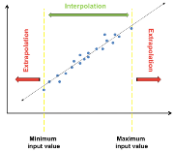
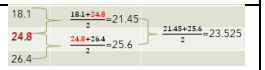
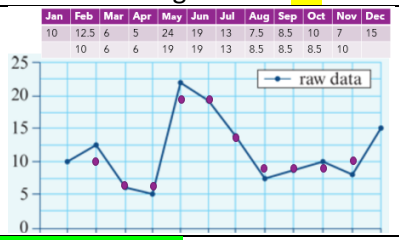
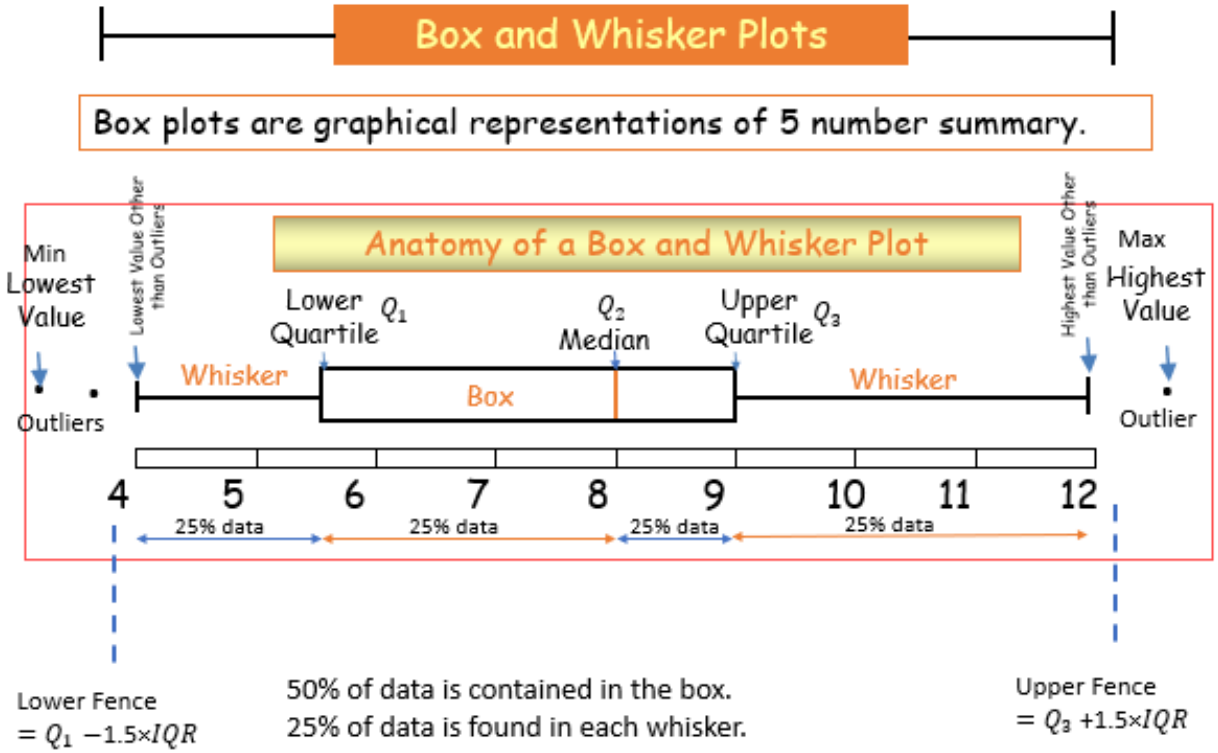
Topics	Data Types 1A		Display/Analyse Tools		Report/Explain/Interpret/Describe		
Univariate Data Chapter 2	Categorical variables 1B	Nominal data Ordinal data	Bar chart, Pie Chart, Frequency Table 1a, Segmented bar chart		Mode/ Modal Value Frequency types, Frequency % = $\frac{Count}{Total\ Count} \times 100\%$		
	Numerical variables 1C	Discrete data	Boxplots 1c 1.2, Grouped Frequency Tables	Stem plot 1b, dot plot 1b 1.2	Shape → Centre → Spread → Outliers → 1F	Symmetric Mean $\bar{x}$ 1.1 Standard Deviation S 1G 1.1  68-95-99.7% rule 1H 1I Z-score=Z=$\frac{x-\bar{x}}{S}$ 1I $x = \bar{x} + Z * S$	Skewed 1.1 Median M or Q_2 IQR, Range $\begin{cases} \text{Lower Fence} = Q_1 - 1.5 * IQR \\ \text{Upper Fence} = Q_3 + 1.5 * IQR \end{cases}$ 1.1 5-figure summary: Min, Q_1, Q_2, Q_3, Max IQR=$Q_3 - Q_1$, Range=Max−Min
		Continuous data		Stem plot 1b, histogram 1b 1.1, log 1.2 log-histogram 1.2			
Bivariate Data Chapter 2	Two categorical variables 2A		Segmented bar chart, two-way frequency table 2a, parallel bar chart 2.1		Mode/ Modal Value Frequency types		
	One categorical, one numerical variable 2B		Back-to-back stem plots, parallel dot plots, parallel box plots 2b 2.2		Shape → Centre → Spread →	Symmetric 1.1 Mean $\bar{x}$ Standard Deviation S	Skewed 1.1 Median M or Q_2 IQR, Range
Chapter 7	Two numerical variables 2C  Interpolation Extrapolation The assumptions for fitting a least squares line 1. the data is numerical 2. the association is linear 3. there are no clear outliers.		Scatterplot 2c 3.2 Explanatory variable → explains/predicts → Response variable residual = actual data value y − predicted 3.3 value y residual = $y - \hat{y}$ Nil pattern residual 3.2 plot 2f = Linear relation Curved/ patterned residual plot ≠ linear relation		Strength → Direction → Form → 3A 3B 3C 2e LSRL y=a+bx 3.1 2e slope $b=\frac{rs_y}{s_x}$ 2e y-intercept $a = \bar{y} - b\bar{x}$	Strong/Moderate/Weak (Check r value) 2D 3.1 Positive / Negative Linear / Non-linear Reporting 2d on Coefficient of Determination r^2 3.1 Almost [r^2 in %] of [RV y] can be explained / predicted by [EV x].	

Time Series 4A 4E: LSRL	Features 4A	Moving smoothing 4.1			Seasonal Index S.I. 4D	Deseasonalising 4D
	Trend Cycles Seasonality Structure change Outliers 	Moving Mean 4B 4.1 3/5 moving mean  	2/4 Moving mean  	Moving Median 4C  	S.I.=$\frac{Value\ for\ Season}{Yearly\ Average}$ Yearly Average=$\frac{Sum\ of\ Season\ Values}{No.of\ season\ per\ year}$ **Correct** S.I. = $(\frac{1}{S.I.} - 1) \times 100$ + means ↑, − means ↓	Deseasonalised Figure = $\frac{Actual\ Figure}{S.I.}$ = $Actual\ Figure * \frac{1}{S.I.}$ Actual figure= Deseasonalised Figure * S.I.

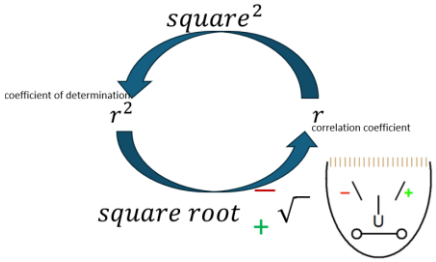
Note: Textbook Summary Notes Section #, Report Instruction Notes #. CAS Instruction Notes #

1E: The Box Plot **1.2**



2D: Correlation 3.1& Causation

$0.75 \leq r \leq 1$ **Strong, positive, linear association**
$0.5 \leq r < 0.75$ **Moderate, positive, linear association**
$0.25 \leq r < 0.5$ **Weak, positive, linear association**
$-0.25 < r < 0.25$ **No association**
$-0.5 < r \leq -0.25$ **Weak, negative, linear association**
$-0.75 < r \leq -0.5$ **Moderate, negative, linear association**
$-1 \leq r \leq -0.75$ **Strong, negative, linear association**



Name of Variation	Symbol of Variation	Equation of Variation	CAS	
			Slope/Equation	Plot/Equation
Direct Variation	$y \propto x$	$y = kx$	**3.1**	$y = kx + c$
Inverse Variation	$y \propto \frac{1}{x}$	$y = \frac{k}{x}$	**3.1**	$y = \frac{k}{x} + c$
Logarithm Variation	$y \propto \log x$	$y = k \log x$	**3.1**	$y = k \log x + c$
Variation involving powers	$y \propto x^2$	$y = kx^2$	**3.1**	$y = kx^2 + c$
Other variations	$y \propto \sqrt{x}$ $y \propto x^3$ $y \propto \frac{1}{x^2}$	$y = k\sqrt{x}$ $y = kx^3$ $y = \frac{k}{x^2}$	**3.1**	

Chapter 9

1A: Types of data

Categorical: characteristics/qualities

- Nominal: grouped according to characteristics
- Ordinal: can be grouped and ordered

Numerical: numbers/quantities

- Discrete: whole numbers, can be counted
- Continuous: is measured

1B: displaying categorical data

Count frequency: number of times the category appears in the data

Percentage frequency: $\frac{count\ frequency}{total\ count} \times 100$

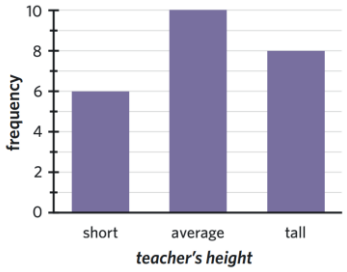
Mode: most frequently occurring value or category

Frequency Table:

teacher's height	frequency	
	number	%
short	6	25.0
average	10	41.7
tall	8	33.3
total	24	100.0

