edges.

**C.** twelve vertices and seven edges.

**D.** ten vertices and thirteen edges.

**E.** nine vertices and eight edges.

*Adapted from VCAA 2018 Exam 1 Networks and decision mathematics Q3*

## Recursion and financial modelling

**14.** Daniel has opened a savings account to buy a new motorbike.

He puts the $5000 that he has saved up in a savings account which pays a 3% interest rate, compounding annually.

**a.** Complete the recurrence relation to model the amount of money, in dollars, in the savings account after $n$ years, $V_n$. (1 MARK)

$V_0 = \square, \quad V_{n+1} = \square \times V_n$

**b.** The amount of money, in dollars, in the account after $n$ years, $V_n$, can be determined using a rule. Complete the rule by writing the appropriate numbers in the boxes provided. (1 MARK)

$V_n = \square^n \times \square$

**c.** How much money will Daniel have in his savings account after 5 years? Round to the nearest cent. (1 MARK)

*Adapted from VCAA 2016 Exam 2 Data analysis Q5*

# 8G Matching problems

**STUDY DESIGN DOT POINTS**

- use of a bipartite graph and its tabular or matrix form to represent a matching problem
- determination of the optimum assignment(s) of people or machines to tasks by inspection or by use of the Hungarian algorithm for larger scale problems



**KEY SKILLS**

During this lesson, you will be:

- representing matching problems using a bipartite graph
- solving matching problems.

**KEY TERMS**

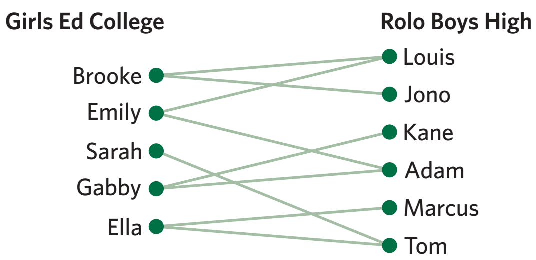
- Bipartite graph
- Hungarian algorithm

There are often situations in which activities or tasks need to be allocated or matched in a way to minimise costs or time, or to maximise scores. An example of this would be the allocation of swimmers to different strokes in a medley relay race to achieve the fastest time. These allocations can be represented and solved using bipartite graphs and the Hungarian algorithm.

## Representing matching problems using a bipartite graph

A **bipartite graph** can be used to show connections between vertices which fall into two separate groups. A vertex cannot be directly connected to another vertex in the same group.

For example, the following bipartite graph shows students from two different schools that have met at a party.




Bipartite graphs are sometimes shown with arrows instead of lines. This is to show the direction of the connection from one vertex to another.

**Worked example 1**

A random survey of individuals' sports preferences was conducted. The results of 4 random individuals from the survey are shown in the following table. Use the information from the table to construct a bipartite graph.

| individual | preferences |
|---|---|
| James | basketball, soccer, tennis |
| Alice | football |
| Harry | soccer, tennis |
| Nathan | soccer, tennis, football |

Continues →

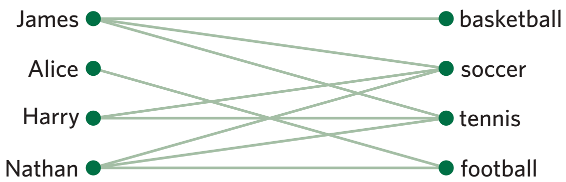
### Explanation

**Step 1:** Draw two sets of dots representing the two sets of groups.

There are 4 different sports, so there will be 4 dots, each representing a sport.


**Step 2:** Draw lines to connect individuals to the sport they like based on the information from the table.

**Answer**




## Solving matching problems

Matching problems are situations in which multiple allocations of tasks or activities can be made. The optimal allocation is the allocation that minimises costs, such as time or money, or maximises a desirable outcome such as scores.

For smaller scale matching problems, the optimal allocation can be solved using inspection. However, to solve large scale matching problems, the Hungarian algorithm must be used.

See worked example 2

See worked example 3

The **Hungarian algorithm** uses cost matrices to find the optimal allocation, or the allocation which will give the minimum cost.

The following steps can be used to find the optimal allocation for any cost matrix. The table shown gives information relating to the amount of time a group of students would each take to prepare a presentation on a subject in days.

| | art | maths | science | drama |
|---|---|---|---|---|
| **Alice** | 3 | 6 | 8 | 10 |
| **Bonnie** | 7 | 3 | 9 | 11 |
| **Carl** | 8 | 6 | 2 | 4 |
| **Danny** | 4 | 7 | 8 | 7 |

1. Locate the lowest value in each row. Subtract it from each element in the same row.
2. Find the minimum number of lines required to cover all of the zeros in the table.

   If this number is the same as the number of allocations needed to be made, jump to step 7.

| | art | maths | science | drama |
|---|---|---|---|---|
| **Alice** | 0 | 3 | 5 | 7 |
| **Bonnie** | 4 | 0 | 6 | 8 |
| **Carl** | 6 | 4 | 0 | 2 |
| **Danny** | 0 | 3 | 4 | 3 |

3. Locate the lowest value in each column. This may be a zero. Subtract it from each element in the same column.

4. Find the minimum number of lines required to cover all of the zeros in the table.

   If this number is the same as the number of allocations needed to be made, jump to step 7.

| | art | maths | science | drama |
|---|---|---|---|---|
| **Alice** | 0 | 3 | 5 | 5 |
| **Bonnie** | 4 | 0 | 6 | 6 |
| **Carl** | 6 | 4 | 0 | 0 |
| **Danny** | 0 | 3 | 4 | 1 |

5. Locate the lowest value that is not covered by a line. Subtract it from all uncovered values and add it to all values covered by two lines.
6. Find the minimum number of lines required to cover all of the zeros in the table.

   If it is not the same as the number of allocations to be made, repeat step 5.

| | art | maths | science | drama |
|---|---|---|---|---|
| **Alice** | 0 | 3 | 4 | 4 |
| **Bonnie** | 4 | 0 | 5 | 5 |
| **Carl** | 7 | 5 | 0 | 0 |
| **Danny** | 0 | 3 | 3 | 0 |

7. Allocate the activities based on the location of zeros.

| | art | maths | science | drama |
|---|---|---|---|---|
| **Alice** | 0 | 3 | 4 | 4 |
| **Bonnie** | 4 | 0 | 5 | 5 |
| **Carl** | 7 | 5 | 0 | 0 |
| **Danny** | 0 | 3 | 3 | 0 |

Alice: art, Bonnie: maths, Carl: science, Danny: drama

## Worked example 2

A university group assignment details a writing task on a research topic. The members in the group decide to split up the work based on which member was the most efficient at each task. Using inspection, find the minimum cost of the following cost matrix. The cost values are in hours.

| | planning | researching | writing | editing |
|---|---|---|---|---|
| **Bryan** | 1 | 4 | 7 | 5 |
| **Mason** | 4 | 5 | 3 | 7 |
| **Lily** | 2 | 2 | 3 | 7 |
| **Carmen** | 8 | 7 | 6 | 4 |

### Explanation

**Step 1:** Determine the most efficient individual at each stage.

For the planning stage, Bryan is the most efficient with a time of 1 hour.

For the researching stage, Lily is the most efficient with a time of 2 hours.

For the writing stage, Mason and Lily are the most efficient with a time of 3 hours.

For the editing stage, Carmen is the most efficient with a time of 4 hours.

**Step 2:** Find the optimal allocation.

Bryan: planning, Lily: researching, Mason: writing, Carmen: editing

Both Mason and Lily are the most efficient for the writing stage, however, Lily is the most efficient for the researching stage. This means Lily will be allocated to researching, and Mason will be allocated to writing.

Continues →

**Step 3:** Calculate the minimum cost.

$1 + 2 + 3 + 4 = 10$

### Answer

10 hours

## Worked example 3

Using the Hungarian algorithm, find the minimum cost of the following cost matrix. The cost values are in hours.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 4 | 4 | 2 | 4 |
| **Bryce** | 4 | 3 | 4 | 5 |
| **Cherry** | 2 | 5 | 6 | 3 |
| **Dinnith** | 8 | 7 | 4 | 6 |

### Explanation

**Step 1:** Locate the lowest value in each row. Subtract it from each element in the same row.

| | activity A | activity B | activity C | activity D | number subtracted |
|---|---|---|---|---|---|
| **Amy** | 2 | 2 | 0 | 2 | 2 |
| **Bryce** | 1 | 0 | 1 | 2 | 3 |
| **Cherry** | 0 | 3 | 4 | 1 | 2 |
| **Dinnith** | 4 | 3 | 0 | 2 | 4 |

**Step 2:** Find the minimum number of lines required to cover all of the zeros in the table.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 2 | 2 | 0 | 2 |
| **Bryce** | 1 | 0 | 1 | 2 |
| **Cherry** | 0 | 3 | 4 | 1 |
| **Dinnith** | 4 | 3 | 0 | 2 |

Only three lines are required, but there are four allocations to be made.

Allocation is not yet possible.

**Step 3:** Locate the lowest value in each column. This may be a zero. Subtract it from each element in the same column.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 2 | 2 | 0 | 1 |
| **Bryce** | 1 | 0 | 1 | 1 |
| **Cherry** | 0 | 3 | 4 | 0 |
| **Dinnith** | 4 | 3 | 0 | 1 |
| **number subtracted** | 0 | 0 | 0 | 1 |

Continues →

**Step 4:** Find the minimum number of lines required to cover all of the zeros in the table.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 2 | 2 | 0 | 1 |
| **Bryce** | 1 | 0 | 1 | 1 |
| **Cherry** | 0 | 3 | 4 | 0 |
| **Dinnith** | 4 | 3 | 0 | 1 |

Only three lines are required, but there are four allocations to be made.

Allocation is not yet possible.

**Step 5:** Locate the lowest value that is not covered by a line. Subtract it from all uncovered values and add it to all values covered by two lines.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 1 | 1 | 0 | 0 |
| **Bryce** | 1 | 0 | 2 | 1 |
| **Cherry** | 0 | 3 | 5 | 0 |
| **Dinnith** | 3 | 2 | 0 | 0 |

**Step 6:** Find the minimum number of lines required to cover all of the zeros in the table.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 1 | 1 | 0 | 0 |
| **Bryce** | 1 | 0 | 2 | 1 |
| **Cherry** | 0 | 3 | 5 | 0 |
| **Dinnith** | 3 | 2 | 0 | 0 |

Four lines are required.

Allocation is now possible.

**Step 7:** Allocate the activities based on the location of zeros.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 1 | 1 | 0 | 0 |
| **Bryce** | 1 | 0 | 2 | 1 |
| **Cherry** | 0 | 3 | 5 | 0 |
| **Dinnith** | 3 | 3 | 0 | 0 |

Cherry will complete activity A.

Bryce will complete activity B.

Amy and Dinnith may both complete either activity C or D.

**Step 8:** Calculate the minimum cost using the cost matrix.

| | activity A | activity B | activity C | activity D |
|---|---|---|---|---|
| **Amy** | 4 | 4 | 2 | 4 |
| **Bryce** | 4 | 3 | 4 | 5 |
| **Cherry** | 2 | 5 | 6 | 3 |
| **Dinnith** | 8 | 7 | 4 | 6 |

If Amy completes activity C and Dinnith completes activity D: $2 + 3 + 2 + 6 = 13$

If Amy completes activity D and Dinnith completes activity C: $2 + 3 + 4 + 4 = 13$

## Answer

13 hours

## Exam question breakdown

*VCAA 2016 Exam 1 Networks and decision mathematics Q8*

Five children, Alan, Brianna, Chamath, Deidre and Ewen, are each to be assigned a different job by their teacher. The following table shows the time, in minutes, that each child would take to complete each of the five jobs.

| | Alan | Brianna | Chamth | Deidre | Ewen |
|---|---|---|---|---|---|
| **job 1** | 5 | 8 | 5 | 8 | 7 |
| **job 2** | 5 | 7 | 6 | 7 | 4 |
| **job 3** | 9 | 5 | 7 | 5 | 9 |
| **job 4** | 7 | 7 | 9 | 8 | 5 |
| **job 5** | 4 | 4 | 4 | 4 | 3 |

Continues →

The teacher wants to allocate the jobs so as to minimise the total time taken to complete five jobs. In doing so, she finds that two allocations are possible.

If each child starts their allocated job at the same time, then the first child to finish could be either

**A.** Alan or Brianna

**B.** Brianna or Deidre

**C.** Chamath or Deidre

**D.** Chamth or Ewen

**E.** Deidre or Ewen

## Explanation

**Step 1:** Apply the Hungarian algorithm.

Subtract the lowest value in each row from all elements in the same row and check the number of lines required to cover the zeros. There are 3 lines required to cover all the zeros.

As each column contains a zero, subtracting the lowest value in each column from all elements in the same column has no effect.

Subtract the smallest uncovered number ('1') from all uncovered elements and add it to elements that are covered twice.

| | Alan | Brianna | Chamth | Deidre | Ewen |
|---|---|---|---|---|---|
| **job 1** | 0 | 3 | 0 | 3 | 3 |
| **job 2** | 0 | 2 | 1 | 2 | 0 |
| **job 3** | 4 | 0 | 2 | 0 | 5 |
| **job 4** | 1 | 1 | 3 | 2 | 0 |
| **job 5** | 0 | 0 | 0 | 0 | 0 |

**Step 2:** Allocate the activities based on the location of zeros.

Job 4 must be allocated to Ewen.

As Ewen is allocated to job 4, job 2 must be allocated to Alan.

As Alan is allocated to job 2, job 1 must be allocated to Chamth.

Brianna or Deidre can be allocated to either job 3 or job 5.

**Step 3:** Compare and interpret the two possible allocations.

The two possible allocations are shown in the following table.

| | | Alan | Brianna | Chamath | Deidre | Ewen |
|---|---|---|---|---|---|---|
| **allocation 1** | **job** | 2 | 3 | 1 | 5 | 4 |
| | **completion time** | 5 min | 5 min | 5 min | 4 min | 5 min |
| **allocation 2** | **job** | 2 | 5 | 1 | 3 | 4 |
| | **completion time** | 5 min | 4 min | 5 min | 5 min | 5 min |

The child that finishes their allocated job the fastest could be either Deidre or Brianna

**30%** of students answered this question correctly.

Many students applied the Hungarian algorithm incorrectly. In addition, some students did not recognise that three of the children had a single optimal allocation.

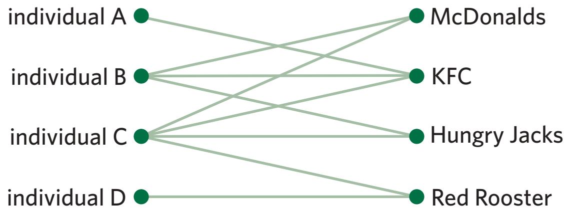
## Answer

B

# 8G Questions

## Representing matching problems using a bipartite graph

**1.** The fast-food preferences of four random individuals are displayed in the following bipartite graph.




Which of the following tables correctly shows the fast-food preferences of the four individuals?

**A.**

| individual | preferences |
|---|---|
| A | KFC |
| B | McDonalds, KFC, Hungry Jacks |
| C | McDonalds, KFC, Hungry Jacks, Red Rooster |
| D | Red Rooster |

**B.**

| individual | preferences |
|---|---|
| A | KFC |
| B | KFC, Hungry Jacks |
| C | McDonalds, KFC, Hungry Jacks |
| D | Red Rooster |

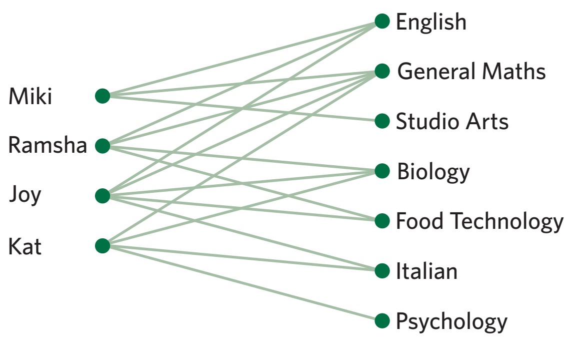
**C.**

| individual | preferences |
|---|---|
| A | KFC |
| B | McDonalds, KFC, Hungry Jacks, Red Rooster |
| C | McDonalds, KFC, Hungry Jacks |
| D | Red Rooster |

**D.**

| individual | preferences |
|---|---|
| A | Red Rooster |
| B | McDonalds, KFC, Hungry Jacks, Red Rooster |
| C | McDonalds, KFC, Hungry Jacks |
| D | KFC |

**2.** Miki, Ramsha, Joy, and Kat are trying to set up a study group for each of their Year 12 subjects. The bipartite graph shows which subjects the students have in common.




**a.** How many of the students study English?

**b.** Which subject has the most students studying it?

**c.** The students want to set up a study group only for subjects that three or more of them are studying. How many study groups will they have?

**d.** Which study groups will Miki and Ramsha be in together?

**3.** Priya, Rhys, Ted, Georgia, Natalia, and Andy were all asked to compete in one event at the school athletics carnival. Each student was asked to write down which events they would be happy to participate in. This is summarised in the following table.

