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Stability for Solutions to Systems of Ordinary Differential Equations

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Conferment

To the pure spirit of my father and to my family.

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Abstract

The aim of this study is to shed light on the system of Ordinary Differential Equations and their solutions and the stability of these solutions and to present some definitions related to them and to provide some examples supported by practical examples .

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CHAPTER 3

Stability of Zero Solution

CHAPTER 4

Stability in the Sense of Lyapunov

Introduction

We begin by reviewing the theory concerning the stability of solutions of differential equation system. This review will begin with study of systems of ordinary differential equations (ODEs), furnishing the relevant stability definitions and the extensive Lyapunov stability theory that is often of use in this area.

It is well-known that the solutions of ordinary differential equations and its systems occur in certain domain. These solutions may not be stable, that is to say it extends outside the domains. But mostly the solutions are stable under specific conditions.

In this research, we will take a study of stability of solutions of the system of ordinary differential equations using the elementary principles and Lyapunov theory.

In this study we will discuss the following chapters:

Chapter 1: Main concepts.

It contains some definitions and introduction to ordinary differential equations and their systems and some solutions to them.

Chapter 2: Stability Solutions for systems of differential equations.

We will address the concept of stability and examine stability and instability in general and will focus on some important concepts.

Chapter 3: Stability of Zero solution.

We will discuss the stability at a fixed point and examine some types of stationary points and the stability of the zero solution.

Chapter 4: Stability in the sense of Lyapunov.

We will present the definitions of stability as defined by Lyapunov and examine some of the theories of stability of his view.

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Chapter 1

Main Concepts

1.1 Definitions

1.2 System of Differential Equation

Main Concepts

This chapter contains some concepts of ordinary differential equation and systems.

The material in this chapter are taken from the following references [2],[4],[9],[10] and [11].

1.1 Definitions

Definition 1.1.1 (Differential Equation)

A differential equation is an equation containing one or more derivatives of a single unknown function.

For a function of one variable, in symbols, a differential equation has the form:

$$f(x) = p\left(x, y, y', \ldots, y^{(n)}\right),$$

where p is any function of indicated inputs y the solution of the differential equation, is a function of x and f is any function of x.

Differential equations involving derivatives of a function of one variable are called *Ordinary Differential Equations*, often abbreviated to ODE, some examples of an ODEs are:

$$x^3 y'' + xy' + (x^2 - v^2)y = 0, v \in [0, \infty), (Bessel\ Equation\ of\ order\ v)$$

$$y' + y = x^2,$$

$$y^{(4)} + 3xy''' + 16\cos(x)\, y'' + e^x y' + 3y = 4,$$

Notice that there is one restriction on the number of independent variable that the unknown function may have for example the equation:

$$\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial y^2}=0,$$

where u is a function of x and y is a perfectly valid differential equation.

This type of differential equation is known as a *Partial Differential Equation*, often abbreviated to PDE, some popular PDEs are:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, (Laplacce\ Equation)$$

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\ , \qquad (Heat\ Equation)$$

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}\ , \qquad (Wave\ Equation)$$

Definition 1.1.2 (Trivial/Nontrivial Solution)

The trivial solution to a differential equation:

$$f(x)=p(x,y,y',y'',\dots,y^{(n)}),$$

is the solution $y=0$ for all x.

Any other type of solution is called nontrivial.

Definition 1.1.3 (General Solution)

The general solution to the differential equation:

$f(x)=p(x,y,y',y'',\dots,y^{(n)})$ is a solution of the from:

$$y = y(x, c_1, c_2, \dots, c_n)$$

where $c_1, c_2, \dots, c_n$ are taken to be arbitrary constants.

Definition 1.1.4 (Order of Differential Equation)

The order of a differential equation is the order of the highest derivative involved in the differential equation.

Definition 1.1.5 (Linear Differential Equation)

An n^{th} order differential equation is said to be linear if it is of the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

where $a_i(x), 0 \leq i \leq n$ and $f(x)$ are given functions of the independent variable x and it is assumed that $a_n(x) \neq 0$.

Otherwise it is said to be nonlinear.