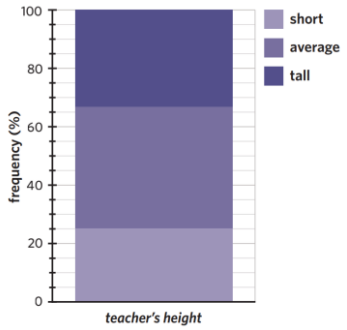
Bar chart:

- Must have gaps between bars



Segmented bar chart:

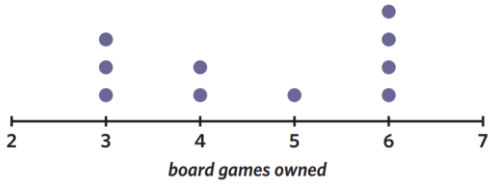
- Can be count or percentage frequency
- Must have a key



1C: Displaying Numerical data

Dot plot

- Discrete data
- Small data sets



Stem and leaf plot

- Needs a key
- Can have class intervals (splitting the stem in two if it is really large)

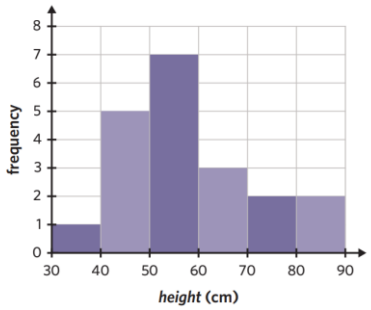
No intervals	With class intervals of 5
Key: 1 | 2 = 1.2	**Key:** 1 | 2 = 12
1 | 3 3 4 6 8	0 | 1 1 2 3 4
2 | 0 4 9	0 | 5 6 6 8 8 9
3 | 1 1 1 4 5 8	1 | 2 3 3
4 | 2	1 | 6 7 7 8 9 9

Grouped frequency tables

height (cm)	frequency	
	number	%
30–<40	1	5
40–<50	5	25
50–<60	7	35
60–<70	3	15
70–<80	2	10
80–<90	2	10
total	20	100

Histogram:

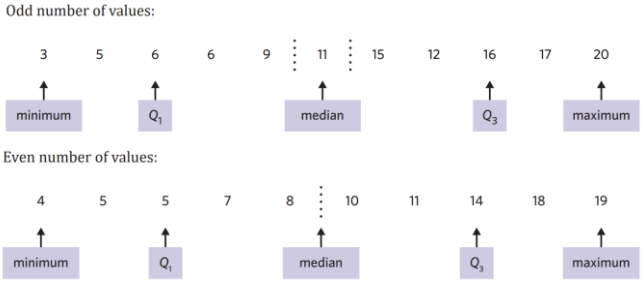
- Continuous data
- Intervals – no gaps between bars
- No gaps between bars
- X-axis markers are always a whole number



1E: the five-number summary and boxplots

5 number summary:

- Minimum: smallest value in data set
- Q1: median of the lower half
- Median: middle value in an ordered data set
- Q3: median of the upper half
- Maximum: largest value in data set



Spread: refers to how variable the data set it

Range = maximum – minimum

Interquartile range: measure of spread of the middle 50% of a data set. Accurate measure of spread when outliers are present.

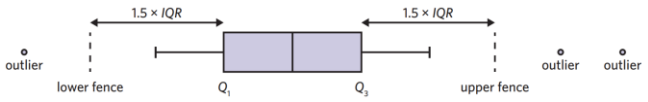
IQR = $Q_3 - Q_1$

Outliers: values which fall outside of what is 'normal'. Outliers are still the minimum and maximum value!

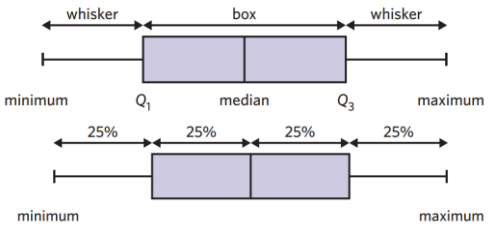
Fence: defines the boundary of what is an outlier. If a value is less than the lower fence or greater than the upper fence it is considered to be an outlier.

$$lower\ fence = Q_1 - (1.5 \times IQR)$$

$$upper\ fence = Q_3 + (1.5 \times IQR)$$

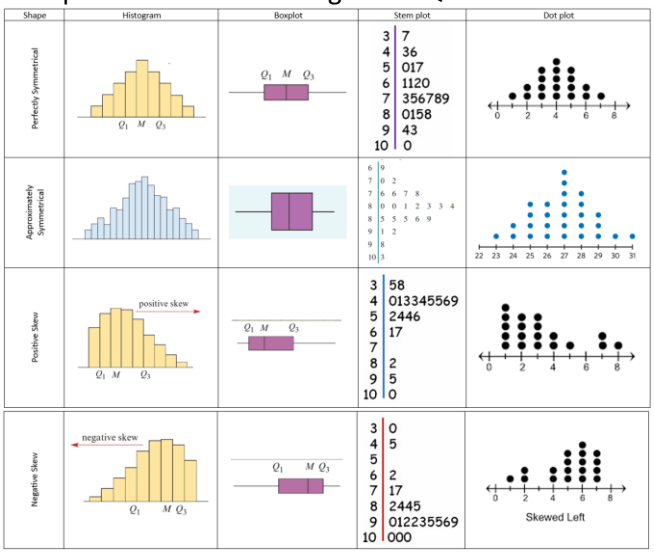


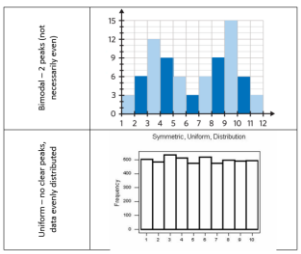
Boxplots:



1F: describing numerical data

- Shape: is the data symmetrical, skewed or have any outliers?
- Centre: What is the median value?
- Spread: What is the range and IQR?



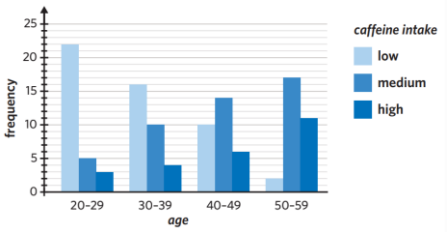


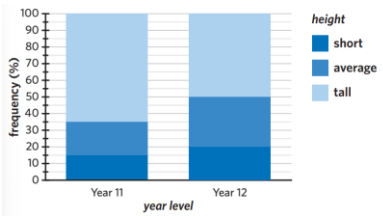
2A: association between 2 variables

Two-way frequency table:

- Columns = EV, Rows = RV
- Percentage frequency is used for greater accuracy when making comparisons if sample sizes are different

Grouped and segmented bar charts:





Describing the association between two variables:

- Whether or not an association between the two variables exists
- Appropriate percentages to support findings

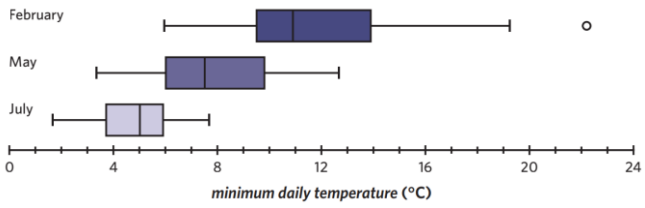
2B: association between numerical and categorical variables

- Back to back stem plot

Key: 3 | 7 = 37 points

Crocodiles				Zebras			
		4	0	2			
8	7	5	2	3	7		
	7	4	4	4			
		8	2	5	2	8	
		3	1	6	0	1	5	9
				7	0	3	5	8
			0	8	1	3	4	

- Parallel boxplot



- Making comparisons: refer to 1F and compare shape, centre and spread of the two categories

2C: association between two numerical variables

Response variable: RV, may be explained or predicted by changes in the explanatory variable.

Explanatory variable: EV, used to explain or predict the changes observed in the response variable.

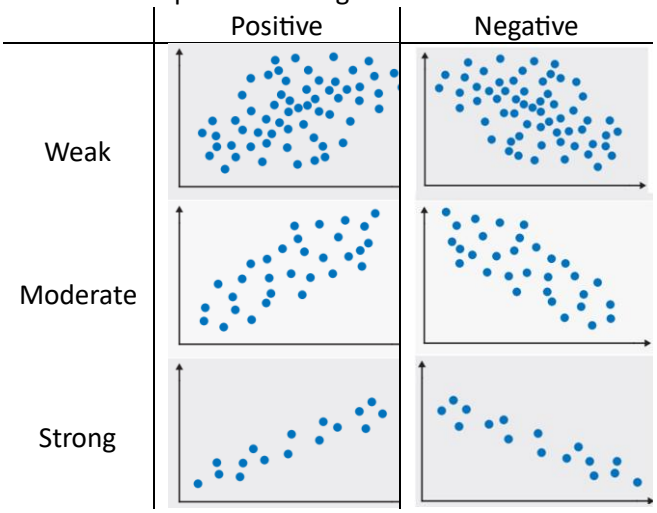
- 'EV explains the RV'

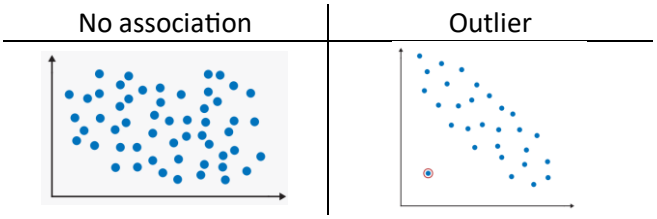
Scatterplots:

- EV = x axis, RV = y axis

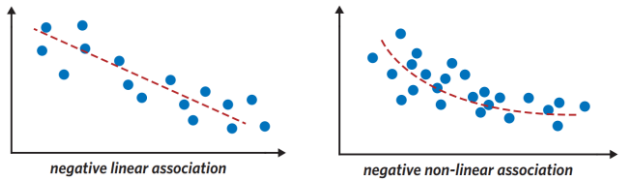
Describing relationship/analysing scatterplots:

- Strength: how close the data points are together
- Direction: positive or negative





- Form: linear (straight) or non-linear (curved)



2D: Correlation and causation

Pearson's correlation coefficient (r): numerical value that determines strength and direction between two numerical variables, assuming:

- Data is linear
- Data is numeric
- No outliers present

$0.75 \leq r \leq 1$	Strong, positive, linear association
$0.5 \leq r < 0.75$	Moderate, positive, linear association
$0.25 \leq r < 0.5$	Weak, positive, linear association
$-0.25 < r < 0.25$	No association
$-0.5 < r \leq -0.25$	Weak, negative, linear association
$-0.75 < r \leq -0.5$	Moderate, negative, linear association
$-1 \leq r \leq -0.75$	Strong, negative, linear association

3A: fitting a least squares regression line

Least squares regression line (LSRL): is the line which creates the minimum sum of the squares of residuals. There are assumptions:

- Data is numerical
- The relationship between variables is linear
- There are no clear outliers present

The line is used to show the general trend in the data and is given by the equation:

$$y = a + bx$$

Intercept Slope

Determining LSRL from a graph: Find the intercept (a) and the slope (b).

