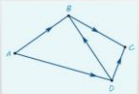
| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Row matrix** | A matrix with only 1 row | $\begin{bmatrix}3 & 2 & 1 & -4\end{bmatrix}$ | 7A |
| **Column matrix** | A matrix with only I column | $\begin{bmatrix}2\\3\end{bmatrix}$ | 7A |
| **Square matrix** | the number of rows equals the number of columns | $\begin{bmatrix}5 & 4\\4 & 2\end{bmatrix}$ $2 \times 2$ | 7A |
| **Zero (Null) matrix** | A matrix with all zero entries | $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ | 7A |
| **Summing matrix** | A row or column matrix in which all the elements are 1.<br>To sum the rows of an m × n matrix, post-multiply the matrix by an n × 1 summing matrix.<br>To sum the columns of an m × n matrix, pre-multiply the matrix by a 1 × m summing matrix. | $\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ $\begin{bmatrix}1 & 1\end{bmatrix}$ $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ | 7C |

| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Transpose of a matrix** | a new matrix that is formed by interchanging the rows and columns. | $A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\\a_{31} & a_{32}\end{bmatrix}$ $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix}a_{11} & a_{21} & a_{31}\\a_{12} & a_{22} & a_{32}\end{bmatrix}$ | 7.3<br>7B |
| **Symmetric matrices** | A matrix **A** is called symmetric if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ | $\begin{bmatrix}2 & 3\\3 & 1\end{bmatrix}$ $\begin{bmatrix}2 & 3 & 4\\3 & 1 & 5\\4 & 5 & 3\end{bmatrix}$ $\begin{bmatrix}1 & 2 & 4 & 6\\2 & 1 & 5 & 7\\4 & 5 & 3 & 8\\6 & 7 & 8 & 5\end{bmatrix}$ | 7.3<br>7A |
| **Diagonal matrices** | if all of the elements off the leading diagonal are zero. | $\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{bmatrix}$ $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 6 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 6\end{bmatrix}$ | 7A |
| **Identity matrices** | This is denoted by the letter **I** and has zero entries except for 1's on the diagonal. | $I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ $I = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$ | 7A |
| **Inverse matrices** | A square matrix A has an inverse if there is a matrix $A^{-1}$ such that: $AA^{-1} = I$ | If $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, then its inverse, $A^{-1}$, is given by $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$ provided $\frac{1}{ad-bc} \neq 0$; that is, provided $\det(A) \neq 0$. | 7.5 |

| Matrices Operations | CAS & Textbook Notes | Matrices Operations | CAS & Textbook Notes |
|---|---|---|---|
| Insert matrix | 7.1 7A | Power of a Matrix | 7B |
| Define a matrix (given a letter name) | 7.2<br>7A | Simultaneous Equations/ Matrices | 7.2 |
| Adding, subtracting, scalar multiplication | 7.2<br>7B | Solving unknow Matrix by given matrix equation | 7.2 |
| Two matrices multiplication | 7C | Constructing a Matrix by given i, j rule | 7A<br>7.6 |

| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Triangular matrices** | 1. An upper triangular matrix: all elements below the leading diagonal are zeros.<br>2. A lower triangular matrix: all elements above the leading diagonal are zeros. | $\begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{bmatrix}$ upper triangular matrix<br>$\begin{bmatrix}1 & 0 & 0 & 0\\3 & 2 & 0 & 0\\6 & 5 & 4 & 0\\0 & 9 & 8 & 7\end{bmatrix}$ lower triangular matrix | 7A |
| **Binary matrices** | A special kind of matrix that has only 1s and zeros as its elements. | $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}$ | 7E |
| **Permutation matrices** | A square binary matrix in which there is only one '1' in each row and column. | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$ $\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$ | 7E |

| NAME & Example | DESCRIPTION | EXAMPLE & key Points | Mathematica |
|---|---|---|---|
| **Communication matrices**<br>Receiver<br>$\begin{array}{cc} & \begin{array}{cccc}E & W & Y & K\end{array}\\ \text{Speaker}\ \begin{array}{c}E\\W\\Y\\K\end{array} & \begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 1 & 1\\0 & 1 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}\end{array}$ | A square binary matrix in which the 1s represent the links in a communication system. | All of the non-zero elements in the leading diagonal of a communication matrix, or its powers, represent redundant links in the matrix. | 7F |
| **Dominance matrices**<br><br> | Is a square binary matrix in which 1s represent one-step dominance between members of a group.<br>Usually an arrow towards one member means it is being dominated.<br>Usually the **row has the winner and the columns, the loser**<br>**D** represents one-step dominance<br>**D²** represents two-step dominance<br>Total dominance scores, **T = D + D²** | $D = \begin{array}{cc} & \begin{array}{cccc}A & B & C & D\end{array}\\ \begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix}0 & 1 & 0 & 1\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 1 & 1 & 0\end{bmatrix}\end{array}$ One-step: 2, 1, 0, 2<br>$D^2 = \begin{array}{cc} & \begin{array}{cccc}A & B & C & D\end{array}\\ \begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix}0 & 1 & 2 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}\end{array}$ Two-step: 3, 0, 0, 1<br>$T = D + D^2 = \begin{array}{cc} & \begin{array}{cccc}A & B & C & D\end{array}\\ \begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix}0 & 2 & 2 & 1\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 1 & 2 & 0\end{bmatrix}\end{array}$ Total: 5, 1, 0, 3 | 7F |

## THE INVERSE OF A MATRIX 7D 7.5

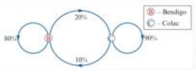
- THE NUMBER $A^{-1}$ IS CALLED THE *MULTIPLICATIVE INVERSE* OF *A* BECAUSE
- $A^{-1}A = I.$
- THE DEFINITION OF THE MULTIPLICATIVE **INVERSE OF A MATRIX** IS SIMILAR.