1.2 System of Differential Equations

Introduction:

A simple problem of the dynamics of a particle can lead to systems of differential equation given forces acting on a particle, find the law of motion, i.e. find the functions $x = x(t), y = y(t)$ and $z = z(t)$ which express the relationship between the coordinates of the moving particle and time. The system of equations which results has the form:

$$\begin{aligned}
\frac{d^2x}{dt^2} &= f\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\
\frac{d^2y}{dt^2} &= g\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right), \\
\frac{d^2z}{dt^2} &= h\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)
\end{aligned} \qquad (1.1)$$

Here x, y and z are the coordinates of the moving particle, t is time, f, g and h are known functions of their arguments.

A system of form (1.1) is known as a canonical system of differential equations.

Definition 1.2.1

System of equations of the first order solved for the derivatives of the required functions:

$$x'_i = f_i(t, x_1, x_2, \dots, x_n), \quad i = 1,2, \dots, n, \qquad (1.2)$$

are called *normal system.*

Definition 1.2.2

The solution of the *normal system* (1.2) on the interval (a, b) of variation of the argument t is any system of n functions:

$$x_1 = x_1(t), x_2 = x_2(t), \dots, x_n = x_n(t), \qquad (1.3)$$

Differentiable on the interval $a < t < b$ which turns the equations of system (1.2) into identities with respect to t on the interval (a, b).

For system (1.2) *Cauchy's problem* is formulated as follows:

Find a solution (1.3) of the system which satisfies, for $t = t_0$ the initial conditions:

$$x_1|_{t=t_0} = x_1^0, x_2|_{t=t_0} = x_2^0, \dots, x_n|_{t=t_0} = x_n^0, \qquad (1.4)$$

Theorem 1.2.1(The Uniqueness and Existence of the Solution of Cauchy's Problem)

Given a normal system of differential equations:

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), \quad i = 1,2, \dots, n,$$

Assume that the functions:

$$f_i(t, x_1, x_2, \dots, x_n), \quad i = 1,2, \dots, n,$$

are defined in a certain n–dimensional domain D of variation of the variables $t, x_1, x_2, \dots, x_n$.

If there is a neighborhood Ω of the point $M_0(t_0, x_1^0, x_2^0, \dots, x_n^0)$ at which the functions f_i are continuous jointly with respect to the arguments and passes bounded partial derivatives with respect to the variables $x_1, x_2, \dots, x_n$, then there is an interval $t_0 - h_0 < t < t_0 + h_0$ of variation of t on which there is the unique solution (1.3) of the normal system satisfying the initial conditions (1.4).

Theorem 1.2.2 (Picard's Theorem)

Suppose that $f(.,.)$ is a continuous function of its argumeats in a region u of the (x, y) plane which contains the rectangle:

$$R = \{(x, y): x_0 < x < X_m , |y - y_0| \leq Y_m\},$$

where $X_m > x_0$ and $Y_m > 0$ are constants.

Suppose also that there exists a positive constant L such that;

$$|f(x, y) - f(x, z)| \leq L|y - z|,$$

holds whenever (x, y) and (x, z) lie in the rectangle R.

Finally, letting:

$$M = \max\{|f(x, y): (x, y) \in R\},$$

Suppose that:

$$M(X_m - x_0) \leq Y_m,$$

Then, there exists a unique continuously differentiable function $x \to y(x)$ defined on the closed interval $[x_0, X_m]$ which satisfies $y' = f(x, y)$ and $y(x_0) = y_0$.

The condition

$$|f(x, y) - f(x, z)| \leq L|y - z|,$$

is called *a Lipchitz condition* and L is the *Lipchitz constant* for f.

Definition 1.2.3

A system of n functions:

$$x_i = x_i(t, c_1, c_2, \dots, c_n), \quad i = 1,2, \dots, n, \tag{1.5}$$

which depend on t and n arbitrary constants $c_1, c_2, \dots, c_n$ is called the *general solution* of the normal systems of equations (1.2) in a domain Ω of the existence and uniqueness of the solution of *Cauchy's problem* if:

1. For any permissible values of $c_1, c_2, \dots, c_n$ the system of function (1.5) turns equations (1.2) into identities.
2. In the domain Ω functions (1.5) solve any Cauchy problem.

The solutions which result from the general solution for specific values of the constants $c_1, c_2, \dots, c_n$ are known as *particular solution*.