Construct a bipartite graph to represent the following information.

| student | events |
|---|---|
| Priya | 100 m, 200 m, high jump |
| Rhys | 100 m, 200 m, high jump |
| Ted | 800 m |
| Georgia | 800 m, javelin |
| Natalia | 100 m, 200 m, triple jump, high jump |
| Andy | high jump, javelin |

## Solving matching problems



**4.** The following table summarises the cost, in dollars, of producing three types of clothing at three different factories.

| | shirts | pants | hoodies |
|---|---|---|---|
| **factory A** | 2 | 3 | 5 |
| **factory B** | 5 | 1 | 5 |
| **factory C** | 6 | 7 | 4 |

Using inspection, which of the following allocations will minimise the total cost of producing the different types of clothing?

**A.** factory A: hoodies, factory B: shirts, factory C: pants

**B.** factory A: pants, factory B: hoodies, factory C: shirts

**C.** factory A: shirts, factory B: hoodies, factory C: pants

**D.** factory A: shirts, factory B: pants, factory C: hoodies

---

**5.** Khan, Henry, Jamie and Mya have been assigned to work with each other on a group project. The project involves four different sections: research, writing, editing, and presentation. Each member of the group will complete one section.

Table 1 shows the time, in minutes, it would take each member of the group to complete each section.

**Table 1**

| | research | writing | editing | presentation |
|---|---|---|---|---|
| **Khan** | 60 | 70 | 65 | 65 |
| **Henry** | 50 | 60 | 40 | 70 |
| **Jamie** | 45 | 60 | 50 | 70 |
| **Mya** | 60 | 55 | 60 | 75 |

The sections will be allocated based on the minimum total time to complete the entire project.

The Hungarian algorithm will be used to make the allocation.

The first step of the Hungarian algorithm is to locate the lowest value in each row and subtract that value from each element in the row. The first step is shown in Table 2, however the values for Mya are missing.

**Table 2**

| | research | writing | editing | presentation |
|---|---|---|---|---|
| **Khan** | 0 | 10 | 5 | 5 |
| **Henry** | 10 | 20 | 0 | 30 |
| **Jamie** | 0 | 15 | 5 | 25 |
| **Mya** | | | | |

**a.** Complete Table 2.

**b.** Explain why it is not yet possible to allocate sections of the project.

The steps of the Hungarian algorithm have continued to be applied until allocation is possible.

**Table 3**

| | research | writing | editing | presentation |
|---|---|---|---|---|
| **Khan** | 0 | 10 | 5 | 0 |
| **Henry** | 10 | 20 | 0 | 25 |
| **Jamie** | 0 | 15 | 5 | 20 |
| **Mya** | 5 | 0 | 5 | 15 |

c. Explain why Jamie must complete the research section instead of Khan.

d. Determine the final allocation for each section. If there is more than one possible allocation, provide them all.

e. How long will it take the group to complete the project?

6. Anna, Margot, Tina, and Svetlana have all been asked to perform a solo piece at their end of year dance concert. To keep the concert exciting, they will each perform a different discipline of dance; ballet, jazz, hip hop and contemporary. They have been ranked from 1 to 4 on their recent performances in each discipline. The best dancer achieves a ranking of 1 for that discipline.

| | ballet | jazz | hip hop | contemporary |
|---|---|---|---|---|
| **Anna** | 3 | 1 | 2 | 2 |
| **Margot** | 4 | 3 | 1 | 4 |
| **Tina** | 2 | 2 | 3 | 3 |
| **Svetlana** | 1 | 4 | 4 | 1 |

Use the Hungarian algorithm to determine which discipline each dancer should perform. If there is more than one possible allocation, provide them all.

7. The Question Creation Team (QCT) at Edrolo wanted to minimise mistakes in their textbook. They decided to have each QCT member be responsible for checking for one of the following mistake types: clarity, grammar, mathematical errors, and other. To decide who was best placed to look for each mistake type, they recorded the number of each type of mistake each QCT member made themselves in one lesson.

| | Sophie | Zephyr | Anna | Cassie |
|---|---|---|---|---|
| **clarity** | 10 | 8 | 21 | 9 |
| **grammar** | 17 | 17 | 11 | 9 |
| **mathematical errors** | 2 | 1 | 4 | 4 |
| **other** | 9 | 4 | 5 | 8 |

Use the Hungarian algorithm to allocate the QCT members to a mistake type. The allocation should be based on the minimum number of mistakes possible. If there is more than one possible allocation, provide them all.

## Joining it all together

8. There are four builders available to complete four tasks, A, B, C and D, for a small construction project. All builders are qualified to complete all tasks.

a. Create a bipartite graph showing each of the builders and the tasks they are able to complete.

The building company wishes to complete the project for the lowest total price possible. In order to complete the project on time, each builder must complete a different task. The table below shows the cost, in dollars, for each worker to complete each task.

| | task A | task B | task C | task D |
|---|---|---|---|---|
| **builder 1** | 100 | 80 | 120 | 90 |
| **builder 2** | 130 | 110 | 100 | 110 |
| **builder 3** | 120 | 110 | 120 | 80 |
| **builder 4** | 90 | 80 | 130 | 100 |

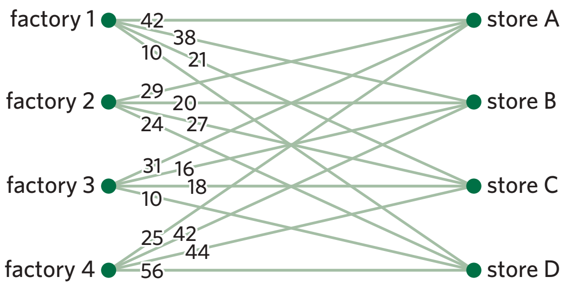
The first step of the Hungarian algorithm involves locating the lowest value in each row and subtracting that value from each element in the row.

b. Complete the first step of the Hungarian algorithm.

c. Allocate each of the builders to a task which will provide the lowest total cost for the building company.

d. Calculate the cost of this allocation.

**9.** Bedazzled Bathers has four different production factories. Four retail stores wish to stock bathers from Bedazzled Bathers. The distance, in kilometres, from each production factory to each retail store is shown in the following bipartite graph.

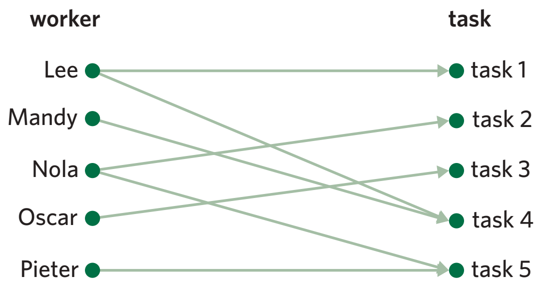



Each factory only makes enough bathers to stock one retail store.

Calculate the lowest possible total distance for all retail stores to receive bathers from a different factory.

## Exam practice

**10.** Lee, Mandy, Nola, Oscar and Pieter are each to be allocated one particular task at work. The following bipartite graph shows which task(s), 1–5, each person is able to complete.




Each person completes a different task.

Task 4 must be completed by

**A.** Lee

**B.** Mandy

**C.** Nola

**D.** Oscar

**E.** Pieter

*VCAA 2016 Exam 1 Networks and decision mathematics Q1*

**97%** of students answered this question correctly.

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**11.** A cricket team has 11 players who are each assigned to a batting position.

Three of the new players, Alex, Bo and Cameron, can bat in position 1, 2, or 3.

The following table shows the average scores, in runs, for each player for the batting positions 1, 2 and 3.

| | | batting position | | |
|---|---|---|---|---|
| | | **1** | **2** | **3** |
| **player** | **Alex** | 22 | 24 | 24 |
| | **Bo** | 25 | 25 | 21 |
| | **Cameron** | 24 | 25 | 19 |

Each player will be assigned to one batting position.

To which position should each player be assigned to maximise the team's score? (1 MARK)

*VCAA 2020 Exam 2 Networks and decision mathematics Q2*

**57%** of students answered this question correctly.

**12.** Bai joins his friend Agatha, Colin and Diane when he arrives for a holiday in Seatown.

Each person will plan one tour that the group will take.

Table 1 shows the time, in minutes it would take each person to plan each of the four tours.

**Table 1**

| | Agatha | Bai | Colin | Diane |
|---|---|---|---|---|
| **tour 1** | 13 | 7 | 13 | 12 |
| **tour 2** | 14 | 9 | 8 | 7 |
| **tour 3** | 19 | 25 | 21 | 18 |
| **tour 4** | 10 | 7 | 11 | 10 |

The aim is to minimise the total time it takes to plan the four tours.

Agatha applies the Hungarian algorithm to Table 1 to produce Table 2.

Table 2 shows the final result of all her steps of the Hungarian algorithm.

**Table 2**

| | Agatha | Bai | Colin | Diane |
|---|---|---|---|---|
| **tour 1** | 3 | 0 | 3 | 3 |
| **tour 2** | 6 | 4 | 0 | 0 |
| **tour 3** | 0 | 9 | 2 | 0 |
| **tour 4** | 0 | 0 | 1 | 1 |

In Table 2 there is a zero in the column for Colin.

When all values in the table are considered, what conclusion about minimum total planning time can be made from this zero? (1 MARK)

*VCAA 2017 Exam 2 Networks and decision mathematics Q2a*

**37%** of students answered this question correctly.

---

**13.** Annie, Buddhi, Chuck and Dorothy work in a factory.

Today each worker will complete one of four tasks, 1, 2, 3 and 4.

The usual completion times for Annie, Chuck and Dorothy are shown in the following table.

Buddhi takes 3 minutes for task 3.

He takes $k$ minutes for each other task.

| | task 1 | task 2 | task 3 | task 4 |
|---|---|---|---|---|
| **Annie** | 7 | 3 | 8 | 2 |
| **Buddhi** | $k$ | $k$ | 3 | $k$ |
| **Chuck** | 5 | 6 | 9 | 2 |
| **Dorothy** | 4 | 8 | 5 | 3 |

Today the factory supervisor allocates the tasks as follows:

- Task 1 to Dorothy
- Task 2 to Annie
- Task 3 to Buddhi
- Task 4 to Chuck

This allocation will achieve the minimum total completion time if the value of $k$ is at least

**A.** 0 **B.** 1 **C.** 2
**D.** 3 **E.** 4

*VCAA 2018 Exam 1 Networks and decision mathematics Q8*

**24%** of students answered this question correctly.

## Questions from multiple lessons

### Recursion and financial modelling

**14.** Four lines of an amortisation table for an annuity investment are shown in the following table.

The interest rate for this investment remains constant, but the payment value may vary.

| payment number | payment | interest | principal addition | balance of investment |
|---|---|---|---|---|
| 27 | 250 | 114.64 | 364.64 | 14 695.14 |
| 28 | 250 | 117.56 | 367.56 | 15 062.71 |
| 29 | 250 | 120.50 | 370.50 | 15 433.21 |
| 30 | | | | 16 000.00 |

The balance of the investment after payment number 30 is $16 000.

The value of payment number 30 is closest to

**A.** $123 **B.** $250 **C.** $373 **D.** $443 **E.** $567

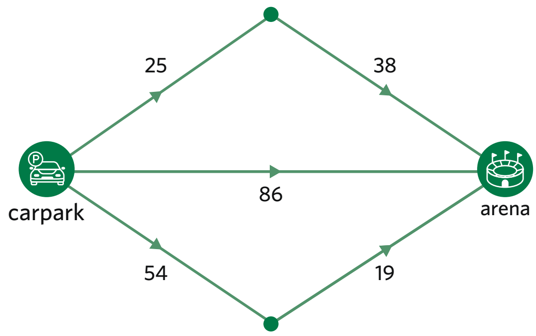
*Adapted from VCAA 2017 Exam 1 Data analysis Q23*

### Networks and decision mathematics

**15.** The following directed graph shows the flow of traffic, in people per minute, in a system of paths connecting a car park to an arena.

The maximum flow, in people per minute, from the car park to the arena is

**A.** 73
**B.** 86
**C.** 130
**D.** 143
**E.** 178




*Adapted from VCAA 2016 Exam 1 Networks and decision mathematics Q2*

### Networks and decision mathematics

**16.** In Venice, Italy, a gondola rider takes tourists to nine destinations along canals labelled as vertices A to I on the following graph.

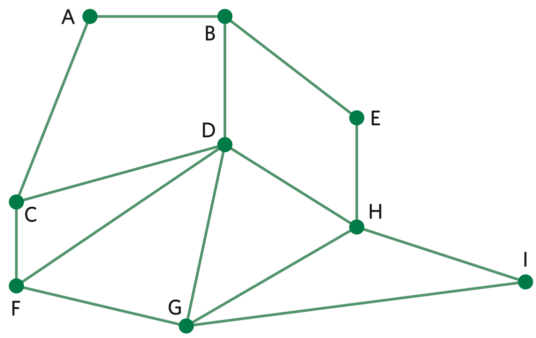
**a.** Which one of the vertices on the graph has a degree of 5? (1 MARK)

**b.** For this graph, an Eulerian trail does not currently exist.

For an Eulerian trail to exist, what is the minimum number of extra edges that the graph would require? (1 MARK)

**c.** The gondola rider has taken tourists to D and continues the tour by following a Hamiltonian path from D.

Draw a possible Hamiltonian path for the gondola rider. (1 MARK)




*Adapted from VCAA 2018 Exam 2 Networks and decision mathematics Q2*

# 8H Activity networks and precedence tables

STUDY DESIGN DOT POINT

- construction of an activity network from a precedence table (or equivalent) including the use of dummy activities where necessary



**KEY SKILLS**

During this lesson, you will be:

- interpreting precedence tables
- constructing activity networks
- constructing activity networks with dummy activities.

**KEY TERMS**

- Immediate predecessor
- Precedence table
- Activity network
- Dummy activity

Networks can be used to visualise the workflow of large projects when the individual activities within it are known, as well as the order in which activities need to be completed. Modelling projects using networks is useful for assigning tasks with the goal of increasing efficiency.

## Interpreting precedence tables

In projects that require multiple activities to be completed, an **immediate predecessor** is an activity that must be completed directly before another activity can begin. For example, when getting dressed, putting on socks is an immediate predecessor to putting on shoes.

A **precedence table** displays all of the activities within a project, and each of their immediate predecessors.

### Worked example 1

Consider the following precedence table.

| activity | immediate predecessor(s) |
|---|---|
| $A$ | – |
| $B$ | $A$ |
| $C$ | $A$ |
| $D$ | $B$ |
| $E$ | $A, D$ |
| $F$ | $E$ |
| $G$ | $F$ |
| $H$ | $C$ |

What activities need to be completed before activity $F$ can begin?

#### Explanation

**Step 1:** Determine the immediate predecessors of activity $F$.

The immediate predecessor of activity $F$ is activity $E$.

Continues →

**Step 2:** Continue to determine the immediate predecessors of activities that occur before *F*.

The immediate predecessors of activity *E* are activities *A* and *D*.

The immediate predecessor of activity *D* is activity *B*.

The immediate predecessor of activity *B* is activity *A*.

Activity *A* has no immediate predecessors.

**Answer**

*A*, *B*, *D*, *E*

## Constructing activity networks

An **activity network** is a visual display of the information in a precedence table. Each activity is represented by a directed edge. The vertices remain unlabelled, except for the vertices that represent the start and finish of the project.

See worked example 2

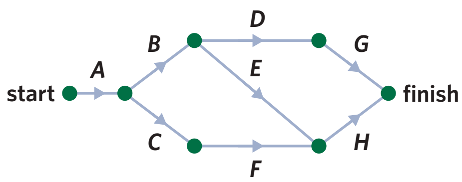
Activities with no immediate predecessors are represented as edges leading from the start vertex. Activities that are not immediate predecessors to any other activities are represented by edges leading to the finish vertex.

A precedence table can also be created from an activity network.

See worked example 3

For example, the following precedence table and activity network display the same project.

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *B* |
| *E* | *B* |
| *F* | *C* |
| *G* | *D* |
| *H* | *E*, *F* |




### Worked example 2

Construct an activity network from the following precedence table.

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | *A* |
| *D* | *A* |
| *E* | *B* |
| *F* | *B* |
| *G* | *D*, *E* |
| *H* | *C*, *G* |
| *I* | *F*, *H* |

Continues →

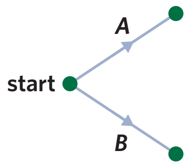
## Explanation

**Step 1:** Determine the starting activities.

The starting activities are those with no immediate predecessors.

Activities *A* and *B* have no immediate predecessors, so they are the starting activities.

Draw these activities as edges connected to the start vertex.

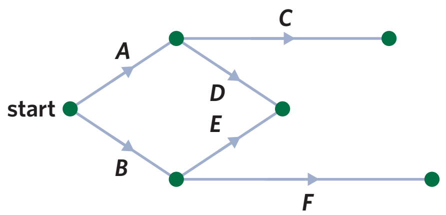



**Step 2:** Draw all activities with *A* or *B* as their immediate predecessor.

Activities *C* and *D* have activity *A* as their immediate predecessor.

Activities *E* and *F* have activity *B* as their immediate predecessor.

Activity *G* has activities *D* and *E* as its immediate predecessors, so activities *D* and *E* must meet at a single vertex.



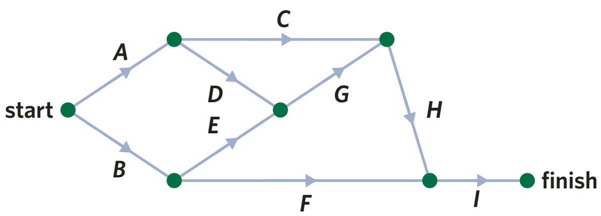

**Step 3:** Continue to draw activities based on their immediate predecessor(s).

Activity *G* has activities *D* and *E* as its immediate predecessors. Activity *H* has activities *C* and *G* as its immediate predecessors, so activity *G* must start at the end of activities *D* and *E* and reach the same vertex as activity *C*, which will lead to activity *H*.