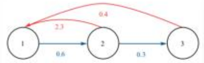
**Definition of the Inverse of a Square Matrix**

Let $A$ be an $n \times n$ matrix and let $I_n$ be the $n \times n$ identity matrix. If there exists a matrix $A^{-1}$ such that

$$AA^{-1} = I_n = A^{-1}A$$

then $A^{-1}$ is called the **inverse** of $A$. The symbol $A^{-1}$ is read "$A$ inverse."

| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Transition matrices** | Used to describe the way in which transitions are made between two states.<br>Recurrence Relation:<br>$S_0$ = initial value, $S_{n+1} = T * S_n$<br>Explicit Rule:<br>$S_n = T^n * S_0$<br>Steady State: determine values for a long run<br>$S = T^{50} * S_0 = T^{51} * S_0$ | <br><br>Rented in<br>$\begin{array}{cc} \begin{array}{cc}\text{Bendigo} & \text{Colac}\end{array} & \\ \begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix} & \begin{array}{c}\text{Bendigo}\\\text{Colac}\end{array}\end{array}$ Returned to | 7G<br>7H<br>7I |
| **Transition matrices**<br>**With Extra (Forcing, Culling)** | Recurrence Relation:<br>$S_0$ = initial value, $S_{n+1} = T * S_n + B$<br>Steady State: determine values for a long run<br>$S = T^{50} * S_0 + B = T^{51} * S_0 + B$ | | 7G<br>7H<br>7I |

| NAME | DESCRIPTION | EXAMPLE | Mathematica |
|---|---|---|---|
| **Leslie matrices** | Model of population growth that is very popular in population ecology.<br>Recurrence Relation:<br>$S_0$ = initial value, $S_{n+1} = L * S_n$<br>Explicit Rule:<br>$S_n = L^n * S_0$<br>Long term (Limiting) Behaviour: The proportion of the population in each age group does not change from one time period to the next. This happens if we can find a real number k such that $L * S_{n+1} = k * S_n$ for some sufficiently large n.<br>$L * S_{51} = K * S_{50}$ | <br><br>An $m \times m$ **Leslie matrix** has the form<br>$L = \begin{bmatrix}b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m\\s_1 & 0 & 0 & \cdots & 0 & 0\\0 & s_2 & 0 & \cdots & 0 & 0\\0 & 0 & s_3 & \cdots & 0 & 0\\\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\0 & 0 & 0 & \cdots & s_{m-1} & 0\end{bmatrix}$<br>where:<br>- $m$ is the number of age groups being considered<br>- $s_i$, the survival rate, is the proportion of the population in age group $i$ that progress to age group $i + 1$<br>- $b_i$, the birth rate, is the average number of female offspring from a mother in age group $i$ during one time period. | |

## THE INVERSE OF A MATRIX 7D 7.5

- THIS SECTION FURTHER DEVELOPS THE ALGEBRA OF MATRICES. TO BEGIN, CONSIDER THE REAL NUMBER EQUATION
  $AX = B.$
- TO SOLVE THIS EQUATION FOR $X$, MULTIPLY EACH SIDE OF THE EQUATION BY $A^{-1}$ (PROVIDED THAT $A \neq 0$).
- $AX = B$
- $(A^{-1}A)X = A^{-1}B$
- $(I)X = A^{-1}B$
- $X = A^{-1}B$

## 7A: introduction to matrices

A matrix is an array of numerical values

- Values are arranged into ROWS and COLUMNS

**ROWS:** are horizontal

- Number rows from top to bottom

**COLUMNS:** are vertical

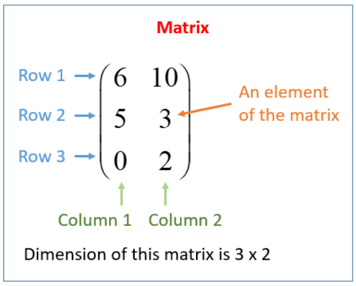
- Number columns from left to right

**ORDER OF A MATRIX:**

- Way to describe the dimensions (size) of a matrix

**ORDER = ROWS × COLUMNS**

- × is 'by'




**Types of Matrices:**

| | | |
|---|---|---|
| Column: only one column, any number of rows | Row: only one row, any number of columns | Zero: every element is '0'. Can be any size |
| $\begin{bmatrix} 42 \\ 56 \\ 74 \end{bmatrix}$ | $\begin{bmatrix} 8 & 17 & 42 & 52 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ |

**Square matrices:** have the same number of rows and columns.

| | |
|---|---|
| Diagonal matrix: all values apart from the leading diagonal are zero | $\begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ |
| Identity matrix: diagonal matrix, where the leading diagonals elements are all '1' | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Symmetric matrix: unchanged by transposition, the elements above the leading diagonal are a mirror image of the elements below | $\begin{bmatrix} 3 & 6 & 5 \\ 6 & 0 & 1 \\ 5 & 1 & 2 \end{bmatrix}$ |
| Upper triangular matrix: all elements below the leading diagonal are zero | $\begin{bmatrix} 4 & 5 & 7 \\ 0 & 1 & 6 \\ 0 & 0 & 8 \end{bmatrix}$ |
| Lower triangular matrix: all elements above the leading diagonal are zero | $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 4 & 2 \end{bmatrix}$ |

**Elements and notation:**

- Matrices are labelled with a capital letter
- The values within a matrix are called **elements**
- Elements are labelled with lowercase letters
- For matrix A, element $a_{mn}$ refers to the entry in the $m^{th}$ row and $n^{th}$ column

| | |
|---|---|
| $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ | $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 1 & 9 \end{bmatrix}$ Eg. Element $a_{21} = 3$ |

$x^{ij}$ – matrices can be constructed using element rules. Eg. Matrix C is a 2×2 matrix with the element rule $c_{ij} = i+j$. Create the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad A = \begin{bmatrix} 1+1 & 1+2 \\ 2+1 & 2+2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

## 7B Operations with matrices

**ADDITION AND SUBTRACTION:**

- Matrices must have the same order
- Add/subtract elements in the same position

**Matrix addition**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} a+e & b+g \\ c+f & d+h \end{bmatrix}$$

**Matrix subtraction**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} a-e & b-g \\ c-f & d-h \end{bmatrix}$$

**SCALAR MULTIPLICATION:**

- Multiply each element in the matrix by a scalar

$$k \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} k \times a & k \times b \\ k \times c & k \times d \end{bmatrix}$$

**Transpose:** swapping a matrices rows and columns. The transpose of matrix A is $A^T$. First row becomes first columns, etc.