- Intercept: read directly from the graph when the EV is 0
- Slope: choose two points on the line that you can clearly ready the coordinates. Use the rule:

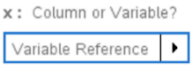
$$b = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

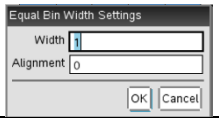
Drawing the LSRL on a graph: Sub in the first value on the x-axis and the last value on the x-axis into the equation. Plot the two points, join the line using a ruler.

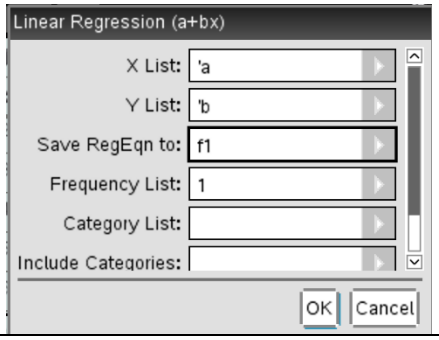
3B: Interpreting LSRL: use the following statements, fill in EV and RV and values of a and b.
y-intercept: when the **EV** is 0, the **RV** is **a**.
Slope: for every one-unit increase in the **EV**, the **RV increases/decreases** by **b**. (If b is positive, increases, if b is negative, decreases)

Making predictions: the LSRL can be used to predict the value of the RV from the EV.
Interpolation: predicting within the range of data.
Extrapolation: predicting outside the range of data. Less reliable.
How to predict: sub the EV value that you are predicting for into the LSRL equation to predict the RV.

CAS reference sheet: Data

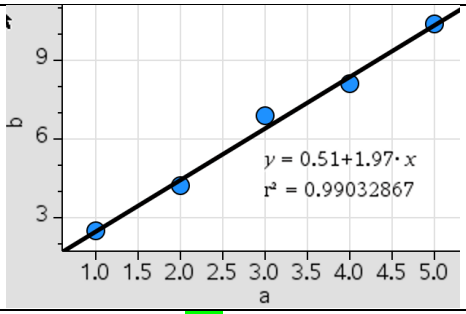
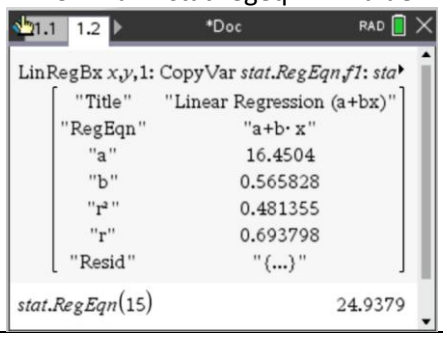
<table>
<tr>
<td>1.1 Univariate Data Statistics: to find the 5 number summary, mean and standard deviation</td>
<td>
1. Ctrl n > 4, Enter data in ‘list and spreadsheets’ (ensure a title)
2. Menu > 4 > 1 > 1 OR ctrl i > 1 > menu > 6 > 1 > 1 in a new page
3. Select the title you gave the data in the ‘X-List’


<table>
<tr><td>MinX</td><td>1.</td></tr>
<tr><td>Q₁X</td><td>1.5</td></tr>
<tr><td>MedianX...</td><td>3.</td></tr>
<tr><td>Q₃X</td><td>4.5</td></tr>
<tr><td>MaxX</td><td>5.</td></tr>
</table>
<table>
<tr><td>$\bar{x}$</td><td>3.</td></tr>
<tr><td>Σx</td><td>15.</td></tr>
<tr><td>Σx^2</td><td>55.</td></tr>
<tr><td>sx := sn-...</td><td>1.58113...</td></tr>
</table>
</td>
</tr>
<tr>
<td>1.2 Graphing Univariate Data: dot plot, boxplots, histograms</td>
<td>
1. Ctrl n > 4, Enter data in ‘list and spreadsheets’ (ensure a title)
2. Ctrl doc OR ctrl i OR home menu > 5 to open a new page and select ‘data and statistics’
3. Click to add data to the x-axis
4. Menu > 1: choose the plot type you are after
Histogram settings:
5. Menu > 2 > 2 > 2 > 1
6. Width is where you put your interval size. Alignment is where the graph will start on the x-axis
7. Zoom: menu > 5 > 2
8. Frequency or Percent: menu > 2 > 2 > 1


</td>
</tr>
<tr>
<td>2.1 Graphing Bivariate Data: Grouped Bar Chart</td>
<td>
1. Create first Bar Chart
Menu > 2 > 9 > select second set of data
</td>
</tr>
<tr>
<td>2.2 Graphing Bivariate Data: parallel boxplots</td>
<td>
2. Create first boxplot
3. Menu > 2 > 5 > select second set of data
</td>
</tr>
<tr>
<td>3.1 Bivariate Data Statistics: a, b, r, r^2</td>
<td>
1. Ctrl n > 4, Enter data in ‘list and spreadsheets’(ensure you give both columns a title)
2. Menu > 4 > 1 > 4 (Spreadsheets) OR ctrl i > 1 > Menu > 6 > 1 > 4 (Calculator)
3. Select your EV in the ‘X-List’ and your RV in the ‘Y-List’


<table>
<tr><td>RegEqn</td><td>a+b*x</td></tr>
<tr><td>a</td><td>0.51</td></tr>
<tr><td>b</td><td>1.97</td></tr>
<tr><td>r^2</td><td>0.99032...</td></tr>
<tr><td>r</td><td>0.99515...</td></tr>
</table>
</td>
</tr>
<tr>
<td>3.2 Scatterplots</td>
<td>
1. Ctrl n > 4, Enter data in ‘list and spreadsheets’ (ensure you give both columns a title)
2. Ctrl doc OR ctrl i OR home menu > 5 to open a new page and select ‘data and statistics’
3. Click to add the EV to the x-axis and the RV to the y-axis

Scatterplot Least Squares Regression Line:
4. Menu 4 > 6 > 2 OR ctrl i > 1 > Menu > 6 > 1 > 4 (Calculator)
</td>
</tr>
</table>

	
3.3 Prediction by LSRL	1. Above 3.2 step1-3 to entre EV and RV. 2. ctrl i > 1 > Menu > 6 > 1 > 4 (Calculator) 3. var > stat.regeqn > x value ↲ 
4.2 Solve	1. Calculator Page 2. Menu > 3 > 1 3. You write inside the brackets, there must be an equals sign (=), you must choose a variable (x) and then use the same variable at the end and do 'comma variable' (,x) $\text{solve}(10=x+2,x) \quad x=8.$
4.3 Back to top of List	Ctrl > 7

1a. Univariate Categorical Data: Frequency Table

The **[types of categories]** of **[total frequency]** **[frequency type]** were classified as **[list of categories].**

Modal Category

The majority of **[frequency type], [modal percentage],** were found to be **[modal category].**
Of the remaining **[frequency types], [percentage X]** were found to be **[category X],** while **[percentage Y]** were found to be **[category Y],** and while **[etc.].**

Equal Categories

The **[frequency types]** all had roughly the same percentages where **[category X]** had **[percentage X], [category Y]** had **[percentage Y], [etc.].**

1b. Univariate Numerical Data: Histogram, Dot Plot, Stem Plot

The shape of the distribution is **[symmetric/positively skewed/negatively skewed]**
The distribution has a **[standard dev./range/IQR]** of **[value]**
The distribution has a **[mean/median/mode]** of **[value]**
The distribution **[has #/has no]** outliers.

Refer to 1F: Describing numerical data

1c. Univariate Numerical Data: Box Plot

The distribution is **[positively skewed/negatively skewed]** with **[outliers/no outliers]**. The distribution is centred at **[value],** the median value. The spread of the distribution, is measured by the IQR, is **[value]** and, as measured by the range **[value].** If outliers present: There are **[value]** many outliers: **[list of outliers]**

2a. Bivariate Data (Both Categorical): Two-way Frequency Table

Worked example: Is there an association between interest in sports and age group?
Yes, the percentage of males with a high level of interest in sport steadily decreases with age group from **56.5 %** for the 'under 18 years' age group, to **35.0%** for the '36-50 years' age group.

Interest in sport	*Age group (%)* *Under 18 years*	*19–25 years*	*26–35 years*	*36–50 years*
High	56.5	50.2	40.7	35.0
Medium	30.1	34.4	36.8	45.8
Low	13.4	13.4	22.5	20.3
Total	100.0	100.0	100.0	100.0

2b. Bivariate Data (one categorical, One Numerical): Comparing two boxplots:

The distributions at **[variable name]** are **[symmetric/positively skewed/negatively skewed]** for both **[boxplot variables]**. There **[are/are no]** outliers. The median **[variable name]** is higher for **[boxplot 1], (M= value),** than **[boxplot 2], (M= value).** The IQR is also greater for **[boxplot 1], (IQR= value),** than **[boxplot 2], (IQR= value).** The range of **[variable name]** is also greater for **[boxplot 1], (R= value),** than **[boxplot 2], (R= value).** Yes, there is an association between **[variable name1] [variable name 2]** as their median change from **(M= value) to (M= value).**

2c. Bivariate Data (Both Numerical): Scatter Plot

There is a **[strong/moderate/weak], [positive/negative], [linear/non-linear]** relationship between **[response variable y]** and **[explanatory variable x]**. There **[are/are no]** clear outliers.