Activity *I* has activities *F* and *H* as its immediate predecessors, so activity *H* must start at the end of activities *C* and *G* and reach the same vertex as activity *F*.

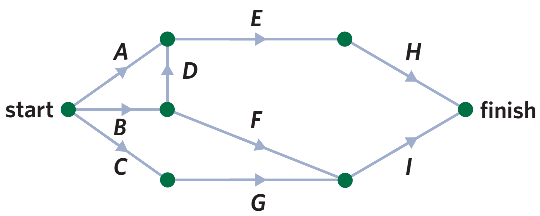
Activity *I* is not an immediate predecessor for any other activity, so it will connect to the finish vertex.

### Answer




## Worked example 3

Construct a precedence table from the following activity network.




Continues →

**Explanation**

**Step 1:** Construct a table.

The table will need 2 columns and a row for each activity.

Label each of the columns 'activity' and 'immediate predecessor(s)' respectively. Fill in the activity column with the names of each activity.

| activity | immediate predecessor(s) |
|---|---|
| *A* | |
| *B* | |
| *C* | |
| *D* | |
| *E* | |
| *F* | |
| *G* | |
| *H* | |
| *I* | |

**Step 2:** Enter a dash for each activity with no immediate predecessors.

Activities *A*, *B* and *C* start at the start vertex, so they have no immediate predecessors.

**Step 3:** Fill in the rest of the table.

The immediate predecessor of activity *D* is activity *B*.

The immediate predecessors of activity *E* are activities *A* and *D*.

The immediate predecessor of activity *F* is activity *B*.

The immediate predecessor of activity *G* is activity *C*.

The immediate predecessor of activity *H* is activity *E*.

The immediate predecessors of activity *I* are activities *F* and *G*.

**Answer**

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | – |
| *D* | *B* |
| *E* | *A*, *D* |
| *F* | *B* |
| *G* | *C* |
| *H* | *E* |
| *I* | *F*, *G* |

## Constructing activity networks with dummy activities

When activities in a project share some, but not all, of the same immediate predecessors, it becomes impossible to draw an activity network correctly. In these circumstances, a **dummy activity** can be used to show the additional predecessors.

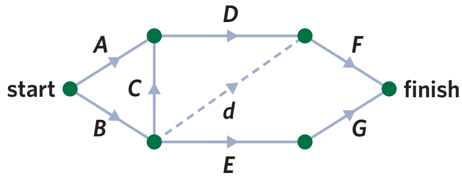
A dummy activity is drawn from the end of the shared immediate predecessor to the start of the activity that has an additional immediate predecessor.

Dummy activities are drawn using dotted lines, and are often denoted 'dummy' or '*d*'.

Note: A dummy activity does not actually exist as a real activity. It is merely a directed edge to properly show the immediate predecessors in an activity network.

For example, the following precedence table and activity network display the same project. The dummy activity runs from the end of activity *B* (shared immediate predecessor) to the start of activity *F* (activity with an additional immediate predecessor).

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | *B* |
| *D* | *A*, *C* |
| *E* | *B* |
| *F* | *B*, *D* |
| *G* | *E* |




## Worked example 4

Consider the following precedence table.

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *A* |
| *E* | *B* |
| *F* | *C* |
| *G* | *D* |
| *H* | *E*, *F* |
| *I* | *C*, *G* |

**a.** Complete the following sentence.

A dummy activity needs to be drawn from the end of activity ________ to the start of activity ________.

### Explanation

**Step 1:** Find activities that share some, but not all, of the same immediate predecessors.

Activity *F* has activity *C* as its immediate predecessor.

Activity *I* has activites *C* and *G* as its immediate predecessors.

**Step 2:** Find the activity that the dummy activity should start at.

A dummy activity is drawn from the end of the shared immediate predecessor.

The shared immediate predecessor is activity *C*.

**Step 3:** Find the activity that the dummy activity should end at.

A dummy activity will be drawn to the start of the activity with an additional immediate predecessor.

The activity with an additional shared immediate predecessor is activity *I*.

### Answer

A dummy activity needs to be drawn from the end of activity *C* to the start of activity *I*.

Continues →

**b.** Construct an activity network from the precedence table.

### Explanation

**Step 1:** Determine the starting activities.

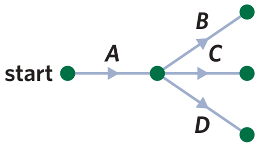
The starting activities are those with no immediate predecessors.

Activity *A* has no immediate predecessor and so is the starting activity.

Draw this as an edge connected to the start vertex.

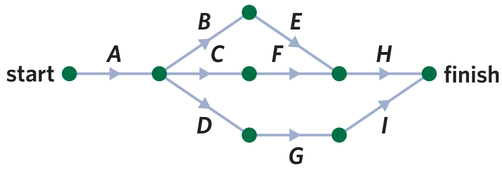

**Step 2:** Draw all activities with *A* as their immediate predecessor.

Activities *B*, *C* and *D* all have activity *A* as their immediate predecessor.




**Step 3:** Continue to draw activities based on their immediate predecessor(s).

Ignore activity *C* as an immediate predecessor of activity *I* as this will be shown as a dummy activity.

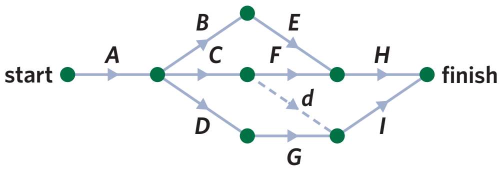



**Step 4:** Draw the dummy activity.

The dummy activity needs to be drawn from the end of activity *C* to the start of activity *I*.

It should be a dotted line denoted ‘*d*’.

### Answer




## Exam question breakdown

*VCAA 2019 Exam 1 Networks and decision mathematics Q7*

A project involves nine activities, *A* to *I*.

The immediate predecessor(s) of each activity is shown in the following table.

A directed network for this project will require a dummy activity.

The dummy activity will be drawn from the end of

**A.** activity *B* to the start of activity *C*.

**B.** activity *B* to the start of activity *E*.

**C.** activity *D* to the start of activity *E*.

**D.** activity *E* to the start of activity *H*.

**E.** activity *E* to the start of activity *F*.

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *B* |
| *E* | *B*, *C* |
| *F* | *D* |
| *G* | *D* |
| *H* | *E*, *F* |
| *I* | *G*, *H* |

### Explanation

**Step 1:** Find activities that share some, but not all, of the same immediate predecessors.

Activity *D* has activity *B* as its immediate predecessor.

Activity *E* has activites *B* and *C* as its immediate predecessors.

**Step 2:** Find the activity that the dummy activity should start at.

A dummy activity is drawn from the end of the shared immediate predecessor.

The shared immediate predecessor is activity *B*.

Continues →

**Step 3:** Find the activity that the dummy activity should end at.

A dummy activity will be drawn to the start of the activity with an additional immediate predecessor.

The activity with an additional shared immediate predecessor is activity *E*.

**Answer**

B

**42%** of students answered this question correctly.

**21%** of students incorrectly chose option A. These students correctly determined that activity *B* was a shared predecessor, and that the dummy should start at activity *B*, but could not determine where the dummy activity ends.

# 8H Questions

## Interpreting precedence tables

**1.** Consider the following precedence table.

What activities need to be completed before activity *E* can begin?

**A.** *C*

**B.** *A*, *C*

**C.** *A*, *B*, *C*

**D.** *A*, *B*, *C*, *D*, *F*

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *B* |
| *E* | *C* |
| *F* | *D* |
| *G* | *F*, *E* |

**2.** Hannah created a precedence table for a DIY project she wanted to complete.

Which activities must be completed before activity *H* can commence?

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *B* |
| *E* | *C* |
| *F* | *C* |
| *G* | *E* |
| *H* | *D*, *E* |
| *I* | *F*, *G* |

## Constructing activity networks

**3.** Consider the following activity network.

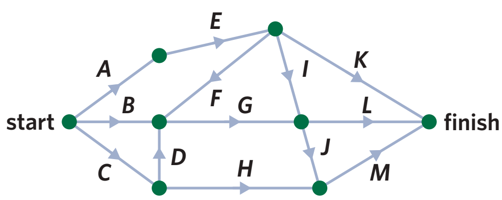
Which activities must be completed before activity G can commence?

**A.** *B*

**B.** *B*, *C*, *D*

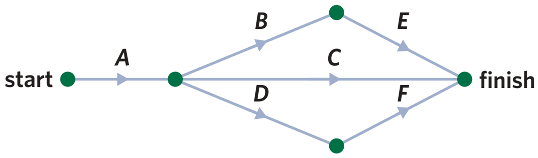
**C.** *A*, *B*, *C*, *D*

**D.** *A*, *B*, *C*, *D*, *E*, *F*

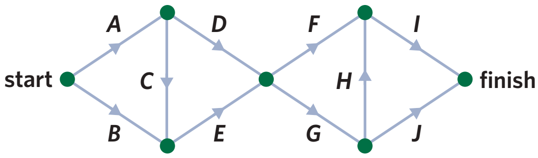


**4.** Construct a precedence table for each of the following activity networks.

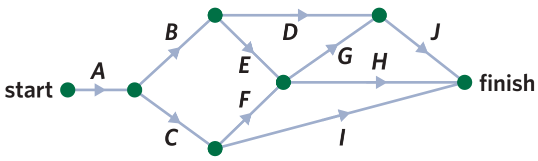
**a.**




**b.**




**c.**




**5.** Construct an activity network for each of the following precedence tables.

**a.**

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | – |
| *D* | *A* |
| *E* | *B* |
| *F* | *C* |
| *G* | *D*, *E* |

**b.**

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | *A* |
| *D* | *C* |
| *E* | *B*, *D* |
| *F* | *B*, *D* |
| *G* | *B*, *D* |
| *H* | *E* |
| *I* | *F* |
| *J* | *G* |

**c.**

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | – |
| *D* | *B* |
| *E* | *A* |
| *F* | *B* |
| *G* | *C*, *D* |
| *H* | *C*, *D* |
| *I* | *E*, *F* |
| *J* | *I*, *G* |
| *K* | *I*, *G* |
| *L* | *H* |
| *M* | *J* |

6. Dustin Martin has multiple strength and conditioning tests that he needs to complete before being considered fit to play in the AFL finals. The tests are shown as activities in the following precedence table.

Construct an activity network for Dustin Martin's strength and conditioning tests.

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | *A* |
| *D* | *A* |
| *E* | *D* |
| *F* | *D* |
| *G* | *B* |
| *H* | *C* |
| *I* | *F*, *G* |
| *J* | *H* |
| *K* | *E*, *I* |

## Constructing activity networks with dummy activities

7. Consider the project displayed in the following precedence table.

A corresponding activity network will include a dummy activity drawn from

**A.** the end of activity *B* to the start of activity *E*.

**B.** the end of activity *B* to the start of activity *H*.

**C.** the end of activity *D* to the start of activity *G*.

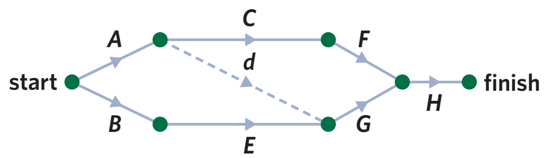
**D.** the end of activity *D* to the start of activity *H*.

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *A* |
| *E* | *B* |
| *F* | *D* |
| *G* | *E* |
| *H* | *B*, *C*, *F* |

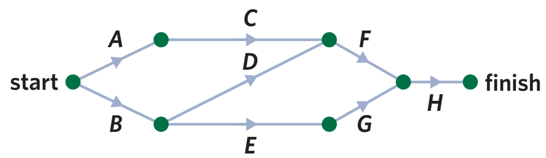
8. Which activity network corresponds to the following precedence table?

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | *A* |
| *D* | *C* |
| *E* | *B* |
| *F* | *C* |
| *G* | *A*, *E* |
| *H* | *F*, *G* |

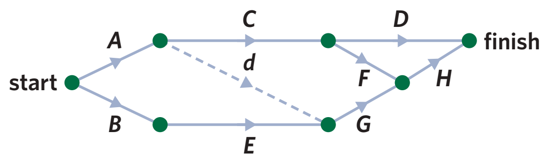
**A.**



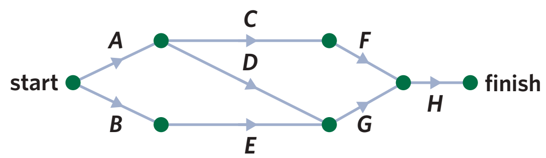

**B.**




**C.**




**D.**



**9.** For each of the following precedence tables, if a corresponding activity network was constructed, how many dummy activities would be drawn?

**a.**

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | – |
| *D* | *A* |
| *E* | *A*, *B* |
| *F* | *C* |
| *G* | *D* |
| *H* | *E* |
| *I* | *E*, *F* |

**b.**

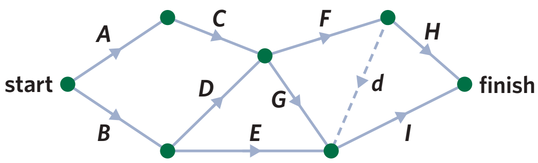
| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | *B* |
| *D* | *B* |
| *E* | *A*, *C* |
| *F* | *D*, *E* |
| *G* | *F* |
| *H* | *F* |

**c.**

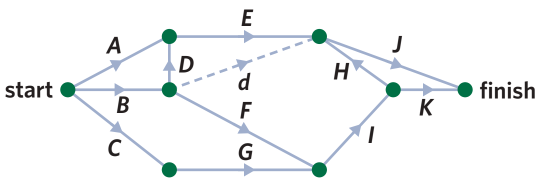
| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | *A* |
| *C* | *A* |
| *D* | *B* |
| *E* | *B* |
| *F* | *C*, *D* |
| *G* | *C*, *D* |
| *H* | *E*, *F* |
| *I* | *F* |
| *J* | *F*, *G* |

**10.** Construct a precedence table for each of the following activity networks.

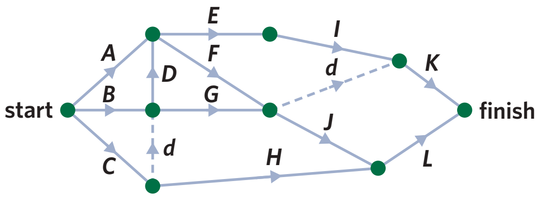
**a.**




**b.**




**c.**



**11.** Construct an activity network for each of the following precedence tables.

**a.**

| activity | immediate predecessor(s) |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | – |
| $D$ | $A$ |
| $E$ | $C$ |
| $F$ | $B, D$ |
| $G$ | $B, D, E$ |

**b.**

| activity | immediate predecessor(s) |
|---|---|
| $A$ | – |
| $B$ | $A$ |
| $C$ | $A$ |
| $D$ | $B$ |
| $E$ | $A, D$ |
| $F$ | $C$ |
| $G$ | $B$ |
| $H$ | $E$ |
| $I$ | $E, F$ |

**c.**

| activity | immediate predecessor(s) |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | $A$ |
| $D$ | $A$ |
| $E$ | $B, C, D$ |
| $F$ | $B, C$ |
| $G$ | $B, C$ |
| $H$ | $F$ |
| $I$ | $F$ |
| $J$ | $G$ |
| $K$ | $E, H$ |
| $L$ | $I, J$ |

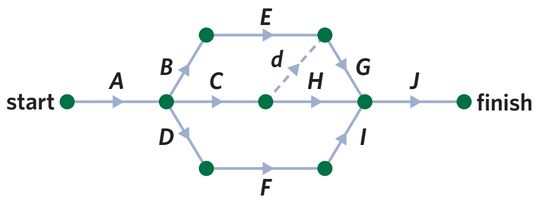
**12.** Nova is creating a Piñata for her daughter's 4$^{th}$ birthday party. The steps required to create the Piñata are shown in the following precedence table.

Construct an activity network for the project.

| activity | immediate predecessor(s) |
|---|---|
| $A$ | – |
| $B$ | – |
| $C$ | – |
| $D$ | $A, B$ |
| $E$ | $B$ |
| $F$ | $B, C$ |
| $G$ | $D, E, F$ |
| $H$ | $G$ |
| $I$ | $E, F$ |
| $J$ | $B, C$ |

## Joining it all together

**13.** Christine is planning on sewing an outfit for her birthday party. She has decided to make a jumpsuit with a matching jacket. The activity network shows all the activities she must complete for the project.