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}^T = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}$$

## 7C: Advanced operations with matrices

**MATRIX MULTIPLICATION:** Involves both multiplication and addition

- **Post multiplication:** AB, matrix A is 'post multiplied' by matrix B
- **Pre multiplication:** AB, matrix B is 'pre multiplied' by matrix A

You must check that matrices can be multiplied first:

- Multiplication criteria: the number of columns in the first matrix MUST EQUAL the number of rows in the second matrix.
  - 'defined' = can be multiplied
  - 'undefined' = cant be multiplied

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix} \qquad BA = \begin{bmatrix} e \\ f \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Order: } 2 \times \underbrace{2 \quad 2}_{\text{equal}} \times 1 \qquad \text{Order: } 2 \times \underbrace{1 \quad 2}_{\text{not equal}} \times 1$$

AB is defined, BA is not defined.

**Matrix product:** the resulting matrix when two or more matrices are multiplied. The size of this matrix is determined by the outside numbers when writing the matrices orders. AB will produce a 2×1

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\text{Order: } \underbrace{2 \times 2 \quad 2 \times 1}_{2 \times 1}$$

**Multiplication by hand:** multiply the rows of the first matrix by the columns of the second matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \times e + b \times g & a \times f + b \times h \\ c \times e + d \times g & c \times f + d \times h \end{bmatrix}$$

**Summing matrices:** a row or column matrix where all elements are 1.

- **To sum the rows** of an m×n matrix, **post multiply** by an n×1 summing matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 1 + 3 \times 1 \\ 4 \times 1 + 5 \times 1 + 6 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

- **To sum the columns** of an m×n matrix, **pre multiply** by a 1×m summing matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 3 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 16 \end{bmatrix}$$

**Raising matrices to a power:** only SQUARE matrices can be raised to a power. The power indicates how many times the matrix is multiplied by itself.

## 7D: Inverse Matrices

**The determinant:**

- Used to identify if an inverse exists
- Must be a positive or negative number
- If the determinant equals ZERO, there is no inverse and the matrix is said to be 'singular'

**Calculating the determinant (2x2 by hand):**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

**The inverse:**

- Only square matrices have an inverse
- Inverse means opposite
- Something multiplied by its inverse is equal to 1, in this case, the IDENTITY MATRIX
- $A^{-1}$ = the inverse of matrix A
- $A \times A^{-1} = I$ (identity matrix)

**Calculating the inverse:**

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- b and c multiply by -1
- a and d swap positions
- Multiply by the scalar so that the answer is just the matrix

****if you are given the inverse of a matrix and need to find the original****

- Put the inverse to the power of -1
- $(A^{-1})^{-1} = A$

## 7E: Binary and Permutation Matrices

**Binary matrix**

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

**Permutation matrix**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Binary:** only has 1's and 0's as its elements (any size)
**Permutation matrix:** a type of binary matrix that has **only one 1 in each row and column.** It is always square and is used to rearrange elements.

- Column permutation: rearranges columns.
  - Post multiply (Q × P)
- Row permutation: rearranges rows.
  - Pre multiply (P× Q)

## 7F: Communication and Dominance Matrices

**Communication:** square binary matrix

- 1's represent a direct line of communication
- Communication goes both ways
- C = one step (direct communication)
- C2 = two step (communication via someone else)
- C + C2 = TC (total communication, showing all one and two step links)

**Dominance:**

- Used to display hierarchy
- One-way connections
- 1's are used for 'winners', 0's for 'losers'
- D = one step
- D2 = two step
- D + D2 = TD (total dominance)
- Directed, arrows point to loser

**Determining dominance when given the sum of one step and two step dominance:**

- One step dominance tells you how many times they won
- Two step dominance tells you the total sum of the one step dominances

## 7G: Introduction to transition matrices

**State matrix $S_n$**

- Column matrix
- Population at a given time

**Initial state matrix**

- Starting population
- $S_0$

**Transition matrix**

- Square matrix
- Studies change over time
- Elements are decimal numbers (0-1) that represent percentages
- Each column must sum to 1
- Other language: 'now/next', 'today/tomorrow', 'from/to'

$$\begin{array}{cc} & \textit{This time} \\ & \begin{array}{cc} P & I \end{array} \\ T = & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{array}{c} P \\ I \end{array} \textit{Next time} \end{array}$$

**Calculating state matrices:**

- Used to find the next state
- Use recursion (step by step)

$$\boxed{S_0 = \textit{initial state matrix}, \quad S_{n+1} = T \times S_n}$$

$S_n$ is the current state matrix
$S_{n+1}$ is the next state matrix
$T$ is the transition matrix.

- Use the rule (find future values, n, faster)

$$\boxed{S_n = T^n \times S_0}$$

*****Note: Finding a previous state:** say you are given T and $S_4$, how could you find a previous state? For example, if you need to find $S_3$ take the inverse of T and multiply is by $S_4$.