2d. The coefficient of determination (r^2):

The coefficient of determination indicates that **[r^2 x 100]** % of the variation in **[response variable]** is explained by the variation in **[explanatory variable]** and **[remaining %]** is explained by other factors and not explained by **[explanatory variable]**.

2e. Least squares line:

The equation of the regression line is: **[response variable]** = **[a]** + **[b]** x **[explanatory variable]**

Slope (b):

On average, **[response variable]** **[increases/decreases]** by **[b units]** for every one **[unit]** increase in **[explanatory variable].**

y- intercept (a):

When **[explanatory variable]** is 0, **[response variable]** is predicted to be **[a units].**

2f. Residual Plot

The residual plot shows a **[random scatter/ curved pattern]** indicating there is a **[linear/non- linear]** relationship between **[response variable]** and **[explanatory variable].**

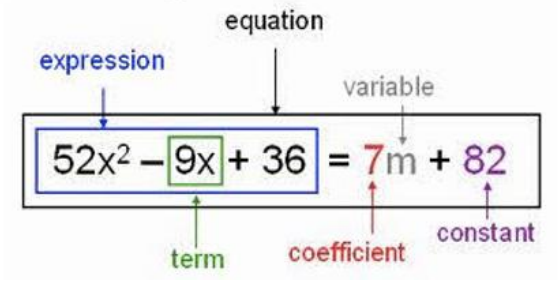
2g. Prediction Reliability

The prediction using **[explanatory variable]** of **[value of EV]** is **[reliable/ unreliable]** as it is **[within/ outside]** the date range and is therefore **[interpolation/extrapolation].**

Chapter 5 Summary Notes

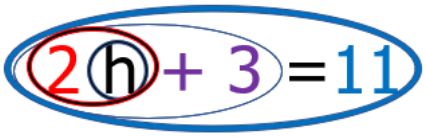
A quick reminder that ...

Sum = +
Difference = -
Product = x
Quotient = ÷



Solving a 2 Step Equation Using Onion Skins

- Solve the Equation : 2h + 3 = 11



Draw the first skin around "h", and then draw more skins radiating outwards, until the whole algebra equation is circled by the final outer skin.

To solve the Equation work from the biggest outer skin, inwards through the smaller skins, applying opposites, until we reach the letter variable.

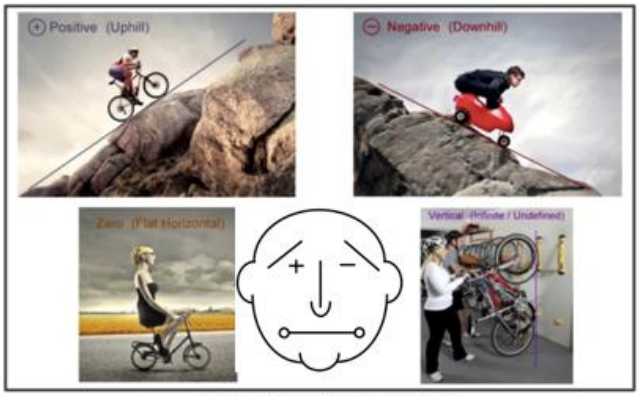
$\frac{11-3}{2}$ → h = 4 ✓

Plotting Linear Graphs: Table of Values

- A **number plane** or **Cartesian plane** includes a vertical y-**axis** and a horizontal x-**axis** intersecting at right angles.
- A point on a number plane has coordinates (x, y).
  - The x-**coordinate** is listed first, followed by the y-**coordinate**.
- The point $(0, 0)$ is called the **origin** (O).
- $(x, y) = \begin{pmatrix} \text{horizontal units from origin} & , & \text{vertical units from origin} \end{pmatrix}$

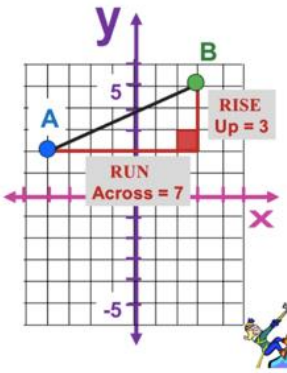
Types of Gradient Slope

There are four types of "Gradient" or "Slope"



Images Purchased from Photozone.com

Gradient Slope – Example One



Find the Gradient between points "A" and "B".

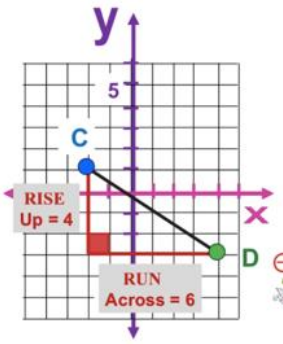
$m = \frac{\text{RISE}}{\text{RUN}}$

$m = \frac{3}{7}$

m = 3/7 ✓

(Uphill Positive Gradient)

Gradient Slope – Example Two



Find the Gradient between points "C" and "D".

$m = \frac{\text{RISE}}{\text{RUN}}$

$m = \frac{4}{6}$

m = 4/6 = -2/3 ✓

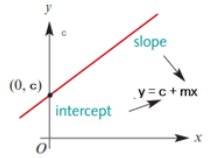
(Downhill Negative Gradient)

General Equation of a Straight Line...

y = c + mx

$y = c + mx$

The 'c' represents the y-intercept of the line: (where the line crosses the y-axis).



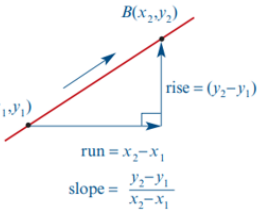
The 'm' represents the gradient/slope of the line: $m = \frac{y_2 - y_1}{x_2 - x_1}$

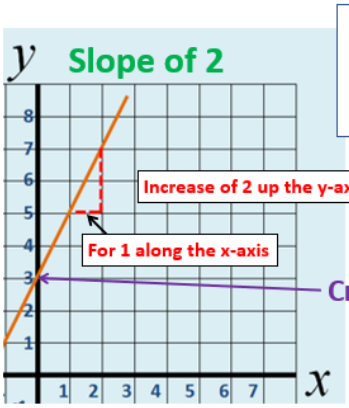
Keywords: The intercept slope form Equation → y = c+mx

How do you find "m" and 'c'?

- **c** is easy: just see where the line crosses the Y axis.
- m (the Slope) needs some calculation:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$





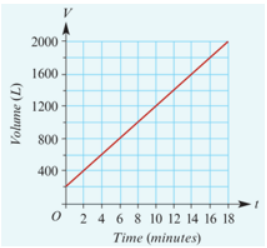
$y = c + mx$ (c: crosses y-axis; m: Slope)

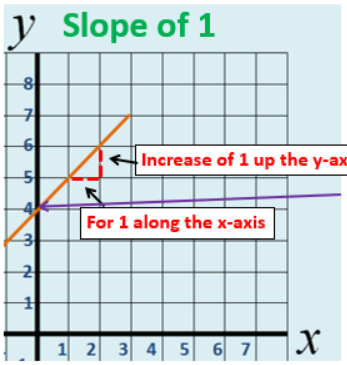
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$

Modelling Real – life

Water is pumped into a partially full tank. The graph gives the volume of water V (in litres) after t minutes.

1.How much water is in the tank at the start t=0?
Read from the graph (when t=0 , V=200).
200 L
2. How much water is in the tank after 10 minutes t=10?
Read from the graph (when t=10 , V=1200).
1200 L
3. The tank holds 2000 L. How long does it take to fill?
Read from the graph (when V=2000 , t=18).
18 minutes





$y = c + mx$ (c: crosses y-axis; m: Slope)

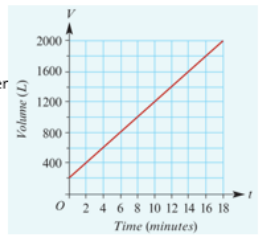
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$

Modelling Real – life

Water is pumped into a partially full tank. The graph gives the volume of water V (in litres) after t minutes.

4. Find the equation of the line in terms of V and t.
Two points (0 , 200) and (18 , 2000)
V=mt+c c=200
$m = \frac{rise}{run} = \frac{2000-200}{18-0} = \frac{1800}{18} = 100$
$\therefore V = 100t + 200$ ($t \geq 0$)
5. Use the equation to calculate the volume of water in the tank after 15 minutes?
V=100(15) +200=1700 L
6. At what rate is the water pumped into the tank; that is, how many litres are pumped into the tank each minute?

The rate = the slope of the graph, 100 L/min



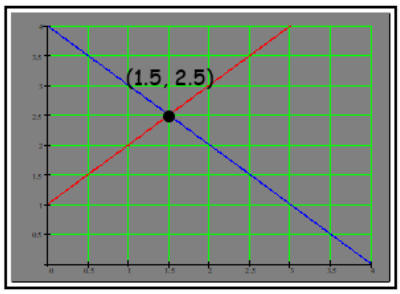
Solve simultaneous Equations

Example 1

Solve the equations

$y = x + 1$
$x + y = 4$

by substitution



x = 1.5

y = 2.5

Constructing a piecewise linear graph model

The amount, C dollars, charged to supply and deliver x m^3 of crushed rock is given by the equations:

$C = 50 + 40x \quad (0 \leq x < 3)$ ①

$C = 80 + 30x \quad (3 \leq x \leq 8)$ ②

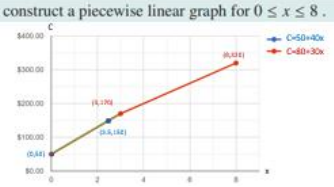
a Use the appropriate equation to determine the cost to supply and deliver the following amounts of crushed rock.

i 2.5 m^3 **ii** 3 m^3 **iii** 6 m^3

50+40×2.5=$150 80+30×3=$170 80+30×6=$260

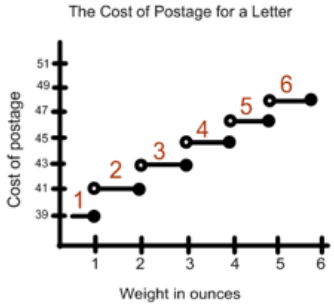
b Use the equations to construct a piecewise linear graph for $0 \leq x \leq 8$.