Definition 1.2.4 (System of Linear Differential Equations)

A system of differential equations is said to be Linear if it is linear with respect to the unknown functions and their derivatives entering into the equations. A system of n *Linear equations* of the first order written in normal form can be written as:

$$\frac{dx_i}{dt} = \sum_{i=1}^{n} a_{ij}(t)x_j + f_i(t), \quad i, j = 1,2, \dots, n, \tag{1.6}$$

and in matrix form as:

$$\frac{dx}{dt} = AX + F, \tag{1.7}$$

where:

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad F = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

$$A = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \cdots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix},$$

Theorem 1.2.3

If all the functions $a_{ij}(t)$ and $f_i(t), i, j = 1,2, \dots, n$ are continuous on the interval $a \leq t \leq b$, then in a sufficiently small neighborhood of every point $M_0(t_0, x_1^0, x_2^0, \dots, x_n^0)$, where $t_0 \in (a, b)$, the conditions of the theorem on the unique existence of the solution of *Cauchy's problem* are fulfilled and consequently, a single integral curve of system (1.6) passes through every such point.

Indeed, in such a case the right-hand sides of system of equations (1.6) are continuous jointly with respect to the arguments $t, x_1, x_2, \dots, x_n$ and their partial derivatives with respect to x_j, $j = 1,2, \dots, n$ and bounded since they are equal to the coefficients $a_{ij}(t)$ continuous on the interval $[a, b]$ we introduce a Linear operator $L = \frac{d}{dt} - A$.

Then we can use a brief notation:

$$L[X] = F, \tag{1.8}$$

for system (1.7), if the matrix F is zero, i.e. $f_i(t) \equiv 0$, $i = 1,2, \dots, n$ on the interval (a, b), then system (1.6) is said to be *homogeneous Linear* and has the form:

$$L[X] = 0, \tag{1.9}$$

Theorem 1.2.4

If $X(t)$ is a solution of homogeneous Linear system $L[X] = 0$, then $cX(t)$ is a solution of that system where c is an arbitrary constant.

Theorem 1.2.5

The sum $X_1(t) + X_2(t)$ of two solutions $X_1(t)$ and $X_2(t)$ of a homogeneous linear system of equations is a solution of that system.

Corollary 1.2.1

A Linear combination $\sum_{i=1}^{m} c_i X_i(t)$ with arbitrary constant coefficients c_i of the solutions $X_1(t), \dots, X_m(t)$ of a homogeneous linear system of differential equations $L[X] = 0$, is a solution of that system.

Theorem 1.2.6

If $\tilde{X}(t)$ is a solution of an inhomogeneous Linear system $L[X] = F$ and $X_0(t)$ is a solution of the corresponding homogeneous system $L[X] = 0$, then the sum $\tilde{X}(t) + X_0(t)$ is a solution of the inhomogeneous system $L[X] = F$.

Indeed, by the hypothesis, $L[\tilde{X}] = F$ and $L[X_0] = 0$. Using the property of additivity of the operator L, we get;

$$L[\tilde{X} + X_0] = L[\tilde{X}] + L[X_0] = F,$$

This means that the sum $\tilde{X} + X_0$ is a solution of the inhomogeneous system of equations $L[X] = F$.

Definition 1.2.5

The vectors $X_1(t), X_2(t), \dots, X_n(t)$ where:

$$X_k(t) = \begin{pmatrix} X_{1k}(t) \\ X_{2k}(t) \\ \vdots \\ X_{nk}(t) \end{pmatrix},$$

are said to be linearly dependent on the interval $a < t < b$ if there are constant numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that:

$$\alpha_1 X_1(t) + \alpha_2 X_2(t) + \cdots + \alpha_n X_n(t) = 0, \qquad (1.10)$$

for $t \in (a, b)$ and at least one of numbers α_i is nonzero. If identity (1.10) is valid only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then the vectors $X_1(t), X_2(t), \dots, X_n(t)$ are said to be Linearly independent on (a, b).

Definition 1.2.6

One vector identity (1.10) is equivalent to n identities;

$$\sum_{k=1}^{n} \alpha_k X_{1k}(t) = 0,$$

$$\sum_{k=1}^{n} \alpha_k X_{2k}(t) = 0,$$

$$\dots$$

$$\sum_{k=1}^{n} \alpha_k X_{nk}(t) = 0,$$

The determinant:

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \cdots & \dots & \dots & \dots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{vmatrix},$$

is known as the wronskian of the system of vectors $X_1(t), X_2(t), \dots, X_n(t)$.