## 7H: The Equilibrium matrix

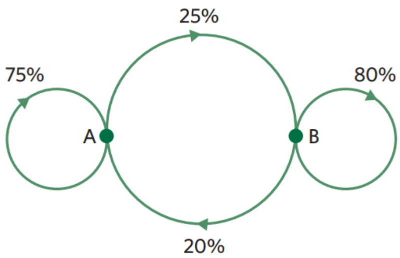
- 'Steady state'
- 'In the long term'
- Over time the populations settle to where there is no visible change
- This is when two consecutive matrices are equal (2dp for accuracy)
- Consecutive means finding $S_{17}$ and $S_{18}$ to be the same

## 7I: Applications of transition matrices

**Transition diagram:** a visual representation of how a transition matrix functions.

$$\begin{array}{c} \text{today} \\ \begin{array}{cc} \text{A} & \text{B} \end{array} \end{array}$$
$$T = \begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix} \begin{array}{c} \text{A} \\ \text{B} \end{array} \text{tomorrow}$$

can be represented by the following transition diagram.



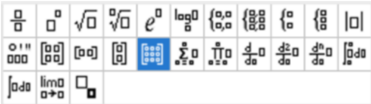

**Culling and Restocking:**

- Populations are subject to change over time
- Considers external forces affecting the population

$$S_{n+1} = TS_n + B$$

**Culling:** reduction/removed, negative number in matrix B
**Restocking:** addition, positive number in matrix B
**Keeping a population constant:** all future state matrices are equal to the initial. This can be determined by calculating $S_0 - S_1$

**CAS reference sheet: Matrices**

| | |
|---|---|
| 7.1 Enter a Matrix | 1. Ctrl N > 1 > \|□\| {⠿ ↵<br>2. ***OR*** Ctrl N > 1 > Ctrl > Menu > 8 ↵<br><br>3. Select No. of rows & columns ↵<br>4. Type each element, using **[tab]** to move. |
| 7.2 Define a Matrix | 1. Name of the matrix > ctrl > \|□\| {⠿ > enter full matrix ↵<br>$a:=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$ $\begin{bmatrix}1.&2.&3.\\4.&5.&6.\\7.&8.&9.\end{bmatrix}$<br>2. ***OR*** Enter full matrix > ctrl var > name of the matrix↵<br>$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\to b$ $\begin{bmatrix}1.&2.&3.\\4.&5.&6.\\7.&8.&9.\end{bmatrix}$ |
| 7.3 Transpose Matrix | 1. Name of the matrix > Menu > 7> 2↵<br>$a^{T}$ $\begin{bmatrix}1.&4.&7.\\2.&5.&8.\\3.&6.&9.\end{bmatrix}$ |
| 7.4 Determinant | 1. Menu > 7 > 3> name of the matrix↵<br>$\det(a)$ 0. |

| | |
|---|---|
| 7.5 Inverse | 1. Name of the matrix > ∧ > **Negation key (-) > 1** ↵<br>$c:=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ $\begin{bmatrix}1.&2.\\3.&4.\end{bmatrix}$<br>$c^{-1}$ $\begin{bmatrix}-2.&1.\\1.5&-0.5\end{bmatrix}$<br>2. Decimal → fraction<br>$\text{exact}(c^{-1})$ $\begin{bmatrix}-2&1\\\frac{3}{2}&\frac{-1}{2}\end{bmatrix}$ |

| | |
|---|---|
| 7.6 Forming matrix by i, j rule | 1. Menu > 7 > 1 >A > ij rule, i, j, row size, column size ↵<br>$\text{constructMat}(2\cdot i+3\cdot j,i,j,2,3)$ $\begin{bmatrix}5.&8.&11.\\7.&10.&13.\end{bmatrix}$ |

## 7A: Intro to matrices

- a matrix is an array of numeric values
- values are arranged into ROWS and COLUMNS

**rows:** are horizontal
number rows from top down
**columns:** are vertical
number columns from left to right

$$\begin{array}{c c} & \text{Column 1} \quad \text{column 2} \\ \begin{array}{c}\text{row 1}\\ \text{row 2}\\ \text{row 3}\end{array} & \begin{bmatrix} 6 & 2 \\ 4 & -7 \\ 3 & 1 \end{bmatrix} \end{array}$$

**the ORDER of a matrix**

- a way of describing the dimensions (size)

ORDER = ROWS × COLUMNS
"by"

$$\begin{array}{c c} \begin{array}{c}1\\ 2\\ 3\end{array} & \begin{bmatrix} 6 & 2 \\ 4 & -7 \\ 3 & 1 \end{bmatrix} \end{array}$$

order = 3 × 2

**types of square matrices**

- have the **SAME number of rows and columns**
- **diagonal matrices:** all values apart from the leading diagonal are zero
- **identity matrix:** diagonal matrix, where the leading diagonal are all '1'

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→leading diagonal

- **symmetric matrix**: unchanged by transposition, ,the elements above the leading diagonal are a mirror image of the elements below.

$$S = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix}$$

- triangular matrices:

upper: all elements below the leading diagonal are zero

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

lower: all elements above the leading diagonal are zero

$$\begin{bmatrix} 7 & 0 & 0 \\ 8 & 9 & 0 \\ 10 & 11 & 12 \end{bmatrix}$$

**other types of matrices:**

- column matrix: has only 1 column $C = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$
- row matrix: has only one row
$R = [7 \quad 8 \quad 11 \quad -5]$
- zero matrix: all elements are "0" $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- *Zero matrix doesn't have to be square.*

**elements and notation**

- matrices are labelled with a capital letter
- the values within a matrix are called elements
- elements are labelled with lower case letters

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$a_{22}$ row 2, column 2

$a_{nm}$ refers to the element in the $n^{th}$ row and $m^{th}$ column

* n=rows, m=columns

example: what is $d_{31}$ ?