50+40×0=$50 (0,50)
50+40×2.5=$150 (2.5,150)
80+30×3=$170 (3,170)
80+30×8=$320 (8,320)



Step graphs

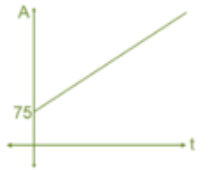
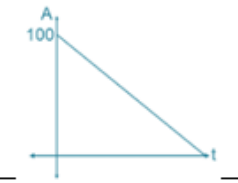
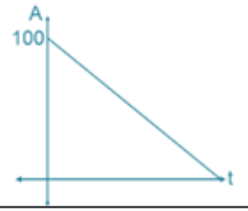
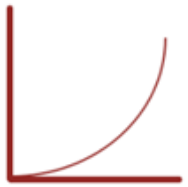
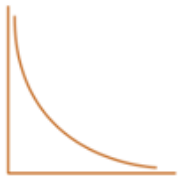
A step graph is a type of piecewise graph that uses line segments where each line segment is horizontal.



Standard compound periods:
1 = yearly
2 = biannually
4 = quarterly
6 = half yearly
12 = monthly
26 = fortnightly
52 = weekly
365 = daily

r = interest rate per annum.
CP/P = compound period. If there are no compound periods, then CP = 1
Recurrence relation: step by step process.
Rule: find values in the future in one step.

D = Common Difference
R = Common Ratio
v_n = Current term
v_{n+1} = Next term

	Recurrence Relation	Application	Explicit Rules	Graphs
Arithmetic Sequences D Common difference	$v_0 = principal, v_{n+1} = v_n + D$ $D = \frac{r}{100*p} \times v_0, \quad r = \frac{D*100*p}{v_0}$	Simple Interest	$v_n = v_0 + n * D$	
	$v_0 = Initial\ Value, v_{n+1} = v_n - D$ $D = \frac{r}{100} \times v_0, \quad r = \frac{D*100}{v_0}$	Flat Rate Depreciation	$v_n = v_0 - n * D$	
	$v_0 = Initial\ Value, v_{n+1} = v_n - D$ $D = Unit\ cost\ in\ dollars$	Unit Cost Depreciation	$v_n = v_0 - n * D$	
Geometric Sequences R Common Ratio Growth Factor Decay factor	$v_0 = principal, v_{n+1} = R * v_n$ $R = 1 + \frac{r}{100*p}, \quad r_a = (R-1) * 100 * p$	Compound Interest	$v_n = R^n * v_0$	
	$v_0 = Initial\ Value, v_{n+1} = R * v_n$ $R = 1 - \frac{r}{100}, \quad r = (1-R) * 100$	Reduced Balance Depreciation	$v_n = R^n * v_0$	

NAME	DESCRIPTION	EXAMPLE	Mathematica
Row matrix	A matrix with only 1 row	$\begin{bmatrix}3 & 2 & 1 & -4\end{bmatrix}$	7A
Column matrix	A matrix with only I column	$\begin{bmatrix}2\\3\end{bmatrix}$	7A
Square matrix	the number of rows equals the number of columns	$\begin{bmatrix}5 & 4\\4 & 2\end{bmatrix}$ 2×2	7A
Zero (Null) matrix	A matrix with all zero entries	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	7A
Summing matrix	A row or column matrix in which all the elements are 1. To sum the rows of an m × n matrix, post-multiply the matrix by an n × 1 summing matrix. To sum the columns of an m × n matrix, pre-multiply the matrix by a 1 × m summing matrix.	$\begin{bmatrix}1\\1\end{bmatrix}\ \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\ \begin{bmatrix}1 & 1\\1 & 1\\1 & 1\end{bmatrix}$	7C

NAME	DESCRIPTION	EXAMPLE	Mathematica
Transpose of a matrix	a new matrix that is formed by interchanging the rows and columns.	$A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\\a_{31} & a_{32}\end{bmatrix}\quad A^T = \begin{bmatrix}a_{11} & a_{21} & a_{31}\\a_{12} & a_{22} & a_{32}\end{bmatrix}$	7.3 7B
Symmetric matrices	A matrix **A** is called symmetric if $\mathbf{A}^T = \mathbf{A}$	$\begin{bmatrix}2 & 3\\3 & 1\end{bmatrix}\ \begin{bmatrix}2 & 3 & 4\\3 & 1 & 5\\4 & 5 & 3\end{bmatrix}\ \begin{bmatrix}1 & 2 & 4 & 6\\2 & 1 & 5 & 7\\4 & 5 & 3 & 8\\6 & 7 & 8 & 5\end{bmatrix}$	7.3 7A
Diagonal matrices	if all of the elements off the leading diagonal are zero.	$\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\ \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{bmatrix}\ \begin{bmatrix}1 & 0 & 0 & 0\\0 & 6 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 6\end{bmatrix}$	7A
Identity matrices	This is denoted by the letter **I** and has zero entries except for 1's on the diagonal.	$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\quad I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\quad I = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$	7A
Inverse matrices	A square matrix A has an inverse if there is a matrix A^{-1} such that: $AA^{-1} = I$	If $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, then its inverse, A^{-1}, is given by $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$ provided $\frac{1}{ad-bc} \neq 0$; that is, provided $\det(A) \neq 0$.	7.5

Matrices Operations	CAS & Textbook Notes	Matrices Operations	CAS & Textbook Notes
Insert matrix	7.1 7A	Power of a Matrix	7B
Define a matrix (given a letter name)	7.2 7A	Simultaneous Equations/ Matrices	7.2
Adding, subtracting, scalar multiplication	7.2 7B	Solving unknow Matrix by given matrix equation	7.2
Two matrices multiplication	7C	Constructing a Matrix by given i, j rule	7A 7.6

NAME	DESCRIPTION	EXAMPLE	Mathematica
Triangular matrices	1. An upper triangular matrix: all elements below the leading diagonal are zeros. 2. A lower triangular matrix: all elements above the leading diagonal are zeros.	$\begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{bmatrix}$ upper triangular matrix $\quad \begin{bmatrix}1 & 0 & 0 & 0\\3 & 2 & 0 & 0\\6 & 5 & 4 & 0\\0 & 9 & 8 & 7\end{bmatrix}$ lower triangular matrix	7A
Binary matrices	A special kind of matrix that has only 1s and zeros as its elements.	$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\ \begin{bmatrix}0\\1\end{bmatrix}\ \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}$	7E
Permutation matrices	A square binary matrix in which there is only one '1' in each row and column.	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\ \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\ \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$	7E

NAME & Example	DESCRIPTION	EXAMPLE & key Points	Mathematica
Communication matrices	A square binary matrix in which the 1s represent the links in a communication system.	All of the non-zero elements in the leading diagonal of a communication matrix, or its powers, represent redundant links in the matrix.	7F
Dominance matrices	Is a square binary matrix in which 1s represent one-step dominance between members of a group. Usually an arrow towards one member means it is being dominated. Usually the **row has the winner and the columns, the loser** **D** represents one-step dominance $\mathbf{D^2}$ represents two-step dominance Total dominance scores, **T = D + D²**	$D = \begin{bmatrix}A & B & C & D & \text{One-step}\\A & 0 & 1 & 0 & 1 & 2\\B & 0 & 0 & 1 & 0 & 1\\C & 0 & 0 & 0 & 0 & 0\\D & 0 & 1 & 1 & 0 & 2\end{bmatrix}$ $D^2 = \begin{bmatrix}A & B & C & D & \text{Two-step}\\A & 0 & 1 & 2 & 0 & 3\\B & 0 & 0 & 0 & 0 & 0\\C & 0 & 0 & 0 & 0 & 0\\D & 0 & 0 & 1 & 0 & 1\end{bmatrix}$ $T = D + D^2 = \begin{bmatrix}A & B & C & D & \text{Total}\\A & 0 & 2 & 2 & 1 & 5\\B & 0 & 0 & 1 & 0 & 1\\C & 0 & 0 & 0 & 0 & 0\\D & 0 & 1 & 2 & 0 & 3\end{bmatrix}$	7F

THE INVERSE OF A MATRIX 7D 7.5

- THE NUMBER A^{-1} IS CALLED THE *MULTIPLICATIVE INVERSE* OF A BECAUSE
- $A^{-1} A = I$.
- THE DEFINITION OF THE MULTIPLICATIVE **INVERSE OF A MATRIX** IS SIMILAR.

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the **inverse** of A. The symbol A^{-1} is read "A inverse."

