BACKTRACKING with spreadsheet

This lesson is designed and presented to support teachers seeking to incorporate important developments in mathematics education into their teaching.

Major emphases of this lesson are:

(a) in mathematics, establishing a firm conceptual base for algebraic equations;

(b) in methodology, using a story line to introduce meaning to an algebraic equation.

FEATURES OF THIS LESSON

1. First-principles focus on concept.
2. Enables quite complex equations to be solved in the first of two lessons.
3. A "story" approach to unravelling order of operations.
4. Forging the link between algebra and reality.

This lesson is an "insert" before formal processess are developed. It functions to delay formal approaches until the links between an algebraic statement and some form of reality are firmly established, for example:

$$\frac{2(x+3)-4}{2}=7$$

The lesson attempts to describe the left side of that equation in meaningful terms. In this case it is simply a summary of a story of what happened to an unknown number (the x). The plot is unravelled through "first-principles", the realm of reality. The answer is then returned to the world of algebra

SUMMARY (two periods)

1. The teacher discovers a number pupils have secretly selected by having the pupils use diagrams, thereby introducing backtracking.

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2. The class then uses the same process to discover a number selected by the teacher.

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3. Practice session.

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4. The story is created from algebra, using order of operations.

Period two

▼

5. Worksheet for practice.

PREREQUISITES

Some experience with algebraic symbols and knowledge of order of operations. All problems in this lesson avoid fractions or integers.

PREPARATION

For the second period each pupil will need a copy of the worksheet.

Comments from trial schools

"The bit of 'magic' used in the introduction got the lesson off to a flying start, and the pupils were very keen to use their numbers or give their operations."
"Simple to prepare and teach."
"Beware of going too fast for your slower pupils."
"An excellent lesson. Pupils enjoyed it and learned much."

PLAN

1. Posing the situation

(a) The teacher invites a class member to write a number on the board. The class sees the number, but the teacher doesn't know what it is. The pupil erases the number from the board.

(b) The teacher (emphasising that the number is unknown to him or her) asks pupils to perform the following operations, keeping the answers to themselves:

Multiply the number by 2: The teacher draws the diagram on the board.

×2

□ → □

Now add 3: The diagram on the board becomes:

×2 +3

□ → □ → □

Now multiply that by 2:

×2 +3 ×2

□ → □ → □ → □

Finally, take away 1:

×2 +3 ×2 −1

□ → □ → □ → □ → □

(c) The teacher now asks the class for the final answer and announces that he or she will be able to "figure out" the starting number.

The series of boxes is now a pictorial record of the "story" of what happened to the starting number. It is deliberately as non symbolic as possible, and there is no *mention of words such as flow chart.*

(d) Pupils copy the "boxes" into their books.

2. The Backtracking

The teacher now demonstrates the backtracking process. "Suppose the final number was 25.

×2 +3 ×2 −1

□ → □ → □ → □ → 25

"If the last thing you did was to take away 1 what number *must* be in the second-last box? The number you had before you took away the 1 *must* have been 26!"

×2 +3 ×2 −1

□ → □ → □ → 26 → 25

The teacher emphasises a desire to work back all the way to the starting number.

The next box (going backwards) *must* have been 13, because that's what was needed to get 26 when multiplied by 2.

Do you remember having 13 at that stage?

The whole process is an attempt to reconstruct the episode pupils have just been through.

×2 +3 ×2 −1

□ → □ → 13 → 26 → 25

Note that there is no *attempt to define the return operations as + 1 and ÷ 2 at this stage. The method is an appeal to the "intuitive logic" of unravelling a plot. The questions are of the form "What number must you have had before . . . [operation]?"*

The teacher similarly backtracks the last two steps and announces the starting number, which pupils verify (hopefully).

×2 +3 ×2 −1

5 → 10 → 13 → 26 → 25

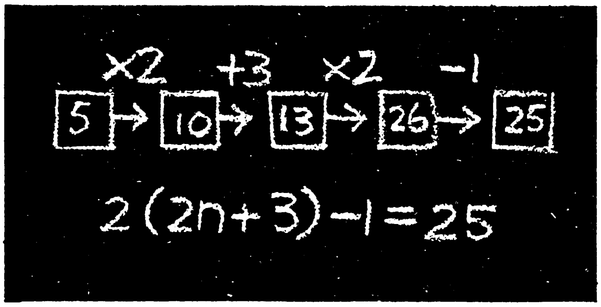
3. Algebraic connection

The teacher announces that a mathematician could summarise our story using *algebra.*

If we use **n** for the number, then, going through the events in our story one by one, the teacher writes the following on the board:

Number:	**n**
which becomes	**2n**
which becomes	**2n + 3**
which becomes	**2 (2n + 3)**
which becomes	**2 (2n + 3) − 1**

The blackboard now looks like this.

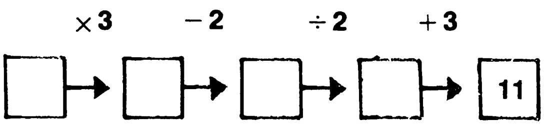


At this stage, however, no emphasis *is given to the algebraic summary. It is just the beginning of a link which will take time to forge.*

Up to this point, the lesson will have taken about fifteen minutes.