$$D = \begin{bmatrix} 6 & 2 \\ 4 & -7 \\ 3 & 1 \end{bmatrix}$$

$d_{31}$
3rd row, 1st column
$d_{31} = 3$

It is relatively easy to turn tables of info into matrices

example: the following table represents the number of people who attend what gym class.

| | weights | aerobics | fitness |
|---|---|---|---|
| male | 16 | 104 | 86 |
| female | 75 | 34 | 94 |

→ matrix = $\begin{bmatrix} 16 & 104 & 86 \\ 75 & 34 & 94 \end{bmatrix}$

- $x_{ij}$
- sometimes the elements of a matrix are determined by a rule relating to the position of an element
- elements in row i, column j → $x_{ij}$

row i, column j

So…

$$X = \begin{bmatrix} x_{ij} & x_{ij} \\ x_{ij} & x_{ij} \end{bmatrix} = \begin{array}{c c} \text{column 1} \quad \text{column 2} & \\ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} & \begin{array}{c}\text{row 1}\\ \text{row 2}\end{array} \end{array}$$

example: the order of matrix X is 2×2, use the following rule to determine the elements:

$x_{ij} = i + 3j$

- $X = \begin{bmatrix} 1+3\times 1 & 1+3\times 2 \\ 2+3\times 1 & 2+3\times 2 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 5 & 8 \end{bmatrix}$

Menu > 7 > 1 >A > ij rule, i, j, row size, column size ↵

$\text{constructMat}(2\cdot i+3\cdot j,i,j,2,3) \quad \begin{bmatrix} 5. & 8. & 11. \\ 7. & 10. & 13. \end{bmatrix}$

## 7B: matrix operations

### Addition and subtraction:

- must have the same order (rows x columns)
- add/subtract elements in the same position

$A = \begin{bmatrix} 4 & 7 \\ 8 & 10 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 3 \\ 0 & 1 \end{bmatrix}$ $c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- A + B $= \begin{bmatrix} \mathbf{4} & \mathbf{7} \\ \mathbf{8} & \mathbf{10} \end{bmatrix} + \begin{bmatrix} \mathbf{6} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} =$

$$\begin{bmatrix} 10 & 10 \\ 8 & 11 \end{bmatrix}$$

- B - A $= \begin{bmatrix} \mathbf{6} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} - \begin{bmatrix} \mathbf{4} & \mathbf{7} \\ \mathbf{8} & \mathbf{10} \end{bmatrix} =$

$$\begin{bmatrix} 2 & -4 \\ -8 & -9 \end{bmatrix}$$

- c + B = undefined

### scalar multiplication:

- multiply each element in the matrix by the scalar.

- 3A=$\mathbf{3}\begin{bmatrix} \mathbf{4} & \mathbf{7} \\ \mathbf{8} & \mathbf{10} \end{bmatrix} =$

x x

$$\begin{bmatrix} 12 & 21 \\ 24 & 30 \end{bmatrix}$$

### transpose:

- switching the rows and columns

- $\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} \end{bmatrix}^{T} = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$

2 x 3 3 x 2

* first row becomes first column.

* second row becomes second column.

## 7C: Matrix multiplication

- involves both matrix multiplication and addition
- you must check the matrices can be multiplied first
- multiplication criteria = the number of columns in the first matrix must equal the number of rows in the second matrix
- "defined" = can be multiplied
- "not defined" = can't be multiplied

Example:

1. $\begin{bmatrix}2&4\\0&1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}16\\3\end{bmatrix}$ $2\times1$

$2\times\underbrace{2\quad2}_{\text{Same number= defined}}\times1$

Column of 1st matrix! Row of 2nd matrix!

These two numbers tell us the size of the product

2. $\begin{bmatrix}1\\3\\2\end{bmatrix}\begin{bmatrix}2&0\\1&3\end{bmatrix}$

$3\times1 \quad 2\times2$

Not the same
∴ Not defined

### Multiplication by hand

- multiply the rows of the first matrix by the columns of the second matrix
- Dot Product
- $\begin{bmatrix}1&0\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}r_1c_1\end{bmatrix}=[(1\times2)+(0\times3)]=[2]$

Example

1. $\begin{bmatrix}1&0\\2&3\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}r_1c_1\\r_2c_1\end{bmatrix}$

$2\times2 \quad 2\times1 \quad 2\times1$

$=\begin{bmatrix}(1\times2)+(0\times3)\\(2\times2)+(3\times3)\end{bmatrix}=\begin{bmatrix}2\\13\end{bmatrix}$

2. $\begin{bmatrix}1&2\\3&4\end{bmatrix}\cdot\begin{bmatrix}8&6\\4&2\end{bmatrix}=\begin{bmatrix}r_1c_1&r_1c_2\\r_2c_1&r_2c_2\end{bmatrix}$

$2\times2 \quad 2\times2 \quad 2\times2$

$=\begin{bmatrix}(1\times8)+(2\times4)&(1\times6)+(2\times2)\\(3\times8)+(4\times4)&(3\times6)+(4\times2)\end{bmatrix}$

$=\begin{bmatrix}16&10\\40&26\end{bmatrix}$

### Summing matrices

- a **row** or **column** matrix where all elements are one (1)

→ to sum the rows of an m×n matrix, **postmultiply** by an n×1 summing matrix

means placing the summing matrix **after** the matrix

example:

$\begin{bmatrix}1&0\\2&3\end{bmatrix}\cdot\begin{bmatrix}\mathbf{1}\\\mathbf{1}\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$

m × n  n × 1

2 × 2  2 × 1

n = 2

→ to sum the columns of an m×n matrix, **premultiply** by a 1×m summing matrix

means placing the summing matrix **before** the matrix

example:

$\begin{bmatrix}\mathbf{1}&\mathbf{1}&\mathbf{1}\end{bmatrix}\cdot\begin{bmatrix}4&2&3\\-1&0&-2\\0&6&1\end{bmatrix}=\begin{bmatrix}3&8&2\end{bmatrix}$