Definition 1.2.7

Given a homogeneous linear system:

$$\frac{dX}{dt} = A(t)X, \tag{1.11}$$

where $A(t)$ is an $n \times m$ matrix with elements $a_{ij}(t)$.

A system of n solutions $x_1(t), x_2(t), \dots, x_n(t)$ of *homogeneous Linear system* (1.11) which are Linearly independent on the interval $a < t < b$ is said to be fundamental.

Theorem 1.2.7

The wronskian $w(t)$ of the system of solutions, fundamental on the interval $a < t < b$, of the homogeneous Linear system (1.11) with coefficients $a_{ij}(t)$ continuos on the interval $a \le t \le b$ is nonzero at all points of the interval (a, b).

Theorem 1.2.8 (The Structure of the General Solution of a Homogeneous Linear System)

The general solution in the domain $a < t < b$, $|x_k| < +\infty$, $k = 1,2,\dots,n$ of the homogeneous Linear system $\frac{dX}{dt} = A(t)X$ with coefficients $a_{ij}(t)$ continuous on the interval $a \le t \le b$, is a Linear combination of n solutions $x_1(t), x_2(t), \dots, x_n(t)$ of system (1.11) Linearly independent on the interval $a < t < b$:

$$X_{g.h} = \sum_{i=1}^{n} c_i X_i(t),$$

where $c_1, c_2, \dots, c_n$ are arbitrary constant numbers.

($X_{g.h}$ is the general solution of the homogeneous linear system).

Definition 1.2.8

The square matrix:

$$X(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \cdots & \dots & \dots & \dots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{vmatrix},$$

whose columns are Linearly independent solutions $X_1(t), X_2(t), \dots, X_n(t)$ of system (1.11) is known as a fundamental matrix of the system, it is easy to verify that the fundamental matrix satisfies the matrix equation:

$$\frac{dX}{dt} = A(t)X(t),$$

If $X(t)$ is the fundamental matrix of system (1.11) then the general solution of the system can be represented as;

$$X(t) = x(t)c, \qquad (1.12)$$

where:

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

is a constant column-matrix with arbitrary elements setting $t = t_0$ in (1.12) we obtain:

$$X(t_0) = x(t_0)c\ , \ \ where\ \ c = x^{-1}(t_0)X(t_0),$$

Consequently,

$$X(t) = x(t)x^{-1}(t_0)X(t_0),$$

the matrix $x(t)x^{-1}(t_0) = k(t, t_0)$ is known as *Cauchy's matrix*. Using it, we can represent the solution of system (1.12) as:

$$X(t) = k(t, t_0)X(t_0), \qquad (1.13)$$

Theorem 1.2.9 (The Structure of the General Solution of an inhomogeneous Linear System of Differential Equations)

The general solution in the domain $a < t < b, |x_k| < +\infty, k = 1,2, \dots, n$ of the inhomogeneous linear system of differential equation:

$$\frac{dX}{dt} = A(t)X + F(t),$$

with coefficients $a_{ij}(t)$, continuous on the interval $a \leq t \leq b$ and the right-hand sides $f_i(t)$ is equal to the sum of the general solution $\sum_{i=1}^{n} c_k X_k(t)$ of the corresponding homogeneous system and some particular solution $\tilde{X}(t)$ of the inhomogeneous system $\frac{dX}{dt} = A(t)X + F(t)$:

$$X_{g.inh} = X_{g.h} + X_{p.inh}$$

($X_{g.inh}$ is the general solution of the inhomogeneous system).

Definition 1.2.9 (System of Linear Differential Equations with Constant Coefficlents)

Let us consider a linear system of differential equations;

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + f_i(t), \qquad i = 1,2, \dots, n,$$

in which all the coefficients a_{ij} $(i, j = 1,2, \dots, n)$ are constant.

It is easier to integrate such a system by reducing it to one equation of a higher order when the latter equation is also linear with constant coefficients. Laplace's method of transformation is another method of integrating systems with constant coefficients.

