Matrices

March 2023

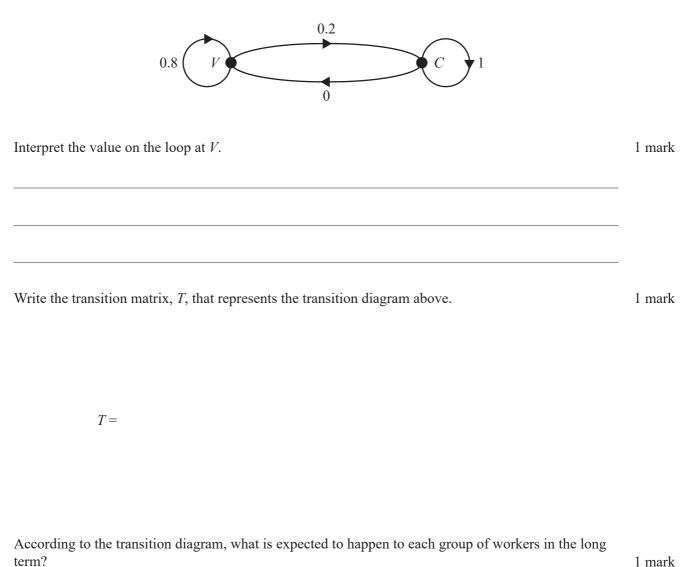
a.

b.

c.

Question 10 (3 marks)

An animal sanctuary is run by both volunteers and employed conservationists. People volunteer (V) at the sanctuary before they become conservationists (C). The transition diagram below shows the way in which employees are expected to transition between groups from month to month.



1 mark

TURN OVER

1 mark

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Question 11 (4 marks)

The sanctuary is trying to increase the population of an endangered fish.

During this process:

- eggs (E) may die (D) or they may live and become baby fish (B)
- baby fish (*B*) may die (*D*) or they may live and become adult fish (*A*)
- adult fish (A) continue to live for a period of time but will eventually die (D).

From year to year, changes to this fish population can be modelled by the recurrence relation

 $S_{n+1} = TS_n,$

where
$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} B \\ B \\ A \\ D \end{bmatrix}$$
 next year

The initial state matrix for this fish population, S_0 , is

$$S_{0} = \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix} \begin{bmatrix} E \\ B \\ A \\ D \end{bmatrix}$$

a. Complete the following multiplication for the row that determines the number of adult fish predicted to be in the population after one year.



b. To take into account the new eggs added to the population when the adult fish begin to breed, the following matrix recurrence relation is used.

$$R_{n+1} = T \times R_n + B,$$
this year
$$E \quad B \quad A \quad D$$
where $T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix} D$
next year,
$$B = \begin{bmatrix} k \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
R_0 = \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

The extra number of eggs expected to be laid by the adult fish each year is represented by k.

i. If
$$R_1 = \begin{bmatrix} 500\\ 4000\\ 650\\ 7150 \end{bmatrix}$$
, show that $k = 500$. 1 mark

ii. Determine how many adult fish there are expected to be after two years.

iii. Which populations, *E*, *B*, *A* and *D*, remain the same in the long term?

1 mark

1 mark

TURN OVER

15

Question 12 (3 marks)

GM 2 (SAMPLE)

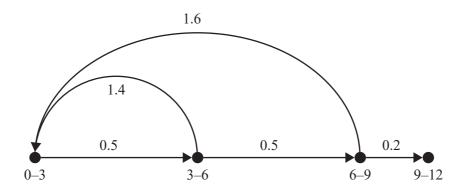
The sanctuary monitors the breeding patterns of the female population of a species of marsupial on a nearby island.

These marsupials have a life span of 12 years and have been categorised into four age groups: 0-3 years, 3-6 years, 6-9 years and 9-12 years.

The Leslie matrix, *L*, that models the breeding patterns for this female marsupial population is as follows.

$$L = \begin{bmatrix} 0 & 1.4 & 1.6 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$$

The same information is presented in the transition diagram below.



a. The following table shows the *birth rate*, for each of the four age groups, the average proportion of offspring born to each age group, and the *survival rate*, the average proportion of offspring to survive to the next age group.

Some values in the table are missing.

Use the Leslie matrix to complete the table below.

	Age group (years)				
	0-3 3-6 6-9 9-1				
Birth rate		1.4		0	
Survival rate	0.5			0	

1 mark

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The initial state matrix for this female marsupial population, S_0 , can be written as follows.