1 × m  m × n

1 × 3  3 × 3

m = 3

### Raising matrices to a Power

only **SQUARE** matrices can be raised to a power

example:

1. $\begin{bmatrix}1&2\\3&4\end{bmatrix}^2=\begin{bmatrix}1&2\\3&4\end{bmatrix}\cdot\begin{bmatrix}1&2\\3&4\end{bmatrix}$

$2\times2 \quad 2\times2$

same # → can be multiplied

2. $\begin{bmatrix}1\\2\\3\end{bmatrix}^3=\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}$ **=undefined**

$3\times1 \quad 3\times1 \quad 3\times1$

not the same #

∴ undefined and can't be multiplied

## 7D: the inverse

the determinant:

- tells us if an inverse exists
- must be a positive or negative number
- if the determinant equals ZERO, there is NO inverse and the matrix is "SINGULAR"

calculating the determinant (2x2)

$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ then det(A) = (a×d) — (b×c)

examples:

$B=\begin{bmatrix}2&3\\3&5\end{bmatrix}$

det(B) =(2×5) — (3×3) =1

:. inverse exists

$C=\begin{bmatrix}2&3\\2&3\end{bmatrix}$

det(C) =(2×3) — (3×2)= 6 — 6 = 0

:. no inverse

the inverse: $\frac{2}{1}\times\frac{1}{2}=\frac{2}{2}=1$

- only square matrices have an inverse
- inverse means opposite
- something multiplied by its inverse is equal to 1 (or the identity matrix)
- $A^{-1}$ = the inverse of A
- $A\times A^{-1}=I(identity)$ $\begin{bmatrix}1&0\\0&1\end{bmatrix}$

calculating the inverse

$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

then $A^{-1}=\frac{1}{det}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

b and c multiply by -1
and
a and d swap places

example

$D=\begin{bmatrix}2&2\\3&4\end{bmatrix}$

det(D) = (2×4) — (2×3) = 2

$\frac{1}{2}\begin{bmatrix}4&-2\\-3&2\end{bmatrix}=\begin{bmatrix}2&-1\\-1.5&1\end{bmatrix}$

* if you are given the inverse of a matrix and need to find the original *

put the inverse to the power of -1

$(A^{-1})^{-1}=A$

inverse  original

CAS

det: menu > 7 > 3

det( ) insert matrix in brackets

inverse: put your matrix in CAS and to the power of -1

## 7E: Binary and Permutation matrices.

binary matrix: only has 1's and 0's as its elements (any size)
permutation matrix: (square only)
- type of binary matrix
*only has one 1 in each row and column
- used to re-arrange elements

column permutation rearranges columns Post multiply (Q × P)

$$\underset{\text{Matrix } A}{\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}} \times \underset{\text{Matrix } P_1}{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}} = \underset{\text{Matrix } B}{\begin{bmatrix}1&3&2\\4&6&5\\7&9&8\end{bmatrix}}$$

row permutation rearranges rows. Pre multiply (P × Q)

$$\underset{\text{Matrix } P_4}{\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}} \times \underset{\text{Matrix A}}{\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}} = \underset{\text{Matrix } B}{\begin{bmatrix}7&8&9\\1&2&3\\4&5&6\end{bmatrix}}$$

examples
binary or permutation?

$$\begin{bmatrix}0\\1\end{bmatrix} \quad \begin{bmatrix}0&0\\0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

binary binary identity binary permutation

create a permutation matrix to rearrange.

1. $\begin{bmatrix}l\\x\\a\\e\end{bmatrix} \to \begin{bmatrix}a\\l\\e\\x\end{bmatrix}$

Rearranging the rows
pre-multiply P × Q

$$\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\begin{bmatrix}l\\x\\a\\e\end{bmatrix} = \begin{bmatrix}r_1c_1\\r_2c_1\\r_3c_1\\r_4c_1\end{bmatrix} = \begin{bmatrix}a\\l\\e\\x\end{bmatrix}$$

4×4 4×1 4×1

P × Q = 4×1

$$= \begin{bmatrix}0*l+0*x+1*a+0*e\\1*l+0*x+0*a+0*e\\0*l+0*x+0*a+1*e\\0*l+1*x+0*a+0*e\end{bmatrix}$$

2. $[u\ \ l\ \ m\ \ p]$ to $[p\ \ l\ \ u\ \ m]$

$$[u\ \ l\ \ m\ \ p]\begin{bmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{bmatrix} = [p\ \ l\ \ u\ \ m]$$

1×4 4×4 1×4

$= [r_1c_1 \quad r_1c_2 \quad r_1c_3 \quad r_1c_4]$

$r_1c_1 = u*0+l*0+m*0+p*1$

$r_1c_2 = u*0+l*1+m*0+p*0$

$r_1c_3 = u*1+l*0+m*0+p*0$

$r_1c_4 = u*0+l*0+m*1+p*0$

Rearranging the columns
post multiply (Q × P)

## 7F: communication + dominance.

communication:
one step: square binary matrix
- 1's represent a direct line of communication
- links go both ways
- C = one step
- $C^2$ = two step (communication via someone else)
- C + $C^2$ = TC (total communication)
showing all one and two step links

Example (undirected)

Jess
Hailey Charlie

$$C = \begin{array}{c} \begin{array}{ccc}C&H&J\end{array} \\ \begin{bmatrix}0&0&1\\0&0&1\\1&1&0\end{bmatrix}\end{array}\begin{array}{c}\\C\\H\\J\end{array}$$

$$C^2 = \begin{array}{c} \begin{array}{ccc}C&H&J\end{array} \\ \begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}\end{array}\begin{array}{c}\\C\\H\\J\end{array}$$