We seek a solution of the system:

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \tag{1.14}$$

in the form:

$$x_1 = \alpha_1 e^{\lambda t}, x_2 = \alpha_2 e^{\lambda t}, \dots, x_n = \alpha_n e^{\lambda t}, \tag{1.15}$$

where $\lambda, a_1, a_2, \dots, \alpha_n$ are constants. Substituting x_k in the form (1.14) into (1.15) cancelling by $e^{\lambda t}$ and transferring all the terms into the same side of the equation we get a system:

$$\begin{aligned} (a_{11} - \lambda)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n &= 0 \\ a_{21}\alpha_1 + (a_{22} - \lambda)\alpha_2 + \cdots + a_{2n}\alpha_n &= 0 \\ &\vdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn} - \lambda)\alpha_n &= 0 \end{aligned}, \tag{1.16}$$

for the system (1.16) of homogeneous linear algebraic equations in n unknowns $\alpha_1, \alpha_2, \dots, \alpha_n$ to have a nontrivial solution, it is necessary and sufficient that its determinant be equal to zero:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0, \tag{1.17}$$

Equation (1.17) is characteristic. Its left-hand side includes a polynomial with respect to λ of degree n. From this equation we find the values of λ for which system (1.16) has nontrivial solutions $\alpha_1, \alpha_2, \dots, \alpha_n$ if all $\lambda_i, i = 1,2, \dots, n$ of the characteristic equation (1.17) are distinct, then successively substituting them into system (1.16) we find the corresponding nontrivial solutions $\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{ni}, i = 1,2, \dots, n$ of the system, and, consequently, find n solutions of the initial system of differential equations (1.14) in the form:

$$x_{1i} = \alpha_{1i} e^{\lambda_i t}, \qquad x_{2i} = \alpha_{2i} e^{\lambda_i t}, \dots, x_{ni} = \alpha_{ni} e^{\lambda_i t}, i = 1,2, \dots, n \qquad (1.18)$$

where the second index indicates the number of a solution and the first index indicates the number of unknown functions. Then particular solutions of the homogeneous linear system (1.14) :

$$X_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, X_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \dots, X_n(t) = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}, \qquad (1.19)$$

constructed in this way form as is easy to verify a fundamental system of solutions of the system consequently, the general solution of the homogeneous system of differential equation (1.14) has the form:

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + \cdots + c_n X_n(t),$$

or the form:

$$x_1(t) = c_1 x_{11}(t) + c_2 x_{12}(t) + \cdots + c_n x_{1n}(t),$$

$$x_2(t) = c_1 x_{21}(t) + c_2 x_{22}(t) + \cdots + c_n x_{2n}(t),$$

$$\dots$$

$$x_n(t) = c_1 x_{n1}(t) + c_2 x_{n2}(t) + \cdots + c_n x_{nn}(t),$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Chapter 2

Stability Solutions for Systems of Differential Equations

2.1 Concept of Stability

2.2 Definitions

2.3 Stability for Linear Systems

2.4 Stability in the First Approximation

Stability Solutions for Systems of Differential Equations

This chapter discusses the concept of stability and contains definitions of stability solutions of systems of differential equations, also discusses stability for linear systems and stability in the first approximation.

The materials in this chapter are taken from the following references [1], [3] and [8].

2.1 Concept of Stability

Let the following system of differential equations be given:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, \dots, x_n, t), \quad i = 1,2, \dots, n \tag{2.1}$$

A solution $\varphi_i(t)$, $i = 1,2, \dots, n$ of system (2.1) satisfying the initial conditions $\varphi_i(t_0) = \varphi_{i_0}$ $i = 1,2, \dots, n$ is said to be a Lyapunov stable solution as $t \to \infty$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that for each solution $x_i(t)$, $i = 1,2, \dots, n$ of system (2.1) whose initial values satisfy the conditions:

$$|x_i(t_0) - \varphi_i| < \delta, \qquad i = 1,2, \dots, n \tag{2.2}$$

the inequalities:

$$|x_i(t) - \varphi_i(t)| < \varepsilon, \qquad i = 1,2, \dots, n \tag{2.3}$$

hold for all $t \geq t_0$.

If for all arbitrarily small $\delta > 0$ inequalities (2.3) fail to hold for at least one solution $x_i(t),\ i = 1,2,\dots,n$ then the solution $\varphi_i(t)$ is said to be unstable.

If under condition (2.2) besides inequalities (2.3) the condition;

$$\lim_{t\to\infty} |x_i(t) - \varphi_i(t)| = 0\,, \qquad i = 1,2,\dots,n \tag{2.4}$$

also holds the solution $\varphi_i(t),\ i = 1,2,\dots,n$ is said be asymptotically stable.