 $S_0 = \begin{bmatrix} 20\\ 40\\ 50\\ 15 \end{bmatrix}$

The recurrence relation to model the breeding patterns of this marsupial population is as follows.

$$S_{n+1} = LS_n$$

- **b.** By calculating S_1 , determine how many marsupials there are aged 0–3 years after one three-year time period.
- **c.** What percentage of this female marsupial population are aged 9–12 years after the second three-year time period? Round your answer to one decimal place.

March 2023

Question 13 (2 marks)

The four cleaners at the sanctuary, Trinh, Koby, Sam and Mon, take turns cleaning four different areas, A, B, C and D.

The cleaning roster for four weeks in February is shown in the table below.

	Trinh	Koby	Sam	Mon
Week 1	А	В	С	D
Week 2	В	С	D	А
Week 3	С	D	А	В
Week 4	D	А	В	С

The information from this table for Week 1 is presented in matrix N.

 $N = \begin{bmatrix} A & B & C & D \end{bmatrix}$

a. When matrix *N* is multiplied by the permutation matrix, *P*, the cleaning roster will change from Week 1 to Week 2.

Write down permutation matrix P in the space provided.

P =

b. The transpose of the permutation matrix, P^T , was used to create a new roster for March. The table below shows the cleaning roster for March. Some values in the table are missing.

Complete the cleaning roster for weeks 2, 3 and 4 in March.

	Trinh	Koby	Sam	Mon
Week 1	А	В	С	D
Week 2				
Week 3				
Week 4				

1 mark

1 mark

4

Matrices

Question 8 (3 marks)

A circus sells three different types of tickets: family (F), adult (A) and child (C). The cost of admission, in dollars, for each ticket type is presented in matrix N below.

$$N = \begin{bmatrix} 36\\15\\8\end{bmatrix} C$$

The element in row *i* and column *j* of matrix *N* is n_{ij} .

a. Which element shows the cost for one child ticket?

b. A family ticket will allow admission for two adults and two children.

Complete the matrix equation below to show that purchasing a family ticket could give families a saving of \$10.

 $\begin{bmatrix} 0 & 2 & 2 \end{bmatrix} \times N - \begin{bmatrix} \dots & \dots & \dots \end{bmatrix} \times N = \begin{bmatrix} 10 \end{bmatrix}$

c. On the opening night, the circus sold 204 family tickets, 162 adult tickets and 176 child tickets.
 The owners of the circus want a 3 × 1 product matrix that displays the revenue for each ticket type: family, adult and child.

This product matrix can be achieved by completing the following matrix multiplication.

$$K \times N = \begin{bmatrix} 7344\\2430\\1408 \end{bmatrix}$$

Write down matrix *K* in the space below.

K =

Question 9 (4 marks)

The circus is held at five different locations, E, F, G, H and I.

The table below shows the total revenue for the ticket sales, rounded to the nearest hundred dollars, for the last 20 performances held at each of the five locations.

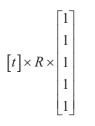
Location	E	F	G	Н	Ι
Ticket sales	\$960 000	\$990 500	\$940100	\$920800	\$901 300

The ticket sales information is presented in matrix R below.

 $R = \begin{bmatrix} 960\ 000 & 990\ 500 & 940\ 100 & 920\ 800 & 901\ 300 \end{bmatrix}$

a. Complete the matrix equation below that calculates the average ticket sales per performance at each of the five locations.
 1 mark

The circus would like to increase its total revenue from the ticket sales from all five locations. The circus will use the following matrix calculation to target the next 20 performances.



b. Determine the value of *t* if the circus would like to increase its revenue from ticket sales by 25%.

The circus moves from one location to the next each month. It rotates through each of the five locations, before starting the cycle again.

The following matrix displays the movement between the five locations.

c. The circus started in town *I*.

What is the order in which the circus will visit the five towns?

The circus plans to add a sixth location, J.

The only change to the cycle is that the circus will be held at location J after location E and before location G.

d. Complete the three columns in the following matrix, showing the new movement between the six locations, *E*, *F*, *G*, *H*, *I* and *J*.