NAME	DESCRIPTION	EXAMPLE	Mathematica
Transition matrices	Used to describe the way in which transitions are made between two states. Recurrence Relation: S_0 = initial value, $S_{n+1} = T * S_n$ Explicit Rule: $S_n = T^n * S_0$ Steady State: determine values for a long run $S = T^{50} * S_0 = T^{51} * S_0$	Ⓑ – Bendigo Ⓒ – Colac 20% 80% 90% 10% Rented in Bendigo Colac $\begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix}$ Bendigo / Colac Returned to	7G 7H 7I
Transition matrices With Extra (Forcing, Culling)	Recurrence Relation: S_0 = initial value, $S_{n+1} = T * S_n + B$ Steady State: determine values for a long run $S = T^{50} * S_0 + B = T^{51} * S_0 + B$		7G 7H 7I

NAME	DESCRIPTION	EXAMPLE	Mathematica
Leslie matrices	Model of population growth that is very popular in population ecology. Recurrence Relation: S_0 = initial value, $S_{n+1} = L * S_n$ Explicit Rule: $S_n = L^n * S_0$ Long term (Limiting) Behaviour: The proportion of the population in each age group does not change from one time period to the next. This happens if we can find a real number k such that $L * S_{n+1} = k * S_n$ for some sufficiently large n. $L * S_{51} = K * S_{50}$	0.4 2.3 1 2 3 0.6 0.3 An $m \times m$ **Leslie matrix** has the form $L = \begin{bmatrix}b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m\\s_1 & 0 & 0 & \cdots & 0 & 0\\0 & s_2 & 0 & \cdots & 0 & 0\\0 & 0 & s_3 & \cdots & 0 & 0\\\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\0 & 0 & 0 & \cdots & s_{m-1} & 0\end{bmatrix}$ where: ▪ m is the number of age groups being considered ▪ s_i, the survival rate, is the proportion of the population in age group i that progress to age group $i+1$ ▪ b_i, the birth rate, is the average number of female offspring from a mother in age group i during one time period.	

THE INVERSE OF A MATRIX 7D 7.5

- THIS SECTION FURTHER DEVELOPS THE ALGEBRA OF MATRICES. TO BEGIN, CONSIDER THE REAL NUMBER EQUATION $AX = B$.
- TO SOLVE THIS EQUATION FOR X, MULTIPLY EACH SIDE OF THE EQUATION BY A^{-1} (PROVIDED THAT $A \neq 0$).
- $AX = B$
- $(A^{-1}A)X = A^{-1}B$
- $(I)X = A^{-1}B$
- $X = A^{-1}B$

7A: introduction to matrices

A matrix is an array of numerical values

- Values are arranged into ROWS and COLUMNS

ROWS: are horizontal

- Number rows from top to bottom

COLUMNS: are vertical

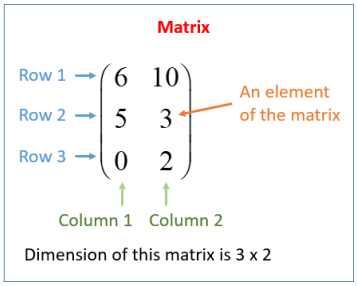
- Number columns from left to right

ORDER OF A MATRIX:

- Way to describe the dimensions (size) of a matrix

ORDER = ROWS × COLUMNS

- × is 'by'



Types of Matrices:

Column: only one column, any number of rows	Row: only one row, any number of columns	Zero: every element is '0'. Can be any size
$\begin{bmatrix}42\\56\\74\end{bmatrix}$	$[8 \;\; 17 \;\; 42 \;\; 52]$	$\begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix}$

Square matrices: have the same number of rows and columns.

Diagonal matrix: all values apart from the leading diagonal are zero	$\begin{bmatrix}25 & 0 & 0\\0 & 9 & 0\\0 & 0 & 16\end{bmatrix}$
Identity matrix: diagonal matrix, where the leading diagonals elements are all '1'	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
Symmetric matrix: unchanged by transposition, the elements above the leading diagonal are a mirror image of the elements below	$\begin{bmatrix}3 & 6 & 5\\6 & 0 & 1\\5 & 1 & 2\end{bmatrix}$
Upper triangular matrix: all elements below the leading diagonal are zero	$\begin{bmatrix}4 & 5 & 7\\0 & 1 & 6\\0 & 0 & 8\end{bmatrix}$
Lower triangular matrix: all elements above the leading diagonal are zero	$\begin{bmatrix}3 & 0 & 0\\2 & 3 & 0\\5 & 4 & 2\end{bmatrix}$

Elements and notation:

- Matrices are labelled with a capital letter
- The values within a matrix are called **elements**
- Elements are labelled with lowercase letters
- For matrix A, element a_{mn} refers to the entry in the m^{th} row and n^{th} column

$A = \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\a_{22} & a_{22} & \cdots & a_{2n}\\\vdots & \vdots & \ddots & \vdots\\a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$	$A = \begin{bmatrix}5 & 2\\3 & 1\\1 & 9\end{bmatrix}$ Eg. Element $a_{21} = 3$

x^{ij} – matrices can be constructed using element rules.
Eg. Matrix C is a 2×2 matrix with the element rule $c_{ij} = i+j$. Create the matrix.

$$A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} \quad A = \begin{bmatrix}1+1 & 1+2\\2+1 & 2+2\end{bmatrix} = \begin{bmatrix}2 & 3\\3 & 4\end{bmatrix}$$

7B Operations with matrices

ADDITION AND SUBTRACTION:

- Matrices must have the same order
- Add/subtract elements in the same position

Matrix addition

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} + \begin{bmatrix}e & g\\f & h\end{bmatrix} = \begin{bmatrix}a+e & b+g\\c+f & d+h\end{bmatrix}$$

Matrix subtraction

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} - \begin{bmatrix}e & g\\f & h\end{bmatrix} = \begin{bmatrix}a-e & b-g\\c-f & d-h\end{bmatrix}$$

SCALAR MULTIPLICATION:

- Multiply each element in the matrix by a scalar

$$k \times \begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}k \times a & k \times b\\k \times c & k \times d\end{bmatrix}$$

Transpose: swapping a matrices rows and columns. The transpose of matrix A is A^T. First row becomes first columns, etc.

$$\begin{bmatrix}a & b & c & d\\e & f & g & h\end{bmatrix}^T = \begin{bmatrix}a & e\\b & f\\c & g\\d & h\end{bmatrix}$$

7C: Advanced operations with matrices

MATRIX MULTIPLICATION: Involves both multiplication and addition

- **Post multiplication:** AB, matrix A is 'post multiplied' by matrix B
- **Pre multiplication:** AB, matrix B is 'pre multiplied' by matrix A

You must check that matrices can be multiplied first:

- Multiplication criteria: the number of columns in the first matrix MUST EQUAL the number of rows in the second matrix.
  - 'defined' = can be multiplied
  - 'undefined' = cant be multiplied

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix} \qquad BA = \begin{bmatrix} e \\ f \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Order: } 2 \times \underbrace{2 \quad 2}_{\text{equal}} \times 1 \qquad \text{Order: } 2 \times \underbrace{1 \quad 2}_{\text{not equal}} \times 1$$

AB is defined, BA is not defined.

Matrix product: the resulting matrix when two or more matrices are multiplied. The size of this matrix is determined by the outside numbers when writing the matrices orders. AB will produce a 2×1

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\text{Order: } \underbrace{2 \times 2 \quad 2 \times 1}_{2 \times 1}$$

Multiplication by hand: multiply the rows of the first matrix by the columns of the second matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \times e + b \times g & a \times f + b \times h \\ c \times e + d \times g & c \times f + d \times h \end{bmatrix}$$

Summing matrices: a row or column matrix where all elements are 1.

- **To sum the rows** of an m×n matrix, **post multiply** by an n×1 summing matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 1 + 3 \times 1 \\ 4 \times 1 + 5 \times 1 + 6 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

- **To sum the columns** of an m×n matrix, **pre multiply** by a 1×m summing matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 3 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 16 \end{bmatrix}$$

Raising matrices to a power: only SQUARE matrices can be raised to a power. The power indicates how many times the matrix is multiplied by itself.

7D: Inverse Matrices

The determinant:

- Used to identify if an inverse exists
- Must be a positive or negative number
- If the determinant equals ZERO, there is no inverse and the matrix is said to be 'singular'

Calculating the determinant (2x2 by hand):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

The inverse:

- Only square matrices have an inverse
- Inverse means opposite
- Something multiplied by its inverse is equal to 1, in this case, the IDENTITY MATRIX
- A^{-1} = the inverse of matrix A
- $A \times A^{-1} = I$ (identity matrix)

Calculating the inverse:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- b and c multiply by -1
- a and d swap positions
- Multiply by the scalar so that the answer is just the matrix

****if you are given the inverse of a matrix and need to find the original****

- Put the inverse to the power of -1
- $(A^{-1})^{-1} = A$

7E: Binary and Permutation Matrices

Binary matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Permutation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Binary: only has 1's and 0's as its elements (any size)
Permutation matrix: a type of binary matrix that has **only one 1 in each row and column.** It is always square and is used to rearrange elements.

- Column permutation: rearranges columns.
  - Post multiply (Q × P)
- Row permutation: rearranges rows.
  - Pre multiply (P× Q)

7F: Communication and Dominance Matrices

Communication: square binary matrix

- 1's represent a direct line of communication
- Communication goes both ways
- C = one step (direct communication)
- C2 = two step (communication via someone else)
- C + C2 = TC (total communication, showing all one and two step links)

Dominance:

- Used to display hierarchy
- One-way connections
- 1's are used for 'winners', 0's for 'losers'
- D = one step
- D2 = two step
- D + D2 = TD (total dominance)
- Directed, arrows point to loser

Determining dominance when given the sum of one step and two step dominance:

- One step dominance tells you how many times they won
- Two step dominance tells you the total sum of the one step dominances

7G: Introduction to transition matrices

State matrix S_n

- Column matrix
- Population at a given time

Initial state matrix

- Starting population
- S_0

Transition matrix

- Square matrix
- Studies change over time
- Elements are decimal numbers (0-1) that represent percentages
- Each column must sum to 1
- Other language: 'now/next', 'today/tomorrow', 'from/to'

$$T = \begin{array}{c} \text{This time} \\ \begin{matrix} P & I \end{matrix} \\ \begin{bmatrix} & \\ & \end{bmatrix} \begin{matrix} P \\ I \end{matrix} \end{array} \text{ Next time}$$

Calculating state matrices:

- Used to find the next state
- Use recursion (step by step)

$$S_0 = \text{initial state matrix}, \quad S_{n+1} = T \times S_n$$

S_n is the current state matrix
S_{n+1} is the next state matrix
T is the transition matrix.