**a.** Construct a precedence table for the project.

Descriptions of activity are shown in the following table.

| activity | description |
|---|---|
| *A* | cut out pattern pieces |
| *B* | sew the top of the jumpsuit |
| *C* | sew the pant legs |
| *D* | sew the body of the jacket |
| *E* | sew the jumpsuit sleeves |
| *F* | sew the jacket sleeves |
| *G* | sew the top and bottom of the jumpsuit together |
| *H* | hem the pant legs |
| *I* | sew a zip onto the jacket |
| *J* | iron |

**b.** What must Christine do immediately before she can sew the top and bottom of the jumpsuit together?

**c.** Which of the following activities can Christine **not** complete before sewing the pant legs?

**A.** Sew a zip onto the jacket

**B.** Sew the top of the jumpsuit

**C.** Sew the jacket sleeves

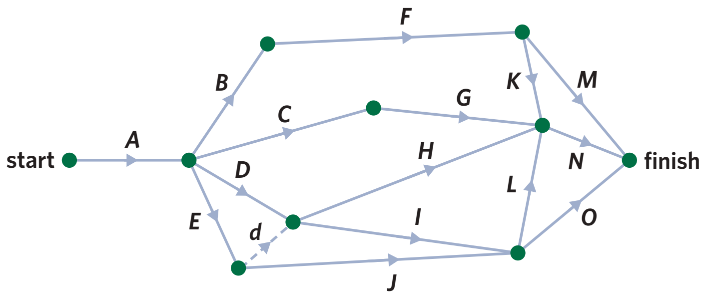
**D.** Sew the top and bottom of the jumpsuit together

## Exam practice

**14.** The rides at a theme park are set up at the beginning of each holiday season.

This project involves activities *A* to *O*.

The following directed network shows these activities.




Write down the two immediate predecessors of activity *I*. (1 MARK)

*Adapted from VCAA 2017 Exam 2 Networks and decision mathematics Q4a*

**67%** of students answered this type of question correctly.

**15.** The Sunny Coast cricket clubroom is undergoing a major works project.

This project involves nine activities: *A* to *I*.

The following table shows the immediate predecessor(s) of each activity.

The information in the table can be used to complete a directed network.

This network will require a dummy activity.

Complete the following sentence by filling in the boxes provided.

This dummy activity could be drawn as a directed edge from the end of

activity [ ] to the start of activity [ ]. (1 MARK)

*Adapted from VCAA 2020 Exam 2 Networks and decision mathematics Q5a*

| activity | immediate predecessor(s) |
|---|---|
| *A* | – |
| *B* | – |
| *C* | *A*, *B* |
| *D* | *C* |
| *E* | *C* |
| *F* | *B* |
| *G* | *D* |
| *H* | *F* |
| *I* | *E*, *G*, *H* |

**26%** of students answered this type of question correctly.

## Questions from multiple lessons

### Data analysis

**16.** The heights of students at Edrolo High are approximately normally distributed with a mean of 170 cm and a standard deviation of 6 cm.

If there are 600 students at the school, the number of students expected to be taller than 164 cm is closest to

**A.** 84 **B.** 408 **C.** 442 **D.** 504 **E.** 507

*Adapted from VCAA 2017NH Exam 1 Data analysis Q9*

### Networks and decision mathematics

**17.** Anton, Belinda, Christian and Daniella all swim together. They compete together in the 4 × 100 mixed medley relay. In the race, each swimmer will complete one of four strokes: backstroke, breaststroke, butterfly and freestyle.

The usual completion times for Anton, Belinda, Christian and Daniella are shown in the following table.

| | backstroke | breaststroke | butterfly | freestyle |
|---|---|---|---|---|
| **Anton** | 61 | 78 | 71 | 51 |
| **Belinda** | *t* | 72 | *t* | *t* |
| **Christian** | 57 | 73 | 82 | 56 |
| **Daniella** | 64 | 79 | 61 | 62 |

Belinda takes 72 seconds to complete the breaststroke.

She takes *t* seconds for each other stroke.

Their coach allocated the strokes for the race as follows:

- Backstroke to Christian
- Breaststroke to Belinda
- Butterfly to Daniella
- Freestyle to Anton

This allocation will achieve the minimum total race completion time if the value of *t* is at least

**A.** 50 **B.** 51 **C.** 56 **D.** 72 **E.** 73

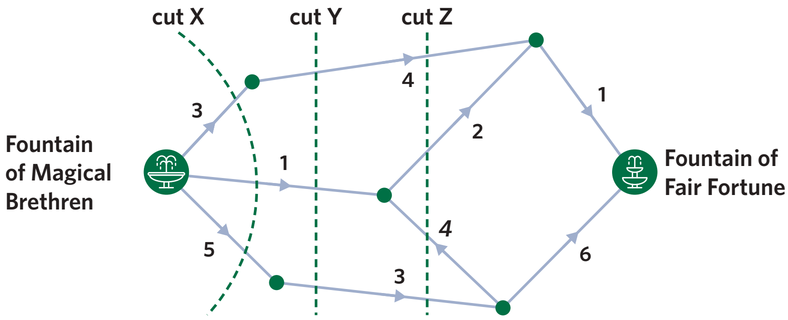
*Adapted from VCAA 2018 Exam 1 Networks and decision mathematics Q8*

## Networks and decision mathematics



**18.** The following graph shows the amount of water flowing through pipes in a series of connected fountains.

The weighting of each edge represents the maximum amount of water (in litres) that can flow through each pipe every second.




**a.** Cut X, shown on the graph, has a capacity of 9 L/s.

Two other cuts are labelled as cut Y and cut Z.

**i.** Write down the capacity of cut Y. (1 MARK)

**ii.** Write down the capacity of cut Z. (1 MARK)

**b.** Determine the maximum volume of water that can be delivered each second from the 'Fountain of Magical Brethren' to the 'Fountain of Fair Fortune'. (1 MARK)

*Adapted from VCAA 2018 Exam 2 Networks and decision mathematics Q1*

# 8I Critical path analysis

**STUDY DESIGN DOT POINTS**

- use of forward and backward scanning to determine the earliest starting times (EST) and latest starting times (LST) for each activity
- use of earliest starting times and latest starting times to identify the critical path in the network and determine the float times for non-critical activities

8A 8B 8C 8D 8E 8F 8G 8H **8I** 8J

**KEY SKILLS**

During this lesson, you will be:

- interpreting weighted precedence tables
- forward scanning to determine the EST and minimum completion time
- backward scanning to determine the LST
- determining the critical path and float times.

**KEY TERMS**

- Scheduling
- Minimum completion time
- Earliest start time (EST)
- Latest start time (LST)
- Critical path
- Float time

Each activity within a project can be assigned a time weighting, or an estimated completion time. The activity network can then be used to find the minimum completion time of the project, which activities need to be prioritised, and where breaks can be taken without reducing the project completion time.

## Interpreting weighted precedence tables

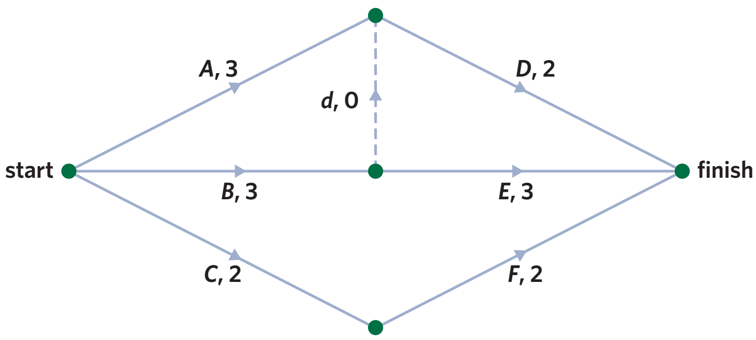
In addition to information about the order of activities, a time weighting, or estimated completion time, for each of the activities may also be known. The allocation of an estimated completion time for each activity is known as **scheduling**.

Scheduling information can be shown in a weighted precedence table by including an additional column with the estimated completion time of each activity. This information can then be used to calculate the minimum time for the entire project to be completed.

Note: As dummy activities are not real activities, they have an estimated completion time of 0.

For example, the following weighted precedence table and activity network represent the same project.

| activity | estimated completion time | immediate predecessor(s) |
|---|---|---|
| *A* | 3 | – |
| *B* | 3 | – |
| *C* | 2 | – |
| *D* | 2 | *A*, *B* |
| *E* | 3 | *B* |
| *F* | 2 | *C* |



## Worked example 1

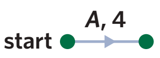
Construct an activity network based on the following weighted precedence table.

| activity | duration (hours) | immediate predecessor(s) |
|---|---|---|
| *A* | 4 | – |
| *B* | 11 | *A* |
| *C* | 5 | *A* |
| *D* | 6 | *A* |
| *E* | 2 | *D* |
| *F* | 3 | *E* |
| *G* | 4 | *B* |
| *H* | 9 | *B*, *C*, *F* |

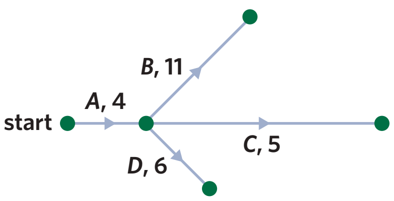
### Explanation

**Step 1:** Draw the start vertex along with the starting activities.

Activity *A* has no immediate predecessor so it is the starting activity. Draw this as an edge connected to the starting vertex, and label its estimated completion time.




**Step 2:** Draw the activities with *A* as their immediate predecessor, and label their estimated completion times.



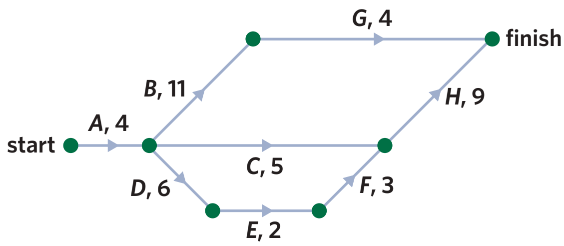

**Step 3:** Continue to draw activities based on their immediate predecessor, and label their estimated completion times.

The predecessor of activity *G* is activity *B*, so draw activity *G* from activity *B*.

The predecessor of activity *E* is activity *D*, so draw activity *E* from activity *D*.

Activity *H* has shared predecessors of *F* and *C*, so draw activity *F* from the end of activity *E* to the end of activity *C*, with *H* following from that vertex.

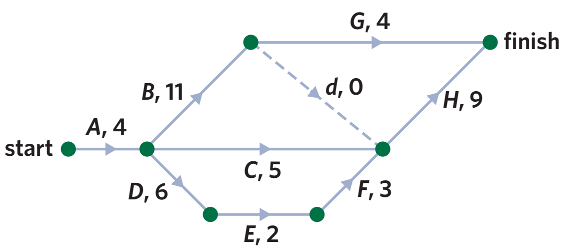
Activities *G* and *H* are not immediate predecessors for any other activity so they will connect to the finish vertex.




**Step 4:** Draw a dummy activity.

Activity *B* is also an immediate predecessor of activity *H*. To complete the activity network, a dummy activity must be drawn from activity *B* to activity *H*. Dummy activities have an estimated completion time of 0.

### Answer



# Forward scanning to determine the EST and minimum completion time

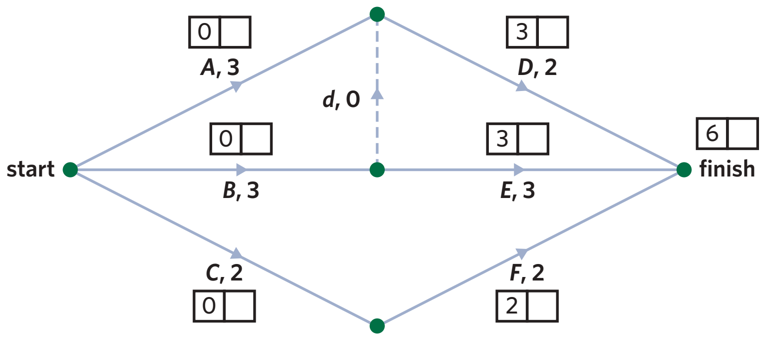
The **minimum completion time** of a weighted activity network is the shortest time in which the project, which is all the activities within the network, can be completed. The **earliest start time (EST)** is the earliest time, relative to the beginning of the project, that an individual activity can start at. To minimise the total completion time for a project, each activity should begin at its EST.

The process for finding the EST of each activity within a project is called forward scanning.

1. Draw a box alongside each activity of the network, as well as the finish vertex. The box should be split into two separate cells.
2. Fill in the EST as zero in the left-hand cell of each box for the starting activities (activities that connect to the start vertex).
3. Continuing filling in the EST in the left-hand cell for all other activities, as well as the finish vertex, working forwards from start to finish.
   - If an activity has one immediate predecessor, its EST can be found by adding the EST and duration of the immediate predecessor.
     $EST = EST \text{ of predecessor} + duration \text{ of predecessor}$
   - If an activity has two or more immediate predecessors, its EST will be the largest value found after calculating the EST using each of the predecessors.

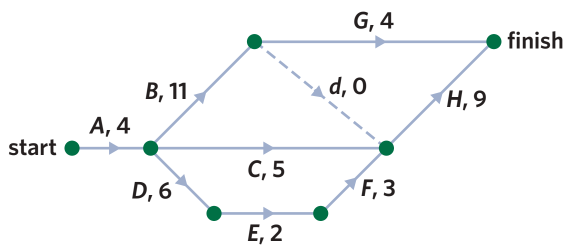
The EST of the finish vertex is the minimum possible completion time for the project.

For example, the following project has a minimum completion time of 6 hours.




## Worked example 2

Consider the following activity network. The estimated completion times are in hours.

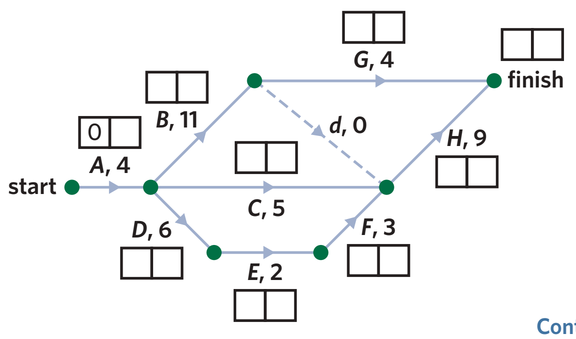



**a.** Find the earliest start time of activity *H*.

### Explanation

**Step 1:** Draw a box, split into two cells, at each activity, excluding the dummy activity.

Fill in the EST as 0 in the left-hand cell of the starting activity (*A*).




Continues →

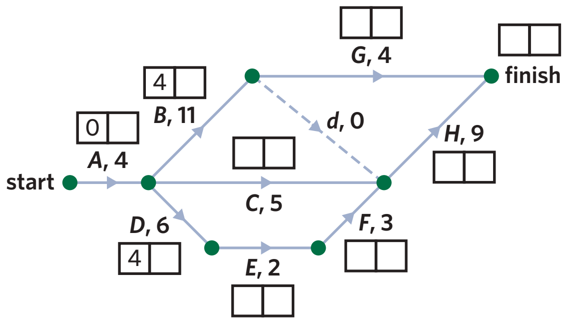
**Step 2:** Fill in the EST of activities *B*, *C* and *D*.

If an activity has one immediate predecessor, its EST can be found by adding the EST and duration of the previous activity.

*EST* = *EST of predecessor* + *duration of predecessor*

The immediate predecessor of activities *B*, *C* and *D* is activity *A*.

*EST* = 0 + 4 = 4

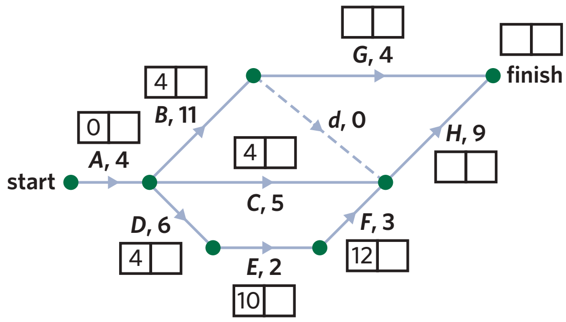



**Step 3:** Find the EST of activities *E* and *F*.

Activities *E* and *F* have only one immediate predecessor.

Activity *E*: *EST* = 4 + 6 = 10

Activity *F*: *EST* = 10 + 2 = 12




**Step 4:** Find the EST of activity *H*.

If an activity has two or more immediate predecessors, its EST will be the largest value found when looking at the predecessors individually.

Using the dummy activity as the predecessor:

*EST* = 4 + 11 + 0 = 15

Using activity *C* as the predecessor:

*EST* = 4 + 5 = 9

Using activity *F* as the predecessor:

*EST* = 12 + 3 = 15

### Answer

15 hours

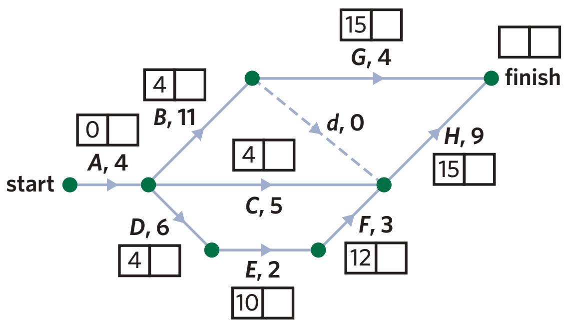
---

**b.** Find the minimum completion time of the project.

### Explanation

**Step 1:** Continuing on from part **a**, find the EST of activity *G*.

*EST* = 4 + 11 = 15




**Step 2:** Find the EST of the finish vertex.

The EST will be the largest value found when looking at the predecessors individually.

Using activity *G* as the predecessor:

*EST* = 15 + 4 = 19

Using activity *H* as the predecessor:

*EST* = 15 + 9 = 24

**Step 3:** Determine the minimum completion time of the project.

The minimum completion time is the EST of the finish vertex.