1 mark

1 mark

Ζ

0 0

this month G H IΕ FJ_____0 _____1 0 E _____0 _____0 _____ F_____0 _____0 ___0 G next month ____ 0 ____ 0 _1 ___ *H* ____ 1 ____ 0 0 ___ *I* $_ 0 _ 0 0 _]J$

Question 10 (3 marks)

Within the circus, there are different types of employees: directors (D), managers (M), performers (P) and sales staff (S). Customers (C) attend the circus.

Communication between the five groups depends on whether they are customers or employees, and on what type of employee they are.

Matrix *G* below shows the communication links between the five groups.

	receiver						
			M				
G = sender	D	0	1	1	1	1	
	M	1	0	1	1	1	
G = sender	P	0	1	0	0	0	
	S	0	1	0	0	1	
	С	0	0	0	1	0	

In this matrix:

- The '1' in row D, column M indicates that the directors can communicate directly with the managers.
- The '0' in row *P*, column *D* indicates that the performers cannot communicate directly with the directors.
- **a.** A customer wants to make a complaint to a director.

What is the shortest communication sequence that will successfully get this complaint to a director? 1 mark

b. Matrix *H* below shows the number of two-step communication links between each group. Sixteen elements in this matrix are missing.

 $H = sender \begin{array}{c|cccc} D & M & P & S & C \\ D & 1 & --- & --- \\ M & 0 & --- & --- \\ M & 0 & --- & --- \\ 0 & --- & --- \\ S & 1 & --- & --- \\ C & 0 & 1 & 0 & 0 & 1 \end{array}$

- i. Complete matrix *H* above by filling in the missing elements.
- ii. What information do elements g_{21} and h_{21} provide about the communication between the circus employees?

TURN OVER

1 mark

Question 11 (2 marks)

The circus requires 180 workers to put on each show.

From one show to the next, workers can either continue working (W) or they can leave the circus (L).

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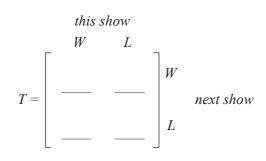
Once workers leave the circus, they do not return.

It is known that 95% of the workers continue working at the circus.

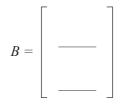
This situation can be modelled by the matrix recurrence relation

$$S_0 = \begin{bmatrix} 180\\0 \end{bmatrix}, \qquad S_{n+1} = TS_n + B$$

a. Write down matrix *T*, the transition matrix, for this recurrence relation.



b. Write down matrix *B* for this recurrence relation to ensure that the circus always has 180 workers. 1 mark



Matrices

Question 9 (3 marks)

Vince works on a construction site.

The amount Vince gets paid depends on the type of shift he works, as shown in the table below.

Shift type	Normal	Overtime	Weekend
Hourly rate of pay (\$ per hour)	36	54	72

This information is shown in matrix R below.

```
R = \begin{bmatrix} 36 & 54 & 72 \end{bmatrix}
```

a. Matrix R^T is the transpose of matrix R.

Write down matrix R^T in the space below.

 $R^T =$

During one week, Vince works 28 hours at the normal rate of pay, 6 hours at the overtime rate of pay, and 8 hours at the weekend rate of pay.

b. Complete the following matrix calculation showing the total amount Vince has been paid for this week.

 $\begin{bmatrix} & & & \\ & & & & \\ & & & \\ & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ &$

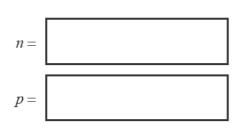
1 mark

Vince will receive \$90 per hour if he works a public holiday shift.

Matrix Q, as calculated below, can be used to show Vince's hourly rate for each type of shift.

$$Q = n \times \begin{bmatrix} 1 & 1.5 & 2 & p \end{bmatrix} \\ = \begin{bmatrix} 36 & 54 & 72 & 90 \end{bmatrix}$$

c. Write the values of n and p in the boxes below.



Question 10 (2 marks)

To access the southern end of the construction site, Vince must enter a security code consisting of five numbers.

The security code is represented by the row matrix W.

The element in row *i* and column *j* of *W* is w_{ij} .

The elements of *W* are determined by the rule $(i - j)^2 + 2j$.

a. Complete the following matrix showing the five numbers in the security code.



To access the northern end of the construction site, Vince enters a different security code, consisting of eight numbers.

This security code is represented by the row matrix X.

The element in row *i* and column *j* of *X* is x_{ij} .

The elements of *X* are also determined by the rule $(i - j)^2 + 2j$.

b. What is the last number in this security code to access the northern end of the construction site?