- Use the rule (find future values, n, faster)

$$S_n = T^n \times S_0$$

***Note: Finding a previous state:** say you are given T and S_4, how could you find a previous state? For example, if you need to find S_3 take the inverse of T and multiply is by S_4.

7H: The Equilibrium matrix

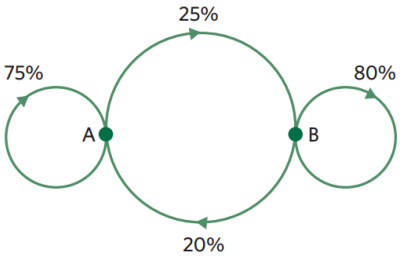
- 'Steady state'
- 'In the long term'
- Over time the populations settle to where there is no visible change
- This is when two consecutive matrices are equal (2dp for accuracy)
- Consecutive means finding S_{17} and S_{18} to be the same

7I: Applications of transition matrices

Transition diagram: a visual representation of how a transition matrix functions.

$$T = \begin{array}{c} \text{today} \\ \begin{matrix} A & B \end{matrix} \\ \begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix} \begin{matrix} A \\ B \end{matrix} \end{array} \text{ tomorrow}$$

can be represented by the following transition diagram.



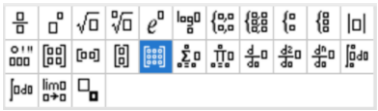
Culling and Restocking:

- Populations are subject to change over time
- Considers external forces affecting the population

$$S_{n+1} = TS_n + B$$

Culling: reduction/removed, negative number in matrix B
Restocking: addition, positive number in matrix B
Keeping a population constant: all future state matrices are equal to the initial. This can be determined by calculating $S_0 - S_1$

CAS reference sheet: Matrices

7.1 Enter a Matrix	1. Ctrl N > 1 > ↵ 2. ***OR*** Ctrl N > 1 > Ctrl > Menu > 8 ↵  3. Select No. of rows & columns ↵ 4. Type each element, using **[tab]** to move.
7.2 Define a Matrix	1. Name of the matrix > ctrl > > enter full matrix ↵ $a:=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$ $\begin{bmatrix}1. & 2. & 3.\\4. & 5. & 6.\\7. & 8. & 9.\end{bmatrix}$ 2. ***OR*** Enter full matrix > ctrl var > name of the matrix↵ $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\rightarrow b$ $\begin{bmatrix}1. & 2. & 3.\\4. & 5. & 6.\\7. & 8. & 9.\end{bmatrix}$
7.3 Transpose Matrix	1. Name of the matrix > Menu > 7> 2↵ $a^{\intercal}$ $\begin{bmatrix}1. & 4. & 7.\\2. & 5. & 8.\\3. & 6. & 9.\end{bmatrix}$
7.4 Determinant	1. Menu > 7 > 3> name of the matrix↵ $\det(a)$ $0.$
7.5 Inverse	1. Name of the matrix > Λ > **Negation key (-) > 1** ↵ $c:=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ $\begin{bmatrix}1. & 2.\\3. & 4.\end{bmatrix}$ c^{-1} $\begin{bmatrix}-2. & 1.\\1.5 & -0.5\end{bmatrix}$ 2. Decimal → fraction $\text{exact}(c^{-1})$ $\begin{bmatrix}-2 & 1\\\frac{3}{2} & \frac{-1}{2}\end{bmatrix}$
7.6 Forming matrix by i, j rule	1. Menu > 7 > 1 >A > ij rule, i, j, row size, column size ↵ $\text{constructMat}(2\cdot i+3\cdot j,i,j,2,3)$ $\begin{bmatrix}5. & 8. & 11.\\7. & 10. & 13.\end{bmatrix}$

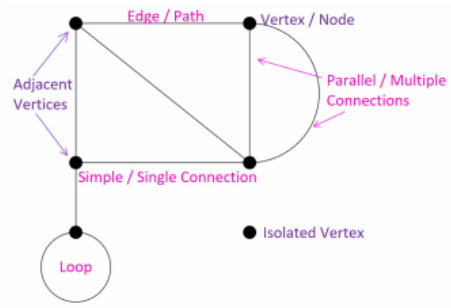
Networks

Vertex (plural Vertices)
An object, represented with a dot.

Edge
A connection between two vertices represented with a line or an arrow.

Loop
An edge that connect from an vertex to itself.

Graph
A collection of vertices that are connected (or not) to each other using edges in some specific way.

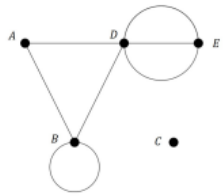


The Degree of a Vertex
The number of edges connected to a vertex.

A loop counts as two edges for the degree.

Vertices are classified as even or odd if their degree is an even number or odd number.

Example



Degrees

$\text{Deg}(A) = 2$
$\text{Deg}(B) = 4$
$\text{Deg}(C) = 0$
$\text{Deg}(D) = 5$
$\text{Deg}(E) = 3$

Handshaking Lemma
Every edge is counted by the degree of two vertices.
Sum of vertex degrees = 2 × number of edges.

A loop counts as one edge in the number of edges.

Direction on Graphs

Undirected Graph
A graph where the edge between two vertices acts in both directions.

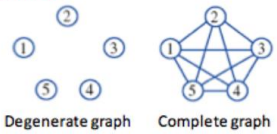


Directed Graph / Digraph
A graph where specific direction is indicated for every edge.
Some vertices may not be reachable from other vertices.



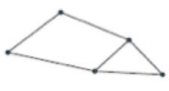
Undirected Graphs

Types of Networks

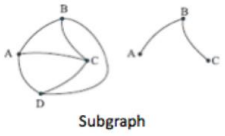


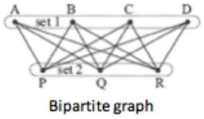










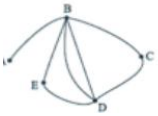


Graph Type	Number of Edges with n vertices
Complete	$\frac{n(n-1)}{2}$
Connected	$n-1$

Adjacency Matrix Representation

Matrix Representation

Networks can be represented using adjacency matrices. The numbers on the leading diagonal represent loops, and all undirected graphs are symmetrical about the leading diagonal.

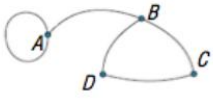


From \ To	A	B	C	D	E
A	0	1	0	0	0
B	1	0	1	2	1
C	0	1	0	1	0
D	0	2	1	0	1
E	0	1	0	1	0

Loops

A loop in an undirected network adds **two** to the degree of a vertex, and adds **one** to the leading diagonal of a matrix. For example:

Node A has degree 3.



$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix A of a graph is an $n \times n$ matrix in which, for example, the entry in row C and column F is the number of edges joining vertices C and F.
A loop is a single edge connecting a vertex to itself.
Loops are counted as one edge.

Planar Graphs & Euler's Formula

Planar Graphs

Planar Graph: A graph that can be drawn in such a way that no two edges meet (or have common points), except at the vertices where they are both incident, is called a planar graph.

- Some graphs can be redrawn to be planar, others not.
- Euler's formula is used to confirm whether graphs are planar or not.
- All simple graphs with **four** or **fewer vertices** are planar.

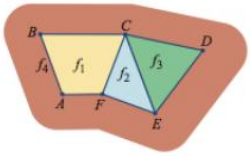


Euler's Formula

Euler's Formula:

$$v - e + f = 2$$

V = Vertices	E = Edges	F = Faces



Consider the connected planar graph opposite. It has 4 faces, 6 vertices and 8 edges.

$v - e + f = 2$

$6 - 8 + 4 = 2$

$\therefore$ *Euler's formula confirms that this graph is a planar graph*

Vertices	Edges	Faces	Prove
$v = e - f + 2$	$e = v + f - 2$	$f = e - v + 2$	$v + f - e = 2$

Euler's Formula/Degree of a Face

$e + 6 \leq 3v$

Example VCAA 2016 Exam 1 Sample Question 6 / VCAA 2014 Exam 1 Question 7

Consider the following four graphs. How many of the four graphs above are planar?






Graph 1: $v = 5,\ e = 8$ — $8 + 6 \leq 3 \times 5$ — $14 \leq 15$ ✔

Graph 2: $v = 5,\ e = 5$ — $5 + 6 \leq 3 \times 5$ — $11 \leq 15$ ✔

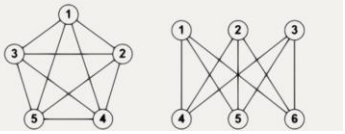
Graph 3: $v = 5,\ e = 7$ — $7 + 6 \leq 3 \times 5$ — $13 \leq 15$ ✔

Graph 4: $v = 5,\ e = 9$ — $9 + 6 \leq 3 \times 5$ — $15 \leq 15$ ✔

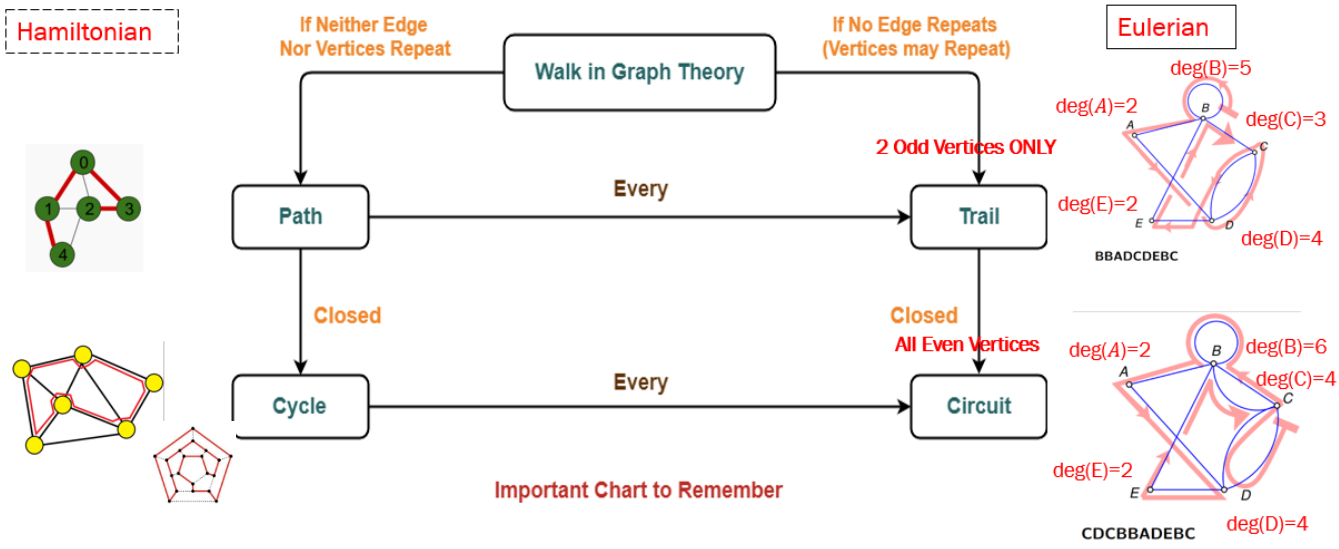
Therefore, all 4 graphs are planar.