### Answer

24 hours

# Backward scanning to determine the LST

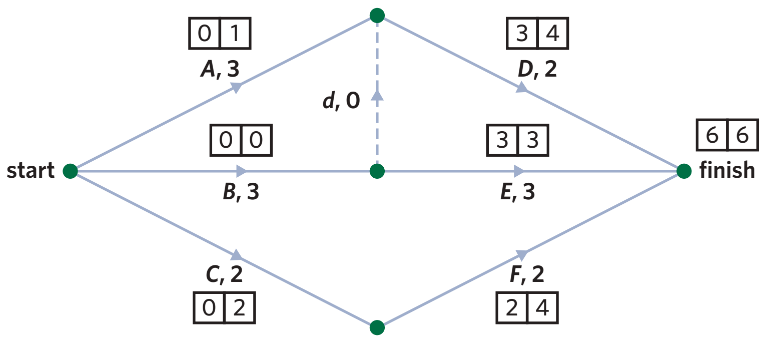
The **latest start time (LST)** is the latest time, relative to the beginning of the project, that an individual activity can start at without increasing the completion time of the entire project. The LST can be greater than the EST because some activities may be able to start after the earliest possible start time and not impact the total minimum completion time for the project.

The process for finding the LST for each activity within a project is called backward scanning.

1. Complete forward scanning.
   - All of the cells on the left should be full.
   - LSTs will be written in the right-hand cell.
2. Fill in the LST of the finish vertex as the minimum possible completion time, which is the same as the EST for the finish vertex.
3. Continue filling in the LST of all other activities, working backwards from finish to start.
   - If an activity has only one activity following it, its LST can be found by subtracting the duration of the activity from the LST of the activity following it.
     *LST = LST of following activity − duration of activity*
   - If an activity has two or more activities following it, its LST will be the smallest value found when looking at each of the activities immediately following it.

Note: The LST of at least one of the starting activities should be zero.

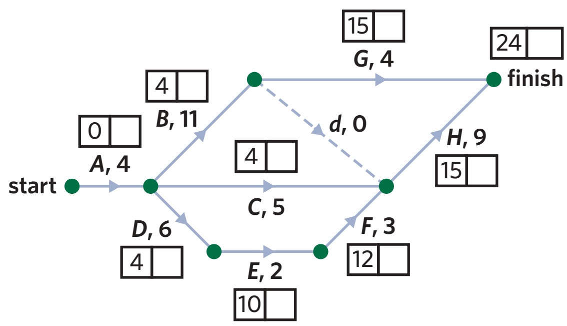
For example, activity *C* in the following project has an LST of 2.




## Worked example 3

Consider the following activity network with the earliest start time for each activity filled in. The estimated completion times are in hours.

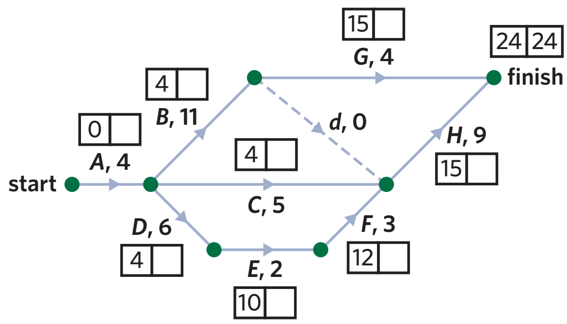
Find the latest start time of activity *B*.




### Explanation

**Step 1:** Fill in the LST of the finish vertex as the minimum possible completion time.

The EST for the finish vertex is 24 hours. This will also be the LST.




Continues →

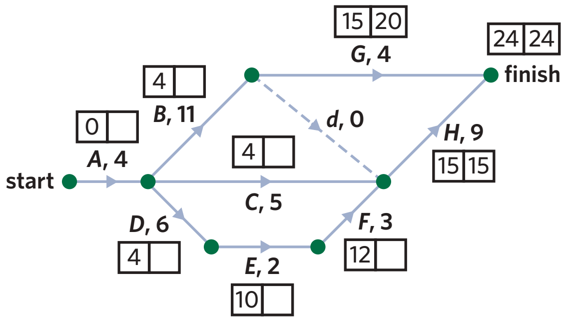
**Step 2:** Fill in the LST of activities *G* and *H*.

Only the finish vertex follows activities *G* and *H*.

$LST = LST\ of\ following\ activity - duration\ of\ activity$

Activity *G*: $LST = 24 - 4 = 20$

Activity *H*: $LST = 24 - 9 = 15$




**Step 3:** Find the LST of activity *B*.

If an activity has two or more activities following it, its LST will be the smallest value found when looking at each of the activities immediately following it.

Using activity *H* as the following activity:

$LST = 15 - 0 - 11 = 4$

Using activity *G* as the following activity:

$LST = 20 - 11 = 9$

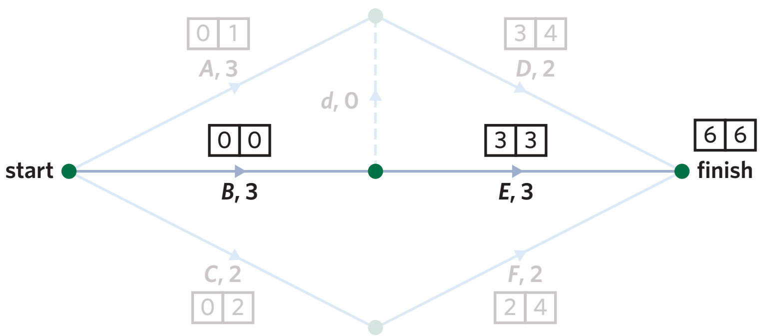
**Answer**

4 hours

# Determining the critical path and float times

An activity is critical when its EST is equal to its LST. A **critical path** is a sequence of critical activities, each of which cannot be delayed without increasing the minimum completion time of the project. It is possible to have multiple critical paths within a project.

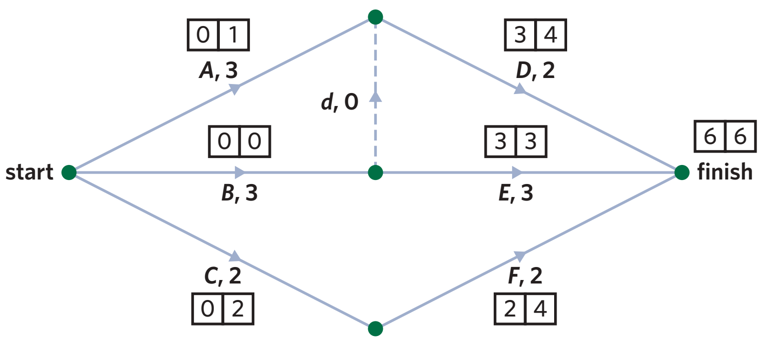
For example, the critical path for the following project is *B*–*E*, as the ESTs and LSTs of each activity are equal. This means that neither activity *B* or *E* can be delayed without increasing the completion time of the project.




The **float time** of an activity is the maximum amount of time that it can be delayed without changing the completion time of the project. It can be calculated by finding the difference between the EST and LST. The float time of a critical activity is 0.

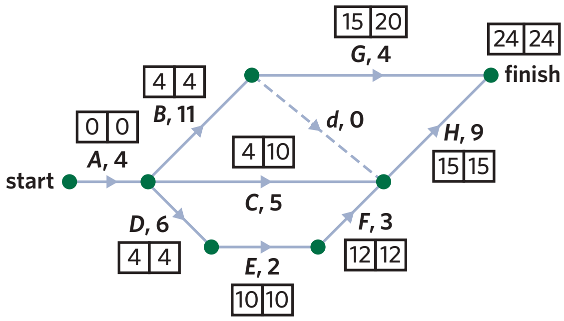
$float\ time = LST - EST$

For example, activity *F* has a float time of $4 - 2 = 2$ in the following project. This means that activity *F* can start up to 2 hours late without impacting the project's minimum completion time.



## Worked example 4

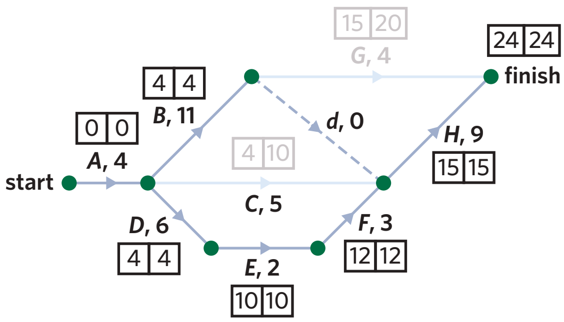
Consider the following activity network with the earliest and latest starting time for each activity shown. The estimated completion times are in hours.




**a.** Determine the critical path(s) of the project.

### Explanation

**Step 1:** Highlight all activities that have an EST and LST that are equal.




**Step 2:** Find all the paths that connect the start vertex to the finish vertex using only critical activities.

### Answer

*A*–*B*–*H* and *A*–*D*–*E*–*F*–*H*

**b.** Determine the float time of activity *C*.

### Explanation

**Step 1:** Determine the EST and LST of activity *C*.

The LST of activity *C* is 10.

The EST of activity *C* is 4.

**Step 2:** Calculate the float time.

$$\begin{aligned} \textit{float time} &= LST - EST \\ &= 10 - 4 \\ &= 6 \end{aligned}$$

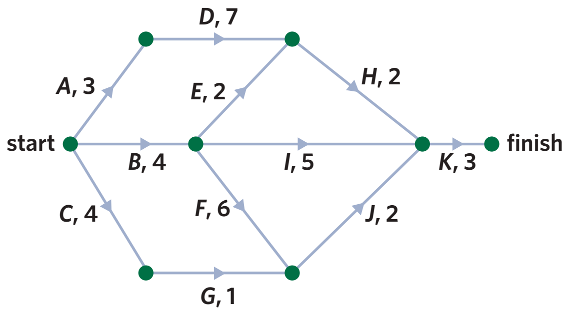
### Answer

6 hours

## Exam question breakdown

*VCAA 2018 Exam 1 Networks and decision mathematics Q5*

The directed network below shows the sequence of 11 activities that are needed to complete a project.
The time, in weeks, that it takes to complete each activity is also shown.



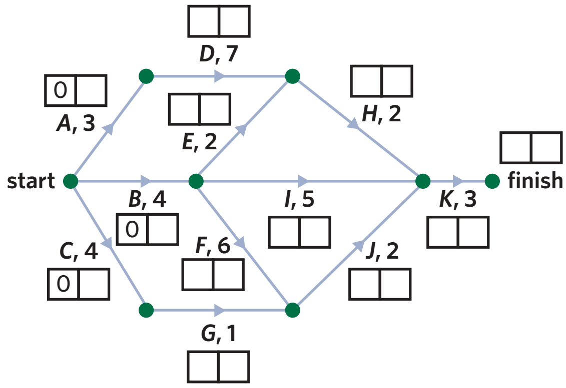

How many of these activities could be delayed without affecting the minimum completion time of the project?

**A.** 3 **B.** 4 **C.** 5 **D.** 6 **E.** 7

### Explanation

**Step 1:** Draw a box, split into two cells, at each activity and the finish vertex.

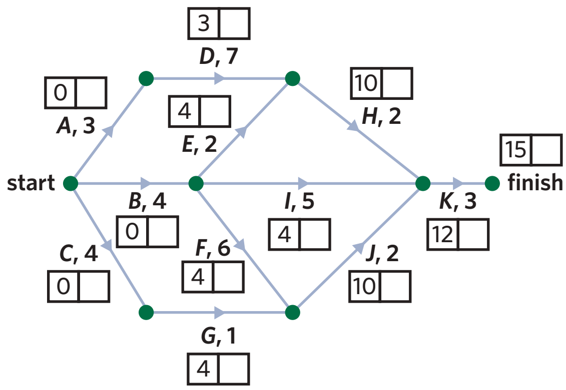
Fill in the EST as 0 in the left-hand cell of each of the starting activities (*A*, *B* and *C*).




**Step 2:** Fill in the EST in the left-hand cell for all remaining activities and the finish vertex.

*EST = EST of predecessor + duration of predecessor*

If an activity has two or more immediate predecessors, its EST will be the largest value found when looking at the predecessors individually.

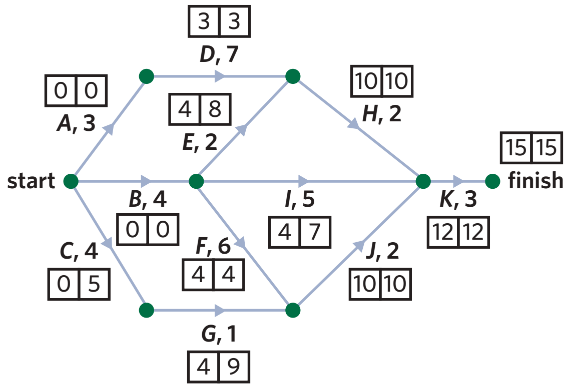



**Step 3:** Fill in the LST in the right-hand cell for all activities and the finish vertex.

*LST = LST of following activity − duration of activity*

The LST of the finish vertex is the same as the EST.

If an activity has two or more activities following it, its LST will be the smallest value found when looking at each of the activities immediately following it.




**Step 4:** Determine the number of non-critical activities.

Non-critical activities have an LST that is larger than its EST.

The non-critical activities are *C*, *E*, *G* and *I*.

### Answer

B

**43%** of students answered this question correctly.

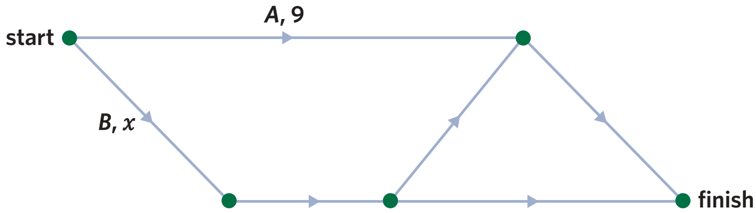
**27%** of students incorrectly chose option A. These students likely found the 2 critical paths, which each had 4 activities, and subtracted 8 from 11 total activities to reach 3 non-critical activities. However, activity *K* is in both critical paths, meaning that there are actually 4 non-critical activities.

# 8I Questions

## Interpreting weighted precedence tables

**1.** A weighted precedence table and its corresponding activity network are shown. Some labels on the activity network are missing.

| activity | duration (days) | immediate predecessor(s) |
|---|---|---|
| *A* | 9 | – |
| *B* | 5 | – |
| *C* | 4 | *B* |
| *D* | 6 | *C* |
| *E* | 2 | *A*, *D* |
| *F* | 3 | *C* |




**a.** What is the value of $x$?

**A.** 2

**B.** 4

**C.** 5

**D.** 6

**b.** Copy the activity network and fill in the missing labels.

**2.** Consider the following weighted precedence table.

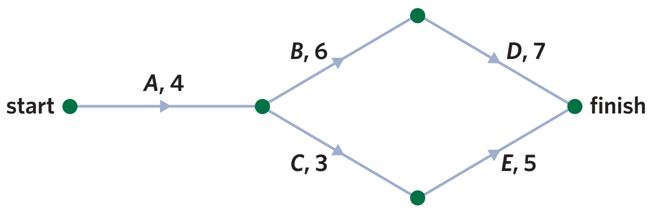
**a.** Draw an activity network based on the table.

**b.** The precedence table has been altered such that activity *F* cannot start until activity *D* has been completed. Show this change on the activity network.

| activity | duration (minutes) | immediate predecessor(s) |
|---|---|---|
| *A* | 4 | – |
| *B* | 13 | *A* |
| *C* | 12 | *A* |
| *D* | 11 | *A* |
| *E* | 4 | *B* |
| *F* | 6 | *C*, *E* |
| *G* | 13 | *D* |
| *H* | 18 | *B* |
| *I* | 15 | *F*, *G* |

## Forward scanning to determine the EST and minimum completion time

**3.** Consider the following activity network.




**a.** The EST of activity *B* is

**A.** 3

**B.** 4

**C.** 6

**D.** 7

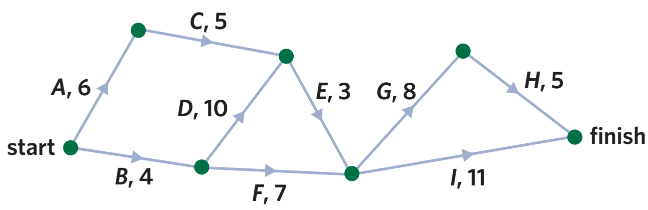
**b.** Determine the EST of activity *E*.

**c.** Determine the minimum completion time of the project.

4. The following activity network displays the time in minutes taken for each of the steps of a DIY craft project.

   a. What is the earliest start time after the beginning of the project that activity *E* can start?

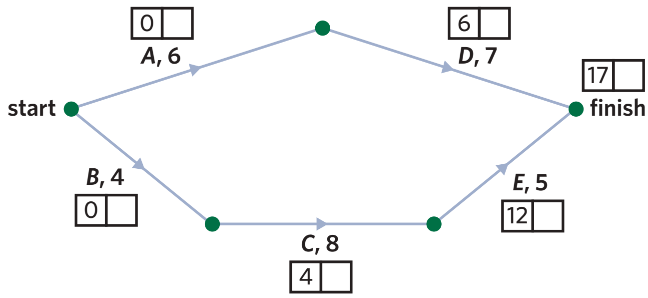
   b. What is the minimum amount of time that it would take to finish the DIY craft project?