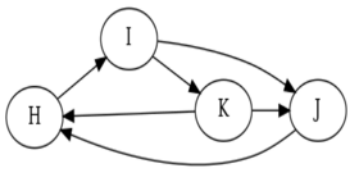
*1 way for H and C to comm via J
*redundant links
"message bounces back"

C + $C^2$ = TC

$$TC = \begin{array}{c} \begin{array}{ccc}C&H&J\end{array} \\ \begin{bmatrix}1&1&1\\1&1&1\\1&1&2\end{bmatrix}\end{array}\begin{array}{c}\\C\\H\\J\end{array} \quad \begin{array}{c}\text{sum rows}\\3\\3\\4\end{array}$$

dominance:
- used to display hierarchy
- one-way connections
- 1's are used for "winners"
- D = one step
- $D^2$ = two step
- D + $D^2$ = TD

directed



arrows point to losers

example: chess tournament results
- H beat I
- I beat J and K
- J beat H
- K beat H and J

who is the most dominant?

$$D = winner \quad \begin{array}{c} \text{Loser} \\ \begin{array}{cccc}H&I&J&K\end{array}\end{array}$$

$$\begin{array}{c}H\\I\\J\\K\end{array}\begin{bmatrix}0&1&0&0\\0&0&1&1\\1&0&0&0\\1&0&1&0\end{bmatrix}\begin{array}{l}1\\2 \to \text{best}\\1\\2 \to \text{best}\end{array}$$

$$D^2 = \begin{array}{c}H\\I\\J\\K\end{array}\begin{bmatrix}0&0&1&1\\2&0&1&0\\0&1&0&1\\1&1&0&0\end{bmatrix}\begin{array}{l}2\\3 \to \text{1st}\\1\\2\end{array}$$

$$TD = \begin{array}{c}H\\I\\J\\K\end{array}\begin{bmatrix}0&1&1&1\\2&0&2&1\\1&1&0&1\\2&1&1&0\end{bmatrix}\begin{array}{l}3 \to \text{3rd}\\5 \to \text{1st}\\3 \to \text{3rd}\\4 \to \text{2nd}\end{array}$$

## 7G: transition matrices

**State matrix:**
- column matrix
- population at a given time

**initial state matrix:**
- initial / starting population

**transition matrix:**
- states change over time
- always SQUARE
- columns MUST add up to 1

**labelling:** now → other language: "today" / "tomorrow" "from" / "to"

$$T = \begin{array}{cc} A & B \end{array}$$
$$T = \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix} \begin{matrix} A \\ B \end{matrix} \quad \text{next}$$

only decimals inside a transition matrix (0-1)

**calculating state matrices:**
- to find the next state
- use recursion (step by step)

$\boxed{S_0 = \text{initial state matrix}, \quad S_{n+1} = T \times S_n}$

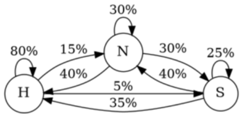
find next state · transition matrix · previous state

**rule:** (find future values faster)

$\boxed{S_n = T^n \times S_0}$

n = time frame interested in.

example: people can be happy (H), neither happy or sad (N) or sad (S)




a. complete T today

$$T = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} H & N & S \end{array}$$
$$T = \begin{bmatrix} 0.8 & 0.4 & 0.35 \\ 0.15 & 0.3 & 0.4 \\ 0.05 & 0.3 & 0.25 \end{bmatrix} \begin{matrix} H \\ N \\ S \end{matrix} \quad \text{tomorrow}$$

0.05 = 5%

b. on a given day 1200 ppl are happy, 600 are neither and 200 are sad, construct $S_0$

$$S_0 = \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix} \begin{matrix} H \\ N \\ S \end{matrix}$$

c. how many people are happy, neither or sad tomorrow?

$$S_1 = T \times S_0 = \begin{bmatrix} 1270 \\ 440 \\ 290 \end{bmatrix} \begin{matrix} H \\ N \\ S \end{matrix}$$

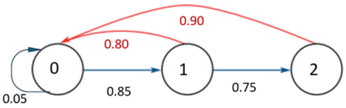
d. how many people are happy, neither or sad in 5 days time?

$$S_5 = T^5 \times S_0 = \begin{bmatrix} 1310 \\ 430 \\ 260 \end{bmatrix} \begin{matrix} H \\ N \\ S \end{matrix}$$

Leslie matrix

| | |
|---|---|
| RR | $S_{n+1} = L \times S_n$ |
| Rule | $S_n = L^n \times S_0$ |

Leslie matrix example.




fighting fish population

age

$$L = \begin{array}{c} \end{array}\begin{array}{ccc} 0 & 1 & 2 \end{array}$$
$$L = \begin{bmatrix} 0.05 & 0.80 & 0.90 \\ 0.85 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \qquad S_0 = \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$$

a. interpret $L_{21}$ 0.85 survival rate of less than 1 year old.

b. Interpret $L_{13}$ 0.90 fertility rate of 2-year-old fish

c. fish population in 2 years?

$$S_2 = L^2 \times S_0 = \begin{bmatrix} 64 \\ 52 \\ 32 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \qquad 64 + 52 + 32 = 148$$

d. Calculate $S_8$

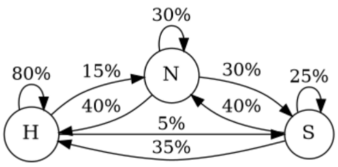
$$S_8 = L^8 \times S_0 = \begin{bmatrix} 128 \\ 97 \\ 66 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$$

## 7H: Equilibrium matrix

- "steady state"
- over time the populations settle to where there is no visible change
- when two **consecutive** matrices are equal (2dp for accuracy)
- consecutive means finding $S_{17}$ and $S_{18}$ to be the same (for example)
- "in the long term"

**example:**

people can be happy (H), neither happy or sad (N) or sad (S)




a. complete T today

$$\begin{array}{ccc} H & N & S \end{array}$$
$$T = \begin{bmatrix} 0.8 & 0.4 & 0.35 \\ 0.15 & 0.3 & 0.4 \\ 0.05 & 0.3 & 0.25 \end{bmatrix} \begin{matrix} H \\ N \\ S \end{matrix} \quad \text{tomorrow}$$

0.05 = 5%

b. on a given day 1200 ppl are happy, 600 are neither and 200 are sad, construct $S_0$

$$S_0 = \begin{bmatrix} 1200 \\ 600 \\ 200 \end{bmatrix} \begin{matrix} H \\ N \\ S \end{matrix}$$

c. In the long term, how many people are expected to be happy??