4. Second problem

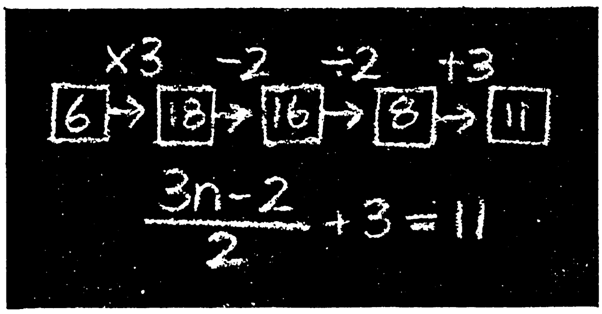
This time the teacher knows the unknown number and the class must work it out. For example, the teacher chooses 6 and asks the class to suggest operations pupils would like to perform on it. (Accept the division operation only if the number divides evenly.) For example:



Pupils record the boxes; when the teacher provides the final answer of 11, pupils "backtrack" as before to find the starting number, which the teacher verifies.

The teacher then, with no undue emphasis, shows how a mathematician would have summarised the story in algebra. The algebra is then presented simply as a form of shorthand.

The board and pupils' books will now look like this:



Note that the first two questions look particularly complex in algebra.

$$2(2n + 3) - 1 = 25$$

$$\frac{3n - 2}{2} + 3 = 11$$

Trial schools report, however, that pupils had no difficulty in solving the problems. The convention of writing 2n to mean 2 × n caused no problems; it is mentioned at this stage but not emphasised.

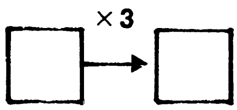
5. Third problem

Let a pupil (or small group) decide the next number. The rest of the class decides what operations to perform, diagrams are drawn, the backtracking occurs to find the unknown number. The teacher then briefly summarises that particular plot in algebra. Probably about five problems can be covered within a lesson. If the class appears ready, towards the end of the lesson the process can be reversed. The teacher announces that here is the algebraic summary of a story.

Can the class recreate the story and find the starting number?
For example:

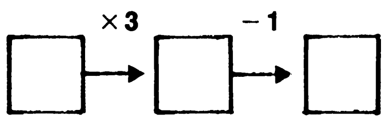
2 (3n − 1) + 1 = 47

Look at the n**; what was the first thing that happened to it?**
(It was multiplied by 3.)



Then what happened?

(We subtracted 1.)



Then what happened? (and so on . . .)

Trial schools found that based on the previous experience, no particular trouble occurred for any *pupils in reconstructing the* order *of the operations.*

Period two

In the second or follow-up period, a mixture of problems can be presented. Some of these start with the story — the pupils write the algebra; others start with the algebra.

(You may use the worksheet supplied.)

The teacher should leave pupils in no doubt as to the ultimate aim — to be able to read and solve the problems, given only the algebra.

Trial schools reported that at the end of the second period "for fun" problems such as

$$\frac{2(3n+1)-4}{5}+1\ -2=28$$

were solved with great ease. After this, typical text-book problems such as

$$x+3=8$$
$$2x+1=13$$

looked indeed trivial.

Trial schools reported clearly it was not the algebra *that causes trouble — it is giving* meaning *to the algebra.*
Once pupils are able to see the algebra as merely a summary of some form of reality, and are able to recreate that reality from the algebra, then difficulties disappeared.

Notes to teachers

1. Backtracking is not used for equations of the type $2x-5=3x+21$. (These need collecting of like terms first, then backtracking.)
2. One school reported success in getting pupils to work in pairs in the practice section. To work in pairs:

   One pupil chooses a number and writes it secretly. The other gives four operations (fewer for less-able pupils) and at the end is given the result.

   Important restriction: The player with the secret number should not accept subtraction if it produces a negative number, or division if it does not divide evenly.

   The second player must use boxes to keep track of what happens to the secret number, then uses backtracking to find the number.

   Players can take turns to hold the secret.

Worksheet attached.

Spreadsheet on the RIME: Algebra CD

The spreadsheet A7-Backtracking may be used to support this lesson. It models the same process.

WORKSHEET

BACKTRACKING

Write in algebra, then backtrack to find the initial number:

1. $\square \xrightarrow{\times 2} \square \xrightarrow{-5} \square \xrightarrow{+3} \boxed{14}$

2. $\square \xrightarrow{-3} \square \xrightarrow{\times 5} \square \xrightarrow{+3} \square \xrightarrow{\div 2} \boxed{4}$

3. $\square \xrightarrow{+2} \square \xrightarrow{\div 3} \square \xrightarrow{+1} \square \xrightarrow{\times 2} \boxed{8}$

4. $\square \xrightarrow{+8} \square \xrightarrow{-4} \square \xrightarrow{\times 5} \square \xrightarrow{-6} \boxed{29}$

5. $\square \xrightarrow{+6} \square \xrightarrow{\div 4} \square \xrightarrow{-7} \square \xrightarrow{+5} \boxed{5}$

Use backtracking diagrams to work out the starting number in each of the following:

6. $(2n-3)\times 3=21$ $\quad \square \quad \square \quad \square \quad \boxed{21}$

7. $\dfrac{7n-2}{10}+2=6$ $\quad \square \quad \square \quad \square \quad \square \quad \boxed{6}$

8. $5(n-2)-3=42$ $\quad \square \quad \square \quad \square \quad \boxed{42}$

9. $\dfrac{n-1}{3}+5=16$ $\quad \square \quad \square \quad \square \quad \square \quad \boxed{16}$

10. $\dfrac{6n+7}{4}+1=12$ $\quad \square \quad \square \quad \square \quad \square \quad \boxed{12}$

11. $2(n+3)=16$

12. $\dfrac{3n-2}{5}=2$

13. $\dfrac{n-3}{2}+6=11$

14. $\dfrac{6n-1}{5}=1$

15. $\dfrac{20n-10}{3}+2=12$