## Backward scanning to determine the LST

5. Consider the following activity network.

   The earliest start time of each activity is filled in.




   a. The LST of activity *C* is

      A. 4

      B. 7

      C. 10

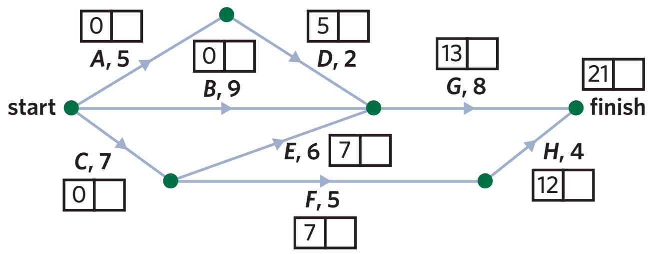
      D. 12

   b. Determine the LST of activity *A*.

---

6. A few students are doing a small group project in class. The activity network shows the time in minutes required for all of the tasks within the project, as well as the earliest start time for each.

   Danny, who has been assigned to activity *D*, decides to go to the bathroom just as the group begins the project. How long can Danny spend on his bathroom break before he holds up the rest of the project?




## Determining the critical path and float times

7. Which one of the following statements about critical paths is true?

   A. Each project has only one critical path.

   B. A critical path must include at least three activities.

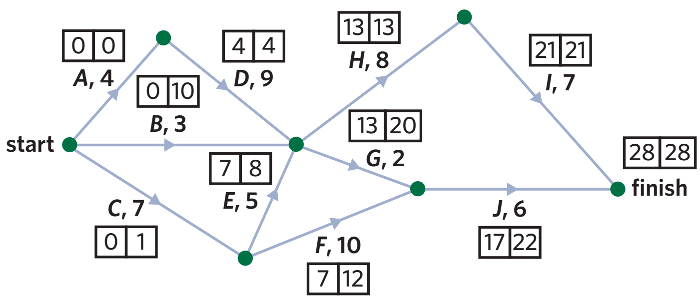
   C. A critical path must include the activity that takes the longest time to complete.

   D. Reducing the estimated completion time of any activity on a critical path for a project will always reduce the minimum completion time for the entire project.

   E. Increasing the estimated completion time of any activity on a critical path for a project will always increase the minimum completion time for the entire project.

---

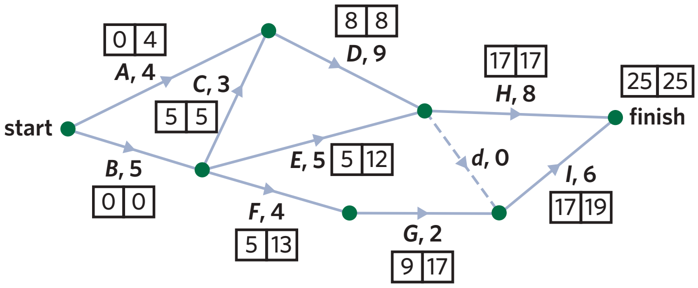
8. Consider the following activity network.




   a. Find the critical path(s) of the activity network.

   b. What is the float time of activity *J*?

9. A group of young film students are editing a short film. The activity network below shows the activities required to complete the film and their duration in hours.




a. Find the critical path(s) of the project.

b. Would the critical path change if there was no dummy activity?

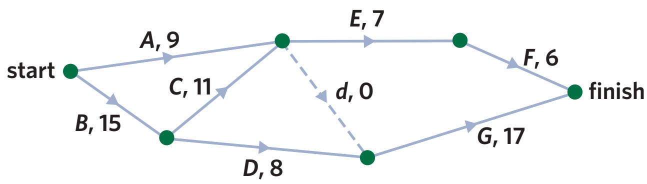
c. Determine the float time of activity *G*.

d. Would the completion time of the short film change if activity *D* was delayed 1 hour?

e. Would the completion time of the short film change if activity *E* was delayed 4 hours?

## Joining it all together

10. Consider the following activity network.




The critical path is

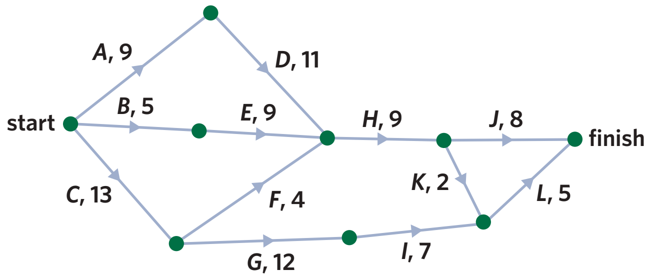
A. *A–E–F*

B. *B–C–G*

C. *B–C–E–F*

D. *B–D–G*

11. The following network has been used to show the activities required to fix a broken laptop. All values are in minutes.



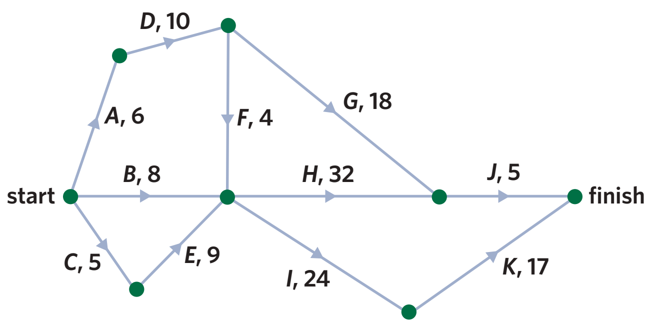

a. What is the least amount of time required to fix the laptop?

b. Determine the float time of activity *G*.

c. Determine the float time of activity *B*.

d. Find the critical path(s) of the project.

12. Jamie Oliver has come up with a new recipe for beef stroganoff. The time (in minutes) needed for each step of the recipe can be represented in the following activity network.




Which of the following statements is true?

A. Activity *J* is critical.

B. Activity *E* has an EST of 11 minutes.

C. Activity *K* has a LST of 44 minutes.

D. Activity *G* has a float time of 4 minutes.

**13.** Amanda is setting up for her 16th birthday party and has created the following precedence table for the project.

**a.** Draw an activity network to represent this information.

**b.** What is the least amount of time that Amanda will need to set up for her party?

**c.** Find the critical path(s) of the project.

Amanda's mum has bought her a surprise cake for her birthday. This alters the activity network slightly because it means activity $H$ cannot start until activity $G$ has been completed.

**d.** Show this change on the activity network.

**e.** How long will Amanda now need to set up for her party?

**f.** What is the new critical path of the project?

| activity | duration (mins) | immediate predecessor(s) |
|---|---|---|
| $A$ | 8 | – |
| $B$ | 10 | – |
| $C$ | 5 | – |
| $D$ | 11 | $A$ |
| $E$ | 12 | $C$ |
| $F$ | 6 | $C$ |
| $G$ | 9 | $B, D, E$ |
| $H$ | 7 | $F$ |
| $I$ | 3 | $G$ |
| $J$ | 8 | $G$ |
| $K$ | 9 | $H, I$ |
| $L$ | 4 | $J, K$ |

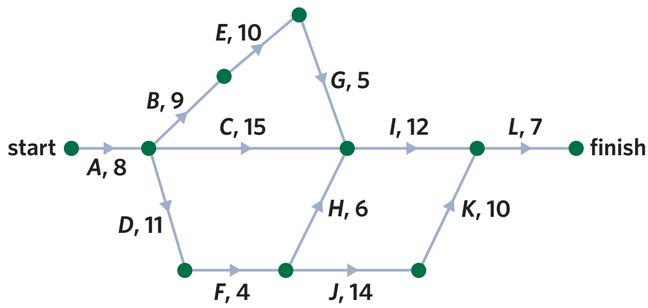
---

**14.** Beyoncé is mid-way through recording her new album 'Ginger Ale'. Her personal assistant has created an activity network for the activities left in the project. All values are in days.

**a.** Find the critical path(s) of the project.

**b.** How long will it take for the album to be finished?

Beyoncé decides at the last minute that she wants to include a bonus track in the album. This requires an extra activity, $M$, to be added to the project. Activity $M$ can begin after the completion of activity $E$ and it must be finished before activity $L$ can begin. Activity $M$ will take 3 weeks.



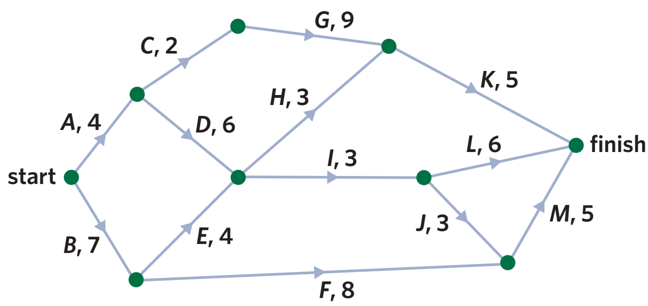

**c.** Draw activity $M$ on the activity network.

**d.** Find the critical path(s) of the project with activity $M$.

**e.** How much longer will the album take to record due to the addition of the bonus track?

## Exam practice

**15.** The following activity network shows the sequence of activities required to complete a project. The number next to each activity in the network is the time it takes to complete that activity, in days.




The minimum completion time for this project, in days, is

**A.** 18

**B.** 19

**C.** 20

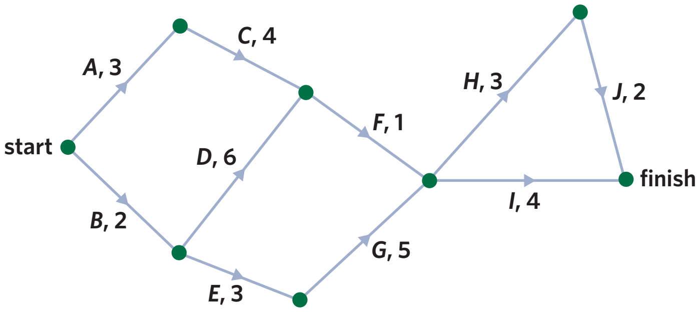
**D.** 21

**E.** 22

*VCAA 2020 Exam 1 Networks and decision mathematics Q6*

**61%** of students answered this question correctly.

**16.** At the Zenith Post Office all computer systems are to be upgraded. This project involves 10 activities, *A* to *J*. The following directed network shows these activities and their completion times, in hours.




**a.** Determine the earliest starting time, in hours, for activity *I*. (1 MARK)

**b.** The minimum completion time for the project is 15 hours. Write down the critical path. (1 MARK)

**c.** Two of the activities have a float time of two hours. Write down these two activities. (1 MARK)

**d.** For the next upgrade, the same project will be repeated but one extra activity will be added. This activity has a duration of one hour, an earliest starting time of five hours and a latest starting time of 12 hours. Complete the following sentence by filling in the boxes provided.

The extra activity could be represented on the network by a directed edge from the end of activity __________ to the start of activity __________. (1 MARK)

*VCAA 2018 Exam 2 Networks and decision mathematics Q3a,d*

Part **a**: **64%** of students answered this question correctly.
Part **b**: **76%** of students answered this question correctly.
Part **c**: **45%** of students answered this question correctly.
Part **d**: **45%** of students answered this question correctly.

---

**17.** The directed graph below shows the sequence of activities required to complete a project. The time taken to complete each activity, in hours, is also shown.

The minimum completion time for this project is 18 hours.

The time taken to complete activity *E* is labelled $x$.

The maximum value of $x$ is

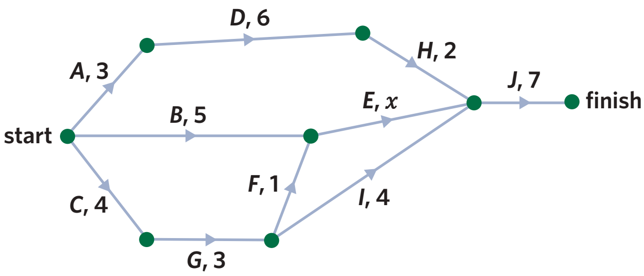
**A.** 2

**B.** 3

**C.** 4

**D.** 5

**E.** 6




*VCAA 2021 Exam 1 Networks and decision mathematics Q6*

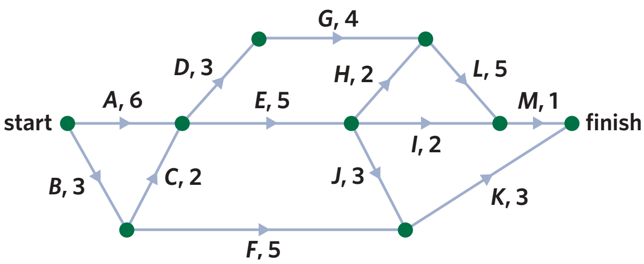
**46%** of students answered this question correctly.

---

**18.** Roadworks planned by the local council require 13 activities to be completed. The following network shows these 13 activities and their completion times in weeks.

**a.** What is the earliest start time, in weeks, of activity K? (1 MARK)

**b.** How many of these activities have zero float time? (1 MARK)




*VCAA 2021 Exam 2 Networks and decision mathematics Q4a,b*

Part **a**: **45%** of students answered this question correctly.
Part **b**: **22%** of students answered this question correctly.

# Questions from multiple lessons

## Recursion and financial modelling

**19.** Anna is planning on taking out a $4000 loan to fund a semester exchange in Copenhagen. She plans to repay it in full after one year.

She has been offered five different loans. Their annual interest rates and compounding periods are provided.

- Loan A: 11.0% p.a, compounding quarterly
- Loan B: 11.8% p.a, compounding quarterly
- Loan C: 11.4% p.a, compounding quarterly
- Loan D: 11.2% p.a, compounding weekly
- Loan E: 10.9% p.a, compounding weekly

Which loan should Anna choose if she wants to minimise the amount of interest paid?

**A.** Loan A **B.** Loan B **C.** Loan C **D.** Loan D **E.** Loan E

*Adapted from VCAA 2018 Exam 1 Recursion and financial modelling Q19*

## Networks and decision mathematics

**20.** The following network shows the distance, in metres, of several streets between Yolanda's office and her house. The intersections of the streets have been labelled from *A* to *H*.

Yolanda wants to work out the shortest path from her office to her house. She could do this using

**A.** Dijkstra's algorithm.

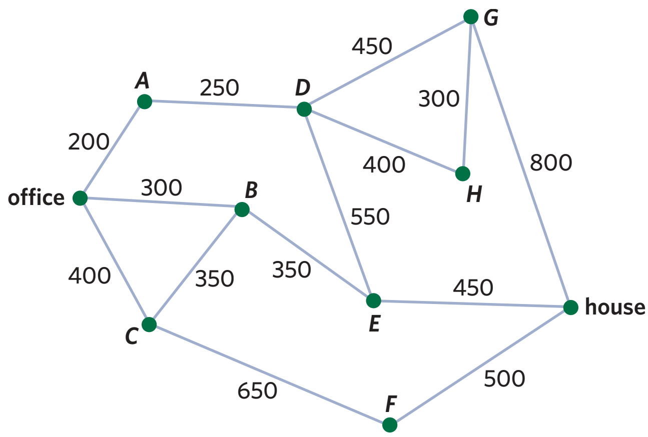
**B.** the Hungarian algorithm.

**C.** Prim's algorithm.

**D.** a minimum cut.

**E.** critical path analysis.

*Adapted from VCAA 2018 Exam 1 Networks and decision mathematics Q2*

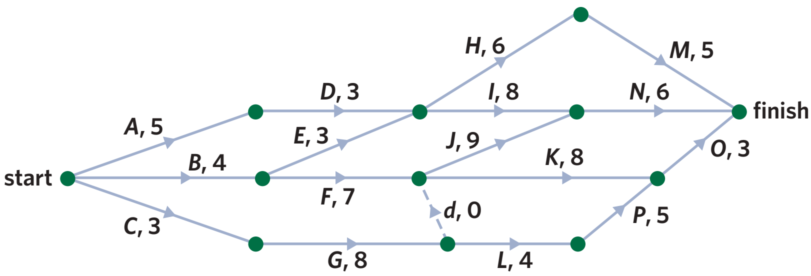



## Networks and decision mathematics

**21.** The tailors at Dior's atelier are constructing an haute couture jumpsuit for Lady Gaga to wear to the 2023 annual MET Gala.

The construction of the jumpsuit requires the completion of 16 activities, labelled from *A* to *P*.

The network to the right shows these activities and each of their completion times, in days.




**a.** What are the two immediate predecessors of activity *J*? (1 MARK)

**b.** The minimum completion time for the project is 26 days.

**i.** This activity network has two critical paths. One of them is *B*–*F*–*J*–*N*. What is the other critical path? (1 MARK)

**ii.** Determine the float time for activity *K*. (1 MARK)

*Adapted from VCAA 2017 Exam 2 Networks and decision mathematics Q4*

# 8J Crashing

STUDY DESIGN DOT POINT

- use of crashing to reduce the completion time of the project or task being modelled

8A 8B 8C 8D 8E 8F 8G 8H 8I 8J

KEY SKILLS

During this lesson, you will be:

- crashing an activity network
- crashing an activity network with restrictions.