1 mark

Question 11 (3 marks)

A population of a native animal species lives near the construction site.

To ensure that the species is protected, information about the initial female population was collected at the beginning of 2023. The birth rates and the survival rates of the females in this population were also recorded.

This species has a life span of 4 years and the information collected has been categorised into four age groups: 0-1 year, 1-2 years, 2-3 years, and 3-4 years.

This information is displayed in the initial population matrix, R_0 , and the Leslie matrix, L, below.

	70		0.4	0.75	0.4	0	
D	80	L =	0.4	0	0	0	
$R_0 =$	90	L =	0	0 0.7	0	0	
	40		0	0	0.5	0	

- a. Using the information above
 - i. complete the following transition diagram.

ii. complete the following table, showing the initial female population, and the predicted female population after one year, for each of the age groups.

1 mark

	Age group						
	0-1 year 1-2 years 2-3 years 3-4 years						
Initial population							
Population after one year							

b. It is predicted that if this species is not protected, the female population of each of the four age groups will rapidly decrease within the next 10 years.

After how many years is it predicted that the total female population of this species will first be half the initial female population?

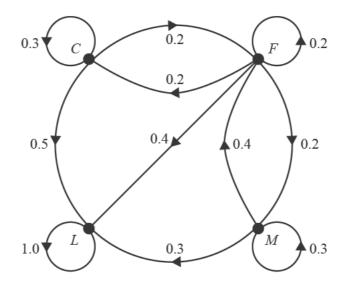
Question 12 (4 marks)

When the construction company established the construction site at the beginning of 2023, it employed 390 staff to work on the site.

The staff comprised 330 construction workers (C), 50 foremen (F) and 10 managers (M).

At the beginning of each year, staff can choose to stay in the same job, move to a different job on the site, or leave the site (L) and not return.

The transition diagram below shows the proportion of staff who are expected to change their job at the site each year.



This situation can be modelled by the recurrence relation

$$S_{n+1} = TS_n$$
, where
 T is the transitional matrix, $S_0 = \begin{bmatrix} 330 \\ 50 \\ 10 \\ 0 \end{bmatrix} \begin{bmatrix} C \\ F \\ M \\ L \end{bmatrix}$ and n is the number of years after 2023.

a. Calculate the predicted percentage decrease in the number of foremen (F) on the site from 2023 to 2025.

b. Determine the total number of staff on the site in the long term.

1 mark

To encourage more construction workers (C) to stay, the construction company has given workers an incentive to move into the job of foreman (F).

Matrix R below shows the ways in which staff are expected to change their jobs from year to year with this new incentive in place.

 $R = \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.4 & 0.2 & 0.4 & 0 \\ 0 & 0.2 & 0.3 & 0 \\ 0.2 & 0.4 & 0.3 & 1 \end{bmatrix} \begin{bmatrix} C \\ F \\ M \\ L \end{bmatrix}$ next year

The site always requires at least 330 construction workers.

To ensure that this happens, the company hires an additional 190 construction workers (C) at the beginning of 2024 and each year thereafter.

The matrix V_{n+1} will then be given by

$$V_{n+1} = RV_n + Z, \text{ where}$$

$$V_0 = \begin{bmatrix} 330 \\ 50 \\ 10 \\ 0 \end{bmatrix} \begin{bmatrix} C \\ F \\ M \\ L \end{bmatrix} = \begin{bmatrix} 190 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C \\ F \\ M \\ L \end{bmatrix} \text{ and } n \text{ is the number of years after 2023.}$$

c. How many more staff are there on the site in 2024 than there were in 2023?

In which year will the number of foremen first be above 200?

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1 mark

1 mark

Matrices

Question 10 (3 marks)

An egg farmer has five barns on their property: I, J, K, L and M. Matrix H below shows the available communication links between the five barns.

$$receiver$$

$$I \quad J \quad K \quad L \quad M$$

$$I \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ L & 0 & 0 & 1 & 0 & 1 \\ M & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

In this matrix:

- The '1' at element h_{25} indicates that barn J can directly communicate to barn M.
- The '0' at element h_{42} indicates that barn L cannot directly communicate to barn J.
- a. Which barn(s) can directly communicate to barn L?
- **b.** State two communication paths for barn J to communicate to barn K.
- **c.** The farmer plans to install one new direct communication link in one barn to receive direct communication from barn *I*.