Kuratowski's Theorem

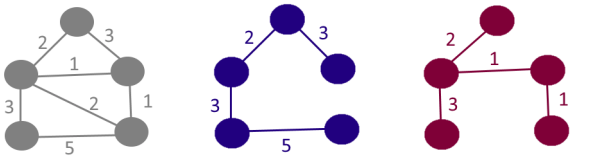
A graph is planar if and only if it does NOT contain a subgraph homeomorphic to K_5 or $K_{3,3}$



Euler & Hamilton



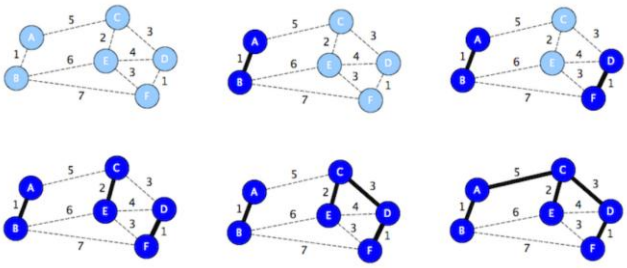
Weighted Graphs



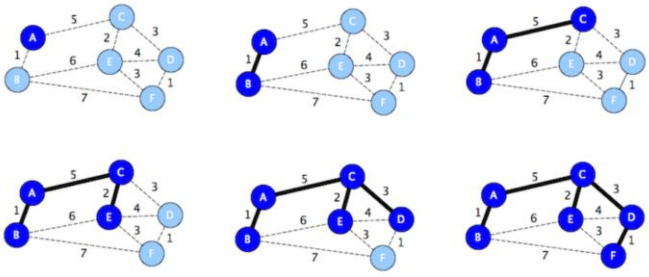
Graph

Spanning Tree
Cost = 13

Minimum Spanning Tree, Cost = 7



Kruskal's Algorithm



Prim's Algorithm

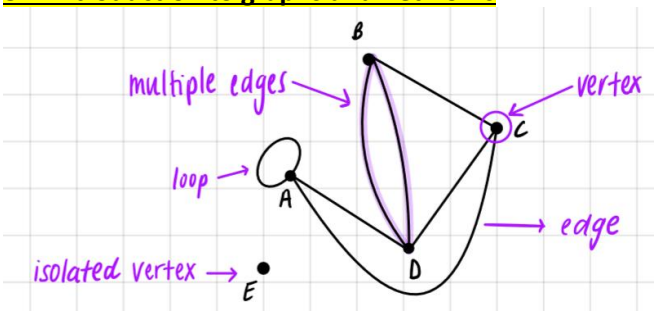
Different Types of Greedy Algorithm

Prim's Minimal Spanning Tree Algorithm
Kruskal's Minimal Spanning Tree algorithm
Dijkstra's Shortest Path Algorithm
Ford-Fulkerson Networks Flows Algorithm
Hungarian Algorithm

Mathematical Terminologies

Undirected Graphs		**Directed Graphs**	
Terminologies	**Algorithm**	**Terminologies**	**Algorithm**
Eulerian trails	Exactly 2 vertices of an odd degree	The Maximum Flow	Ford-Fulkerson Algorithm
Eulerian circuits	All vertices even degree	The Shortest Path	Dijkstra's Algorithm
Hamiltonian paths	Visits all of the vertices in a graph only once	Matching & Allocation Problems	Hungarian Algorithm 8.1
Hamiltonian cycles	Visit All vertices, begin & end @ the same vertex	Critical Path Problems	Forward scanning = Biggest Number Backward scanning = Smallest Number Float = LST—EST
Minimal Spanning Tree	Prim's Algorithm, Kruskal's Algorithm		

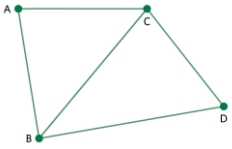
8A: Introduction to graphs and networks



- **Graph:** a diagram used to show connections between groups of things, people of activities.
- **Vertex:** the dots
- **Edge:** line that connects two vertices
- **Isolated vertex:** a vertex that is not connected to an edge
- **Multiple edges:** connects the same vertex using more than one edge
- **Loop:** attach twice to a vertex
- **Degree of a vertex:** the number of edges that attach to a vertex. The degree of vertex D is 4.

Types of graphs:

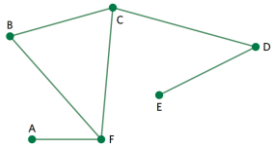
Simple graph: no loops or multiple edges



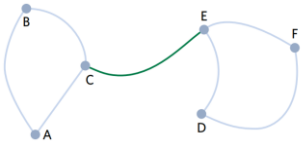
Degenerate graph: all vertices are isolated, there are no edges



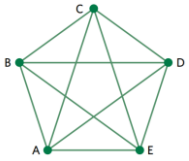
Connected graph: every vertex is connected to every other vertex, either directly or indirectly



Bridge: an edge in a connected graph, that if removed will cause the graph to be **disconnected.** Edge C to E is a bridge

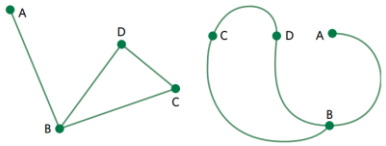


Complete graph: every vertex is connected to every other vertex

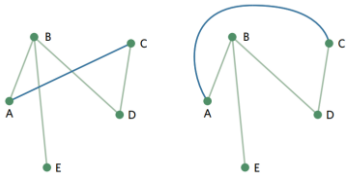


Subgraph: a tree, a part of a larger graph

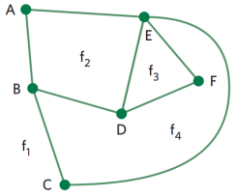
Equivalent/isomorphic graph: graphs that contain the EXACT same information, vertices and the connections between them, they are just drawn differently



Planar graph: a graph that can be re-drawn without any overlapping edges. If it cannot be re-drawn and has overlapping edges, the graph is **non-planar.**



Faces: faces are found in **planar** graphs and are the sections enclosed by edges. There is an infinite face outside of the graph. There is an infinite face outside of the graph.



Euler's rule: there is a relationship between the number of vertices (v), edges (e) and faces (f) in a connected planar graph.

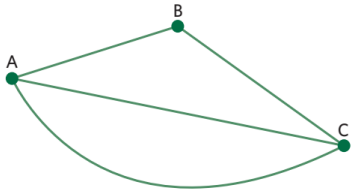
$$v - e + f = 2, \text{where}$$

- v is the number of vertices
- e is the number of edges
- f is the number of faces

8B: Graphs, networks and matrices

Adjacency matrix: a matrix that records the number of connections between vertices. The number of vertices on the graph tells you how many rows and columns you need. A '0' mean no direct connection, a '1' means one connecting edge etc. Ensure you label the matrix with the vertex letters.

$$\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix}$$



Representing directed graphs: this is a network containing arrows on each edge, signalling one direction. These matrices need to be labelled with 'from' and 'to'.

8C: Travelling

Identifying types of walks: A route is a list of the vertices travelled through, in order, when moving from one vertex to another.
Walk: a continuous sequence of edges that pass through any number of vertices, in any order, starting and finishing at any vertex.
Trail: a walk with no repeated edges. The same vertex can be visited multiple times.
Path: a walk with no repeated edges or vertices.
Circuit: a trail beginning and ending at the same vertex.
Cycle: a path beginning and ending at the same vertex.

Eulerian trails: a walk that includes every edge in a graph exactly once. It must:

- Exactly two vertices of odd degree, the rest are even
- Start and finish at a vertices of odd degree

Eulerian circuit: an Eulerian trail that starts and ends at the same vertex. It must:

- Have all vertices of even degree

Hamiltonian paths: a walk that includes every vertex exactly once, with no repeated edges. Every edge does not need to be included.
Hamiltonian cycle: a Hamiltonian path that starts and ends at the same vertex.

8D: Minium connector problems

Weighted graph: numerical information attached to each edge of a graph. 'Weights' often represent time or distance.

Tree: type of connected graph that has no loops, duplicate edges or cycles. It uses the least number of edges to connect the vertices.

- The number of edges in a tree is always one less than the number of vertices
- $e = v - 1$
- A tree can be a subgraph and so not all vertices in the larger graph need to be included.

Spanning tree: a tree which connects ALL vertices in the original graph.

- There are often multiple spanning trees for the one graph

Minimum spanning tree: a spanning tree with the LOWEST TOTAL WEIGHT.

- Prim's algorithm can be used to find the minimum spanning tree.