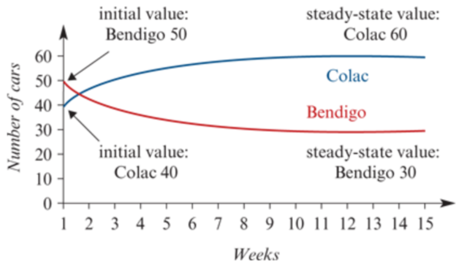
$$S_{15} = S_{16} = \begin{bmatrix} 1312 \\ 429 \\ 259 \end{bmatrix} \begin{matrix} H \\ N \\ S \end{matrix}$$

1312 ppl expected to be happy.

- To investigate a car rental question, we start by listing the state matrices from week 0 to week 15.

| Week | 0 | 1 | 2 | 3 | 4–11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| State matrix | $\begin{bmatrix} 50 \\ 40 \end{bmatrix}$ | $\begin{bmatrix} 44 \\ 46 \end{bmatrix}$ | $\begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$ | $\begin{bmatrix} 36.9 \\ 53.1 \end{bmatrix}$ | … | $\begin{bmatrix} 30.3 \\ 59.7 \end{bmatrix}$ | $\begin{bmatrix} 30.2 \\ 59.8 \end{bmatrix}$ | $\begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$ | $\begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$ |

- In summary, even though the number of cars returned to each location varied from day to day, the numbers at each location eventually settled down to an equilibrium or steady-state solution. In the steady state, the number of cars at each location remained the same.



## 7I Applications of transition matrices

The total population is regularly increased or decreased by a set quantity, for a reason that is not already accounted for, a new model is required.

This model is particularly useful for modelling the population of animals, where humans may choose to add or remove a set number of animals at a regular interval.

Use recursion (step by step)

$$\boxed{S_0 = \text{initial state},\ S_{n+1} = T \times S_n + B}$$

Culling Or restocking matrix

find next state   transition matrix   previous state

where B is a matrix of the same order as $S_n$ and $S_{n+1}$

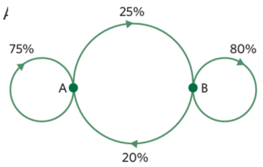
**Culling** is the reduction of an animal population by slaughter. Represents a **subtraction** to the population from one state to the next.

**Restocking** typically refers to replacing stock with a new supply. Represents an **addition** to the population from one state to the next.

**example 1:** A wildlife park has two groups of kangaroos:

Each month:

- 75% of kangaroos in A
- 25% move to B.
- 20% of kangaroos in B move to A.
- 80% stay in B.




a. construct a transition matrix

$$T = \begin{array}{c} \text{this month} \\ \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ A \\ B \end{array} \text{ Next month}$$

b. Suppose this month's population is

$$P_0 = \begin{bmatrix} 200 \\ 300 \end{bmatrix} \begin{array}{l} A \\ B \end{array}$$

Find the population tomorrow.

$P_1 = TP_0$ $\quad P_1 = \begin{bmatrix} 210 \\ 290 \end{bmatrix} \begin{array}{l} A \\ B \end{array}$

c. Unfortunately, a disease outbreak requires wildlife managers to cull 20 animals from A. write a culling matrix.

$$C = \begin{bmatrix} -20 \\ 0 \end{bmatrix} \begin{array}{l} A \\ B \end{array}$$

d. The following month, healthy kangaroos are introduced.

40 kangaroos to enclosure A

25 kangaroos to enclosure B

The restocking matrix is

$$R = \begin{bmatrix} 40 \\ 25 \end{bmatrix} \begin{array}{l} A \\ B \end{array}$$

Find the new population after restocking.

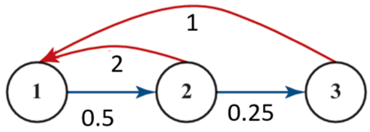
$P_{final} = TP_1 + R$ $\quad P_{final} = \begin{bmatrix} 256 \\ 310 \end{bmatrix} \begin{array}{l} A \\ B \end{array}$

**example 2:** The following life-cycle describes the growth of a rabbit population. Stage 1 = Baby rabbits; Stage 2 = Young rabbits; Stage 3 = Adult rabbits.

- Young rabbits produce **2** baby rabbits each breeding season. Adults produce **1** Baby rabbit.
- 50% of baby rabbits survive to become young rabbits.
- 25 of young rabbits survive to become adults.

a. Write the Leslie matrix and graph for this system.

$$L = \begin{array}{c} \text{now} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ 1 \\ 2 \\ 3 \end{array} \text{ next}$$




b. The initial population is $S_0 = \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$, find $S_3$

$$S_3 = \begin{bmatrix} 241 \\ 114 \\ 27 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$$

c. Find $S_{20}$ $S_{21}$.Verify that the stage distributions of the two population matrices are nearly in the same ratio and hence determine the long-term population growth rate per breeding season.

$S_{20} = \begin{bmatrix} 622 \\ 294 \\ 70 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$ $\quad S_{21} = \begin{bmatrix} 658 \\ 311 \\ 74 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$

$S_{21} \approx 1.057 * S_{20}$

At this stage, the rate of increase is approximately 5.7% for each breeding season.