KEY TERMS

- Crashing

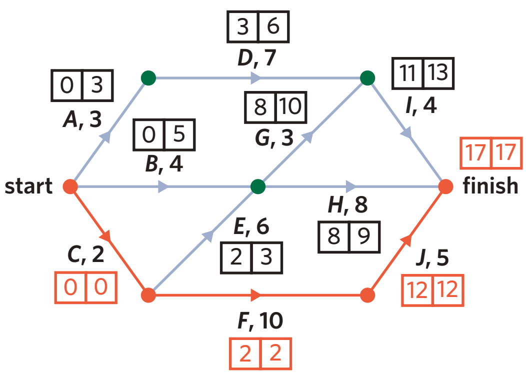
It is often advantageous to complete a project as quickly as possible. Sometimes it is so important that a manager of a project is willing to pay to reduce the completion time. Real life examples of this include hiring more workers, purchasing better equipment or improving technology. However, it is crucial to determine which activities to reduce the duration of, since not all will actually have an impact on a project's completion time.

## Crashing an activity network

It is sometimes possible to reduce the time required to complete certain activities, and thereby lower the total completion time of the project.

If the activity that is being altered is on all critical paths of the project, the minimum completion time of the project will also change. This process of reducing the completion time of a project is known as **crashing**. Crashing can cause the critical path to change if activities on the critical path are altered by more than the minimum float time outside of the critical path.

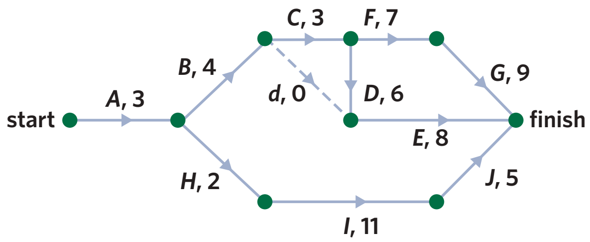
For example, the critical path of the following activity network is currently *C*–*F*–*J*, and the minimum float time outside of the critical path is 1, for activities *E* and *H*.




- If the duration of either activity *F* or *J* is reduced by 1 hour, the minimum completion time of the project will be reduced by 1 hour and there will be a second critical path *C*–*E*–*H*.
- If the duration of either activity *F* or *J* is reduced by 2 hours, the minimum completion time of the project will be reduced by only 1 hour, and *C*–*E*–*H* will be the only critical path.
- Reducing the duration of activity *C* by 2 hours will decrease the minimum completion time of the project by 2 hours, but the critical path will remain the same as the activity is on both *C*–*F*–*J* and *C*–*E*–*H*.

## Worked example 1

Claire is part of a study group that is planning to create revision notes on chapter 8 of this textbook. The activity network shows all the lessons within chapter 8. All values are in hours.




Claire asks her friend Klare to help the study group. With Klare's assistance, the study group can reduce the time it will take to complete revision notes for lesson *C* by 2 hours and lessons *B*, *G* and *I* by 3 hours.

**a.** What is the minimum amount of time it will take the group to complete revision notes for chapter 8 with Klare's help?

### Explanation

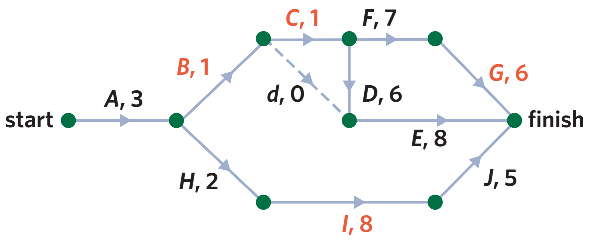
**Step 1:** Reduce the duration of each of the activities by the maximum number of hours.

Activity *B*: $4 - 3 = 1$

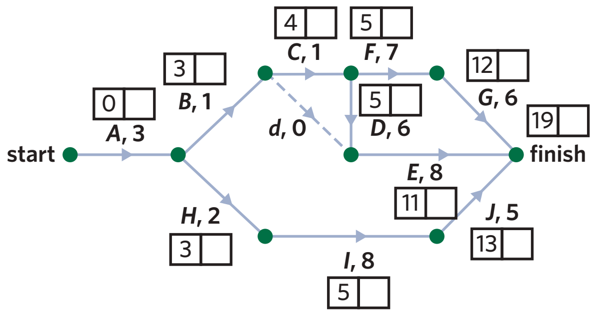
Activity *C*: $3 - 2 = 1$

Activity *G*: $9 - 3 = 6$

Activity *I*: $11 - 3 = 8$




**Step 2:** Use forward scanning to find the new minimum completion time.




The minimum completion time is the EST of the finish vertex.

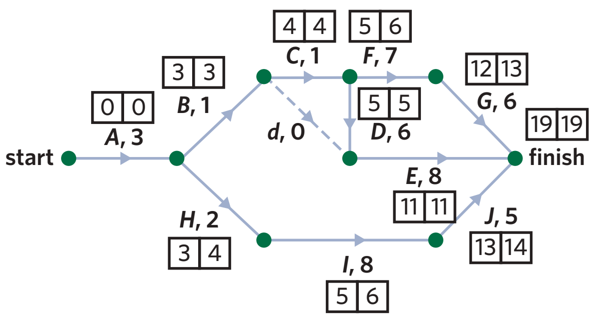
### Answer

19 hours

**b.** What is the critical path of the project?

### Explanation

**Step 1:** Use backward scanning to find the LSTs of each lesson.




**Step 2:** Find the critical path(s).

Critical activities have an EST and LST that are equal.

### Answer

*A–B–C–D–E*

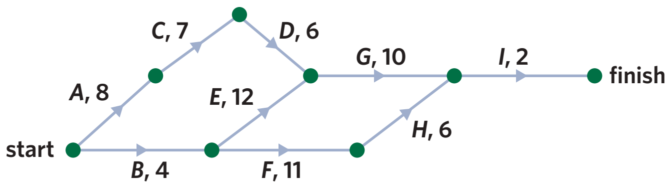
# Crashing an activity network with restrictions

Often a cost may be incurred in order to reduce the completion time of an activity. The amount of money spent to reduce the completion time of a project is balanced against the amount of time it will save, and the most efficient strategy is chosen.

For example, if the manager of a project can choose to reduce the length of two different activities, and both reduce the completion time of a project by an equal amount, it is more efficient to reduce the length of the activity that is cheaper.

## Worked example 2

A landscaping company is preparing to renovate and landscape a garden. The activities that must be completed for the project are represented in the following activity network. All values are in days.




The manager is willing to employ more staff in order to speed up the completion of the project.

The completion times of activities $A$, $B$, $C$, $D$ and $E$ can each be reduced by $150 per day, for up to two days.

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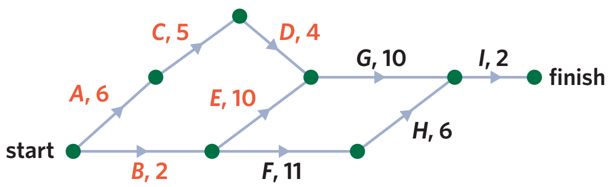
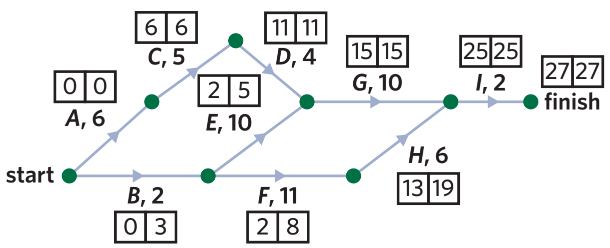
**a.** What is the minimum cost of reducing the minimum completion time by as much as possible?

### Explanation

**Step 1:** Reduce the duration of each of the activities by the maximum number of days.




**Step 2:** Complete forward and backward scanning.




Activities $A$, C and $D$ have float times of 0.

Activities $B$ and $E$ have float times of 3.

**Step 3:** Determine any redundant reduction in the duration of activities.

It is inefficient to pay for the reduction in the duration of activities when there will be float time.

The minimum completion time of the project would not change if the duration of activities $B$ and $E$ were a total of 3 days longer. Therefore, the duration of one of activity $B$ and $E$ does not need to be reduced at all, and the duration of the other only needs to be reduced by 1 day.

**Step 4:** Identify the cost of reducing the minimum completion time by as much as possible.

The duration of activities $A$, $C$ and $D$ should each be reduced by 2 days and the duration of activity $B$ (or $E$) should be reduced by 1 day. This is a total of 7 days.

Each day of reduced time costs $150.

$150 \times 7 = 1050$

### Answer

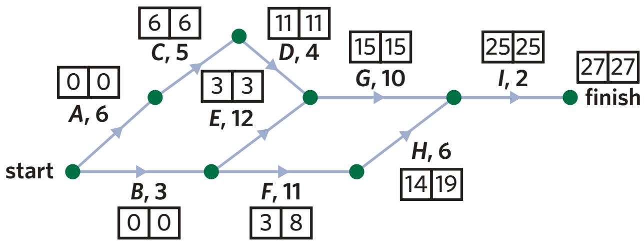
$1050

Continues →

**b.** On top of the current reductions, the manager may also be able to further reduce either the duration of activity *A* by 1 day for $140, the duration of activity *G* by 1 day for $175 or the duration of activity *I* by 1 day for $225. Which activity should the manager reduce the duration of?

### Explanation

**Step 1:** Find the critical path(s) of the project after the initial crashing.




Critical activities have an EST and LST that are equal.

The critical paths are *A*–*C*–*D*–*G*–*I* and *B*–*E*–*G*–*I*.

**Step 2:** Identify whether activities *A*, *G* and *I* are on all of the critical paths.

Activities *G* and *I* are on both critical paths while activity *A* is not. Hence, reducing the duration of activity *A* would not alter the completion time of the project.

**Step 3:** Identify the cheaper option.

It costs $175 to reduce the duration of activity *G* by 1 day and $225 to reduce the duration of activity *I* by 1 day. Both reductions will have the same effect on the minimum completion time of the project. The cheaper option is activity *G*.

### Answer

Activity *G*

## Exam question breakdown

*VCAA 2016 Exam 2 Networks and decision mathematics Q3d*

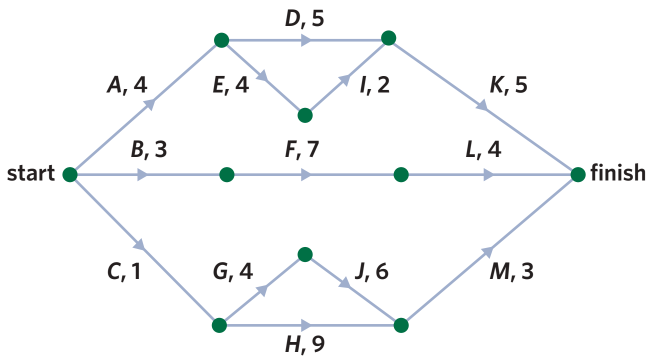
A new skateboard park is to be built in Beachton.

This project involves 13 activities, *A* to *M*.

The directed network shows these activities and their completion times in days.

The completion times for activities *E*, *F*, *G*, *I* and *J* can each be reduced by one day.

The cost of reducing the completion time by one day for these activities is shown in the following table.

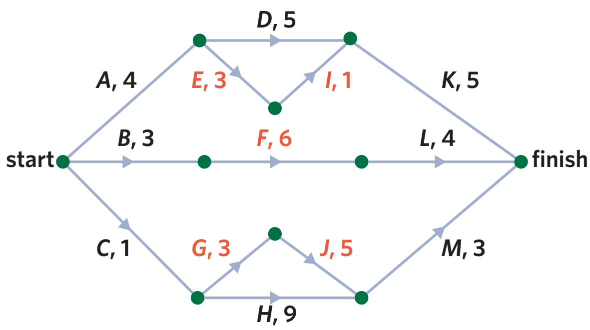



| activity | *E* | *F* | *G* | *I* | *J* |
|---|---|---|---|---|---|
| cost ($) | 3000 | 1000 | 5000 | 2000 | 4000 |

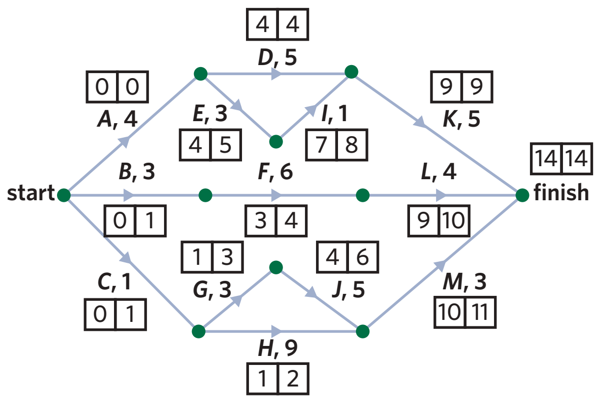
What is the minimum cost to complete the project in the shortest time possible? (1 MARK)

### Explanation

**Step 1:** Reduce the duration of each of the activities by the maximum number of days.




**Step 2:** Complete forward and backward scanning.




Activities *E*, *F* and *I* have a float time of 1.

Activities *G* and *J* have a float time of 2.

Continues →

**Step 3:** Determine any redundant reduction in the duration of activities.

The minimum completion time of the project would not change if the duration of activity $F$ was 1 day longer.

The minimum completion time of the project would not change if the duration of activities $G$ and $J$ were a total of 2 days longer.

Therefore, the durations of each of activity $F$, $G$ and $J$ do not need to be reduced at all.

The minimum completion time of the project would not change if the duration of activities $E$ and $I$ were a total of 1 day longer.

Therefore, the duration of only one of activity $E$ and $I$ needs to be reduced.

**Step 4:** Identify the cheaper option.

It costs \$3000 to reduce the duration of activity $E$ and \$2000 to reduce the duration of activity $I$. The cheaper option is activity $I$.

**Answer**

\$2000

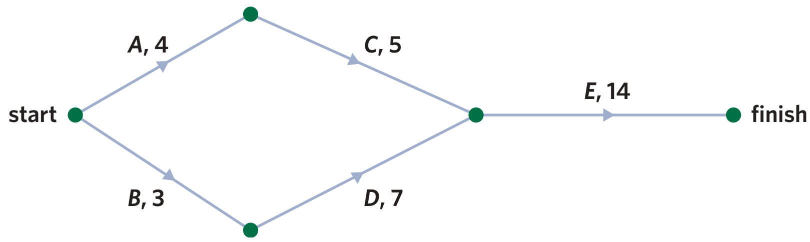
**21%** of students answered this question correctly.

Many students likely did not recognise that $A$-$D$-$K$ becomes a critical path after reducing the duration of activity $I$. This led to answers of \$10 000 as the three new critical paths could be reduced in duration by a further day. Other students reduced the duration of activity $E$ instead of $I$, failing to make the most efficient choice.

# 8J Questions

## Crashing an activity network

**1.** Consider the following activity network. The minimum completion time of the project is 24 days.

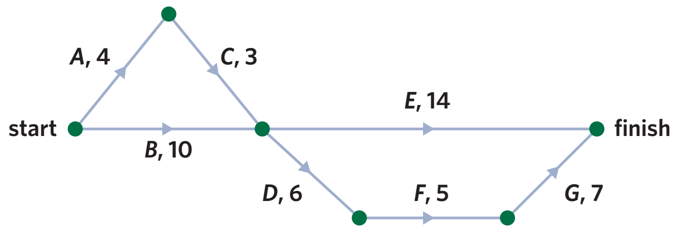



The duration of each activity within the project can be reduced by one day.

To complete this project in 21 days, the minimum number of activities that must have their durations reduced by one day is

**A.** 2 **B.** 3 **C.** 4 **D.** 5

**2.** The activity network for a project is shown. All values are in days.




**a.** Which one of the following activities **cannot** be reduced by 1 day without altering the completion time of the project?

The durations of activities *B* and *G* can be reduced by up to 5 days and the durations of activities *C* and *E* can be reduced by up to 2 days.

**b.** If all are reduced as much as possible, what would be the new minimum completion time?

**c.** If all are reduced as much as possible, what would be the new critical path(s)?

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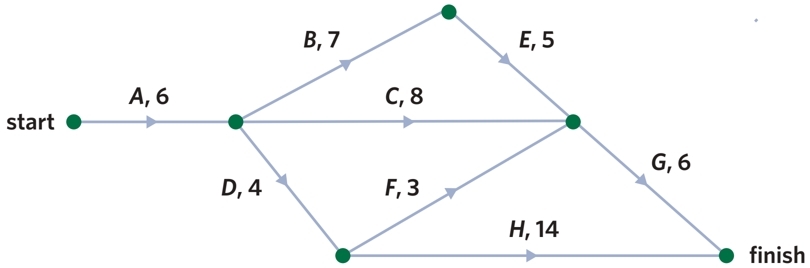
**3.** In order to be more efficient, Alyssa, a Year 12 student, has timed all of her activities as she gets ready for school in the morning. She is able to multitask by doing numerous activities at once. The activity network for her morning activities is shown. All values are in minutes.




**a.** Find the critical path(s).

Alyssa is able to cut down the time it takes to complete activity *B*. She can reduce its duration by 2 minutes.

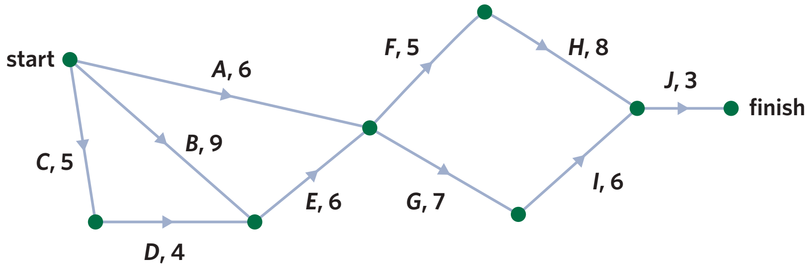
**b.** Will Alyssa be able to get ready for school faster? If not, explain why. If so, how much time has she saved?