In which barn should the new direct communication link be installed to ensure that barn *I* has either a one-step or a two-step communication link with every other barn? 1 mark

Question 11 (2 marks)

The farmer sells four different egg sizes (medium, large, extra large and jumbo) in cartons containing 12 eggs.

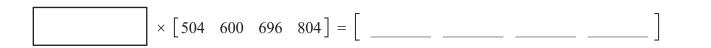
The table below shows the total egg weight, in grams, per carton for the different egg sizes.

egg size	medium	large	extra large	jumbo
total egg weight (grams)	504	600	696	804

This information is also displayed in Matrix *E*.

 $E = \begin{bmatrix} 504 & 600 & 696 & 804 \end{bmatrix}$

a. Complete the following matrix equation that will calculate the average egg weight in a carton for each egg size.



b. On one day, the farmer sold 122 cartons of medium eggs, 148 cartons of large eggs, 80 cartons of extra large eggs, and 52 cartons of jumbo eggs as shown in Matrix *Q*.

 $Q = \begin{bmatrix} 122 & 148 & 80 & 52 \end{bmatrix}$

In the space below, write a matrix calculation using Matrix Q and a column matrix to show that the total number of eggs sold on this day was 4824.

1 mark

Question 12 (4 marks)

Three types of eggs are sold at a local supermarket: cage eggs (C), free-range eggs (F) and barn-laid eggs (B).

Each week customers purchase eggs from one of the three types.

The supermarket is holding a $10\-$ week campaign to encourage more shoppers to purchase free-range eggs.

Let S_n be the state matrix that shows the expected proportion of customers who purchased each type of egg n weeks after the campaign began.

The proportion of customers who purchased each type of egg prior to the beginning of the 10-week campaign is shown in S_0 below.

$$S_0 = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} B$$

The recurrence relation that is used to calculate the expected proportion of customers who purchased each type of egg is given by

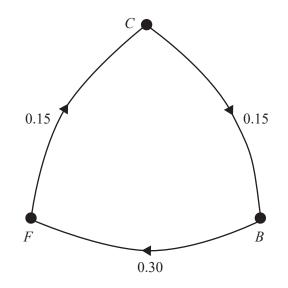
this week

$$C \quad F \quad B$$

$$S_{n+1} = T \times S_n \quad \text{where} \quad T = \begin{bmatrix} 0.65 & 0.15 & 0.25 \\ 0.20 & 0.70 & 0.30 \\ 0.15 & 0.15 & 0.45 \end{bmatrix} B \quad next \text{ week}$$

a. An incomplete transition diagram for Matrix *T* is shown below.

Complete the transition diagram by adding the missing edges and the missing proportions.



- **b.** How many weeks after the campaign begins will more than 44% of the customers first be expected to purchase free-range eggs (F)?
- C. For the duration of the 10-week campaign, 2000 customers purchased eggs at this supermarket each week.

Identify the expected number of additional shoppers who purchased free-range eggs (F) at the end of the campaign compared to the start of the campaign.

Round your answer to the nearest whole number.

d. The farmer believes that their free-range eggs are so superior that shoppers will eventually stop purchasing cage eggs and barn-laid eggs, and only buy their free-range eggs.

Complete column F in the following transition matrix for a scenario in which shoppers will eventually stop purchasing cage eggs and barn-laid eggs, and buy only free-range eggs.

0.15

1 mark

Question 13 (3 marks)

A farmer has four fields for chickens to roam in during the day.

The fields are named North (N), East (E), South (S) and West (W).

The following transition matrix is used to determine the expected proportion of chickens that will either change fields from one day to the next, or stay in the same field on consecutive days.

There are initially $10\,000$ chickens in each field.

 $T = \begin{bmatrix} 0.30 & 0.12 & 0.30 & 0.20 \\ 0.14 & 0.28 & 0.25 & 0.44 \\ 0.24 & 0.40 & 0.12 & 0.17 \\ 0.32 & 0.20 & 0.33 & 0.19 \end{bmatrix} W \quad next \, day$

a. After one day, how many chickens are expected to remain in the same field that they were in initially?

1 mark

b. After two days, 8962 chickens are expected to be in North field.
How many of these 8962 chickens are expected to have been in North field initially?
1 mark

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- **c.** Of the chickens expected to be in East field after three days, what is the percentage of chickens that were also expected to be in East field after both one day and two days?

Round your answer to the nearest whole number.