**c.** What is the new float time of activity *B*?

**d.** Find the new critical path(s).

## Crashing an activity network with restrictions

**4.** Consider the following activity network. All values are in minutes.




The durations of activities *E* and *I* can each be reduced by one minute for $200.

The durations of activities *B*, *D* and *F* can each be reduced by one minute for $250.

If the durations of only three activities can be reduced, it would be best to reduce the durations of activities

**A.** *B*, *D* and *E*.  **B.** *B*, *D* and *F*.  **C.** *E*, *D* and *I*.  **D.** *E*, *F* and *I*.

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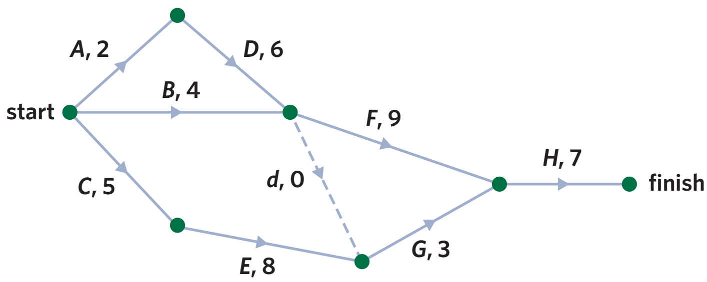
**5.** The organisers of Falls Music Festival have drawn an activity network for the setup of their festival. All values are in days.



a. Find the critical path(s).

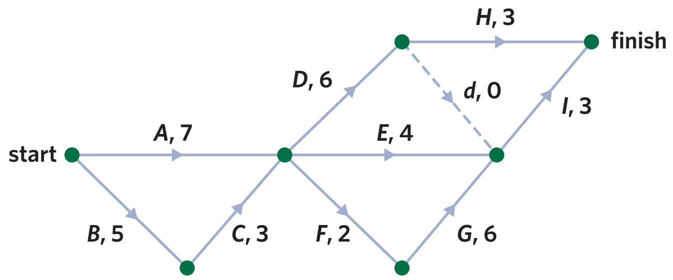
The organisers are able to employ more workers to reduce the overall setup time. They can reduce the duration of activity *D* for $1800 per day and activity *F* for $1300 per day. The duration of the activities can be reduced by a maximum of four days each.

b. What is the minimum cost of reducing the setup time as much as possible?

c. How long would it take the organisers of Falls Music Festival to set up the festival after employing the extra workers?

d. Find the new critical path(s).

**6.** The following activity network shows the steps required to build an automatic farm on Minecraft.




The current minimum completion time is 19 minutes.

Activities *A* and *B* can each be reduced for 9 emeralds per minute.

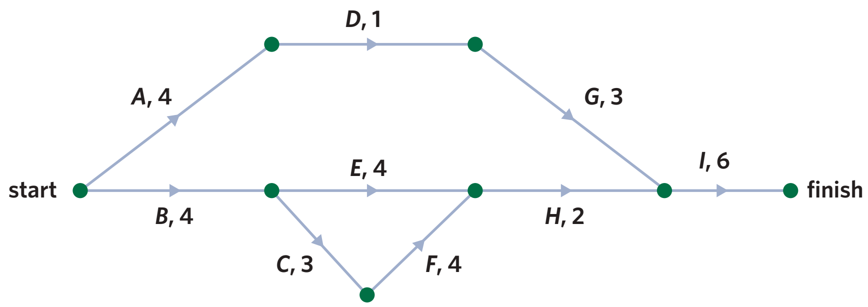
Activities *C* and *D* can each be reduced for 7 emeralds per minute.

Each activity can be reduced by a maximum of 2 minutes.

What is the minimum additional cost of completing the automatic farm in 16 minutes?

## Joining it all together

**7.** There are plans for a new basketball court to be built in West Melbourne. Nine activities have been identified for this project. The following activity network shows the activities and their completion times in weeks.



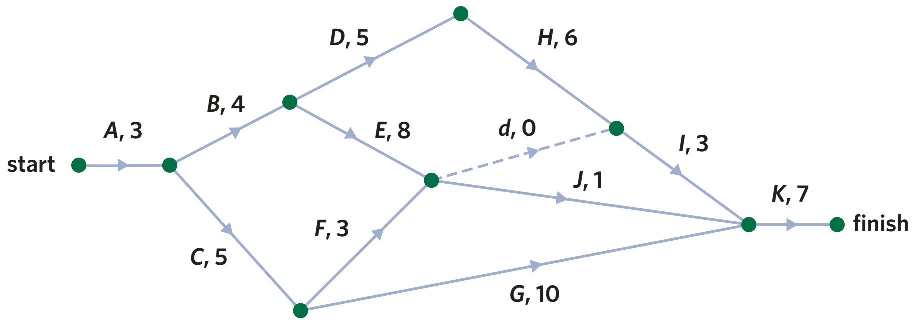

The builders are able to speed up the project. Activities *A*, *C*, *E*, *F* and *G* can be reduced in completion time at an additional cost. The maximum reduction in time for each of the five activities is 2 weeks.

a. Which of these activities, if their durations were reduced individually, would **not** result in an earlier completion of the project?

b. Determine the minimum time, in weeks, that the project can be completed now that certain activities can be reduced in duration.

c. The cost of reducing the time of each activity is $5000 per week. Determine the minimum additional cost for completing the project in this reduced time.

**8.** Hermione is planning on brewing a batch of Polyjuice Potion and has created an activity network for the project. Each activity is in hours.




**a.** What is the minimum brewing time for the Polyjuice Potion?

**b.** Find the critical path(s).

Hermione is able to buy special ingredients from Diagon Alley that speed up the time required for certain activities. Each special ingredient can reduce the duration of one of the activities *D*, *E*, *F*, *G*, *H* and *I* by exactly 2 hours.

**c.** If Hermione purchased all the special ingredients, what would be the new minimum brewing time for the Polyjuice Potion?

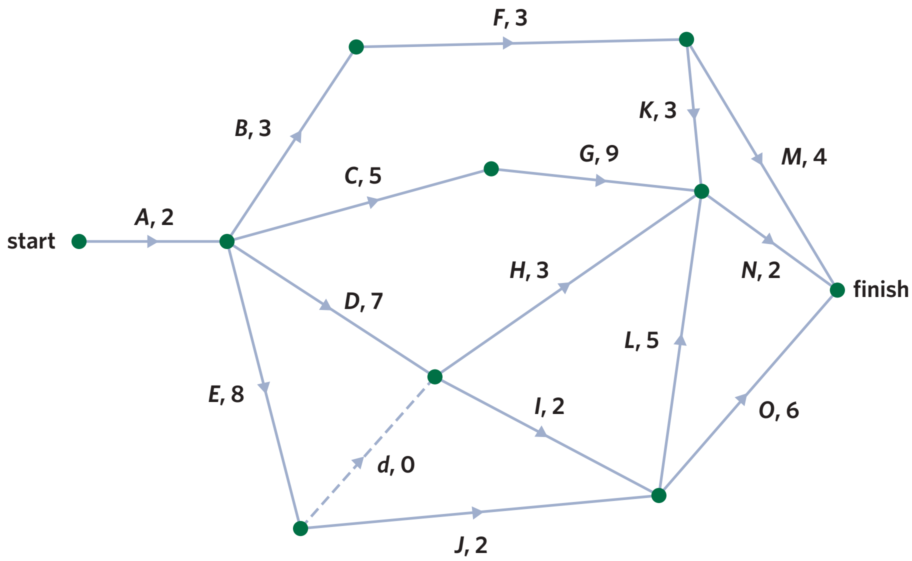
**d.** The cost of each special ingredient is 30 galleons. What is the least Hermione can spend while reducing the brewing time as much as possible?

## Exam practice

**9.** The rides at a theme park are set up at the beginning of each holiday season.

This project involves activities *A* to *O*.

The directed network shows these activities and their completion times in days.




The project could finish earlier if some activities were crashed.

Six activities, *B*, *D*, *G*, *I*, *J* and *L*, can all be reduced by one day.

The cost of this crashing is $1000 per activity.

**a.** What is the minimum number of days in which the project could now be completed? (1 MARK)

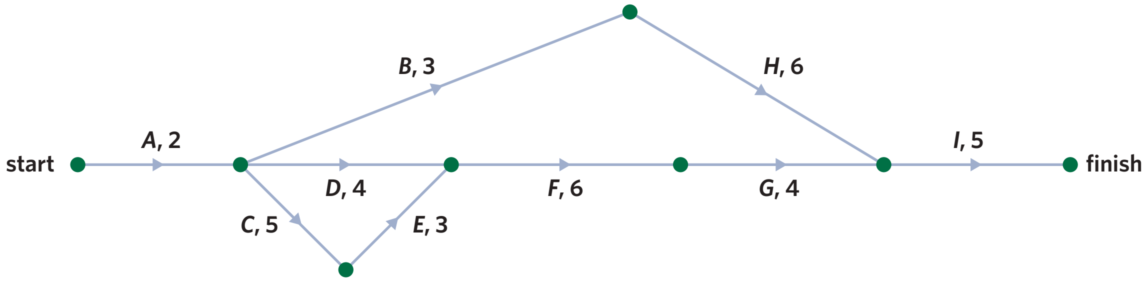
**b.** What is the minimum cost of completing the project in this time? (1 MARK)

*VCAA 2017 Exam 2 Networks and decision mathematics Q4ci,ii*

Part **a**: **35%** of students answered this question correctly.
Part **b**: **15%** of students answered this question correctly.

10. The following directed network shows the sequence of activities, $A$ to $I$, that is required to complete an office renovation.

The time taken to complete each activity, in weeks, is also shown.




The project manager would like to complete the office renovation in less time.

The project manager asks all the workers assigned to activity $H$ to also work on activity $F$.

This will reduce the completion time of activity $F$ to three weeks.

The workers assigned to activity $H$ cannot work on both activity $H$ and activity $F$ at the same time.

No other activity times will be changed.

This change to the network will result in a change to the completion time of the office renovation.

Which one of the following is correct?

**A.** The completion time will be reduced by one week if activity $F$ is completed before activity $H$ is started.

**B.** The completion time will be reduced by three weeks if activity $F$ is completed before activity $H$ is started.

**C.** The completion time will be reduced by one week if activity $H$ is completed before activity $F$ is started.

**D.** The completion time will be reduced by three weeks if activity $H$ is completed before activity $F$ is started.

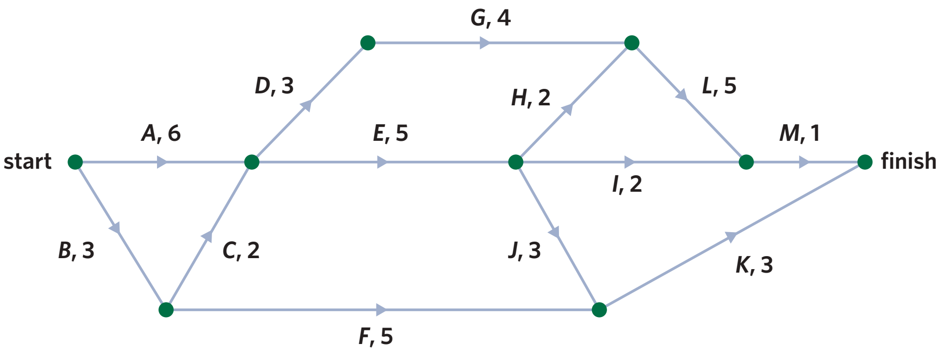
**E.** The completion time will be increased by three weeks if activity $H$ is completed before activity $F$ is started.

*VCAA 2020 Exam 1 Networks and decision mathematics Q10*

**24%** of students answered this question correctly.

11. Roadworks planned by the local council require 13 activities to be completed.

The following network shows these 13 activities and their completion times in weeks.




It is possible to reduce the completion time for activities $A$, $E$, $F$, $L$ and $K$.

The reduction in completion time for each of these five activities will incur an additional cost.

The table shows the five activities that can have their completion time reduced and the associated weekly cost, in dollars.

| **activity** | $A$ | $E$ | $F$ | $L$ | $K$ |
|---|---|---|---|---|---|
| **weekly cost ($)** | 140 000 | 100 000 | 100 000 | 120 000 | 80 000 |

The completion time for each of these five activities can be reduced by a maximum of two weeks.

The overall completion time for the roadworks can be reduced to 16 weeks.

What is the minimum cost, in dollars, of this change in completion time? (1 MARK)

*VCAA 2021 Exam 2 Networks and decision mathematics Q4c*

**9%** of students answered this question correctly.

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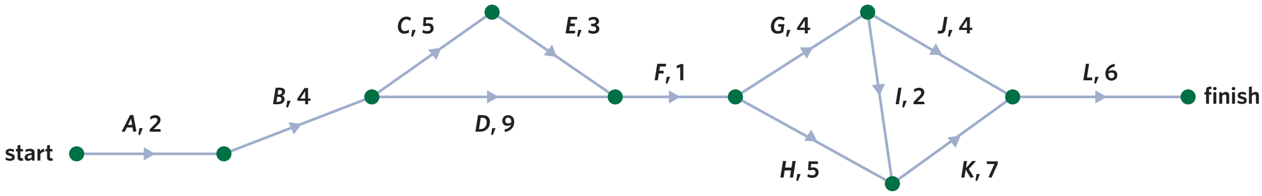
**12.** Fencedale High School is planning to renovate its gymnasium.

This project involves 12 activities, *A* to *L*.

The directed network shows these activities and their completion times, in weeks.




The minimum completion time for the project is 35 weeks.

It is possible to reduce the completion time for activities *C*, *D*, *G*, *H* and *K* by employing more workers.

The reduction in completion time for each of these five activities will incur an additional cost to the school.

The table shows the five activities that can have their completion times reduced and the associated weekly cost, in dollars.

| **activity** | *C* | *D* | *G* | *H* | *K* |
|---|---|---|---|---|---|
| **weekly cost ($)** | 3000 | 2000 | 2500 | 1000 | 4000 |

The completion time for each of these five activities can be reduced by a maximum of two weeks.

Fencedale High School requires the overall completion time for the renovation project to be reduced by four weeks at minimum cost.

Complete the following table, showing the reductions in individual activity completion times that would achieve this. (2 MARKS)

| **activity** | *C* | *D* | *G* | *H* | *K* |
|---|---|---|---|---|---|
| **reduction in completion time (0, 1 or 2 weeks)** | | | | | |

*VCAA 2019 Exam 2 Networks and decision mathematics Q3e*

The average mark on this question was **0.2**.

## Questions from multiple lessons

### Recursion and financial modelling

**13.** Seth purchased a TI-Nspire CX CAS Graphing Calculator for $286.

After three years, he sold the calculator for its estimated value of $58 on Melbourne Trade.

On average, Seth pressed the ‘enter’ button 643 times per school term.

The value of the calculator was depreciated using a unit-cost method of depreciation.

The depreciation in the value of the TI-Nspire CX CAS Graphing Calculator, per press of the ‘enter’ button, is closest to

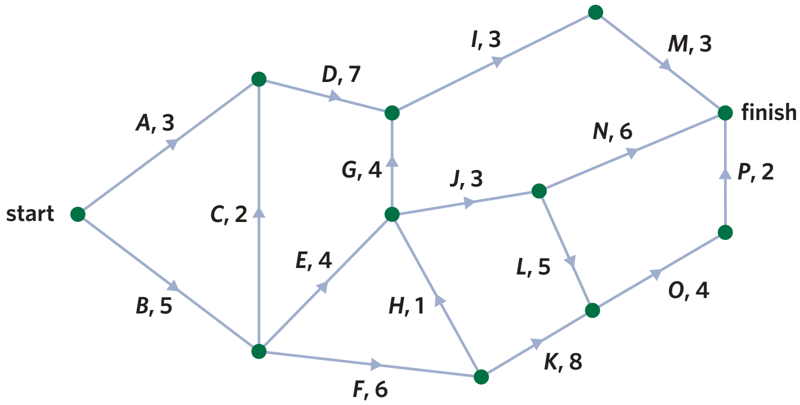
**A.** 2 cents. **B.** 3 cents. **C.** 4 cents. **D.** 10 cents. **E.** 12 cents.

*Adapted from VCAA 2017 Exam 1 Recursion and financial modelling Q21*

## Networks and decision mathematics

**14.** Sophie is making her renowned sticky date pudding.

The completion times, in minutes, for the activities required to make the sticky date pudding are shown in the following activity network.




What is the earliest starting time for activity $O$?

**A.** 16 minutes

**B.** 17 minutes

**C.** 18 minutes

**D.** 19 minutes

**E.** 20 minutes

*Adapted from VCAA 2017 Exam 1 Networks and decision mathematics Q4*

## Data analysis

**15.** In a large population of high school schools, the number of students per school is approximately normally distributed with a mean of 360 students and a standard deviation of 43 students.

**a.** Using the 68–95–99.7% rule, determine the percentage of schools expected to have less than 446 students. (1 MARK)

**b.** Using the 68–95–99.7% rule, determine the number of schools expected to have less than 317 students in a sample of 2000 schools. (1 MARK)

**c.** The standardised number of students in a school is $z = -3.4$. Determine the actual number of students in this school. Round to the nearest whole number. (1 MARK)

*Adapted from VCAA 2017 Exam 2 Data analysis Q1b,c*