Investigating a solution $\varphi_i(t),\ i = 1,2,\dots,n$ of system (2.1) for stability can be reduced to investigating for stability the zero (trivial) solution $x_i \equiv 0,\ i = 1,2,\dots,n$ of some system similar to system (2.1) :

$$\frac{dx_i}{dt} = F_i(x_1, x_2, x_3, \dots, x_n, t), \quad i = 1,2,\dots,n$$

where,

$$F_i(0,0,\dots,0,t) \equiv 0, \qquad i = 1,2,\dots,n$$

A point $x_i = 0,\ i = 1,2,\dots,n$ is said to be a stationary point of system (2.1).

As applied to the stationary point the definitions of stability and instability can be formulated as follows. A stationary point $x_i = 0,\ i = 1,2,\dots,n$ is stable according to Lyapunov if whatever $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $x_i(t),\ i = 1,2,\dots,n$ whose initial date $x_{i_0} = x_i(t_0),\ i = 1,2,\dots,n$ satisfy the condition:

$$|x_{i_0}| < \delta, \qquad i = 1,2,\dots,n \tag{2.5}$$

the inequalities:

$$|x_i(t)| < \varepsilon, \qquad i = 1,2,\dots,n \tag{2.6}$$

hold for all $t \geq t_0$.

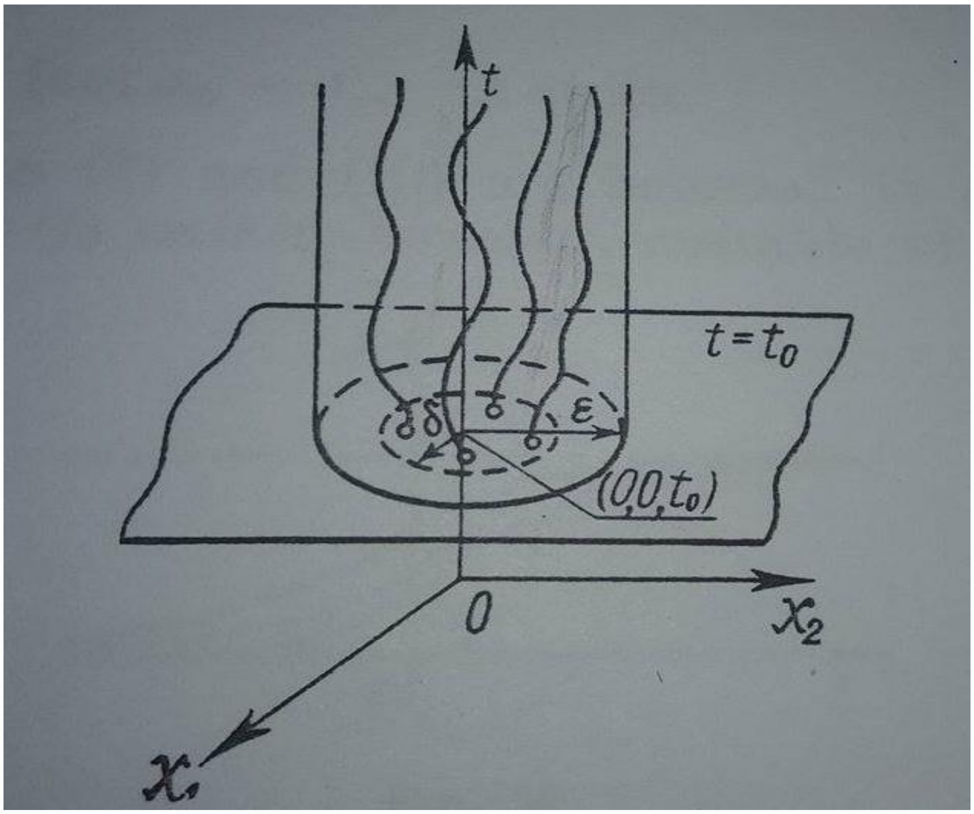


Figure 2.1(Concept of Stability)

Geometrically, for the case $n = 2$ this implies the following. However small a cylinder of radius ε with the Ot axis may be there is a δ–neighbourhood of the point $(0,0,t_0)$ in the plane $t = t_0$ such that all in integral curves:

$$x_1 = x_1(t), \qquad x_2 = x_2(t),$$

emanating from that neighbourhood will remain inside the cylinder for all $t \geq t_0$ (fig. 1).

If besides inequalities (2.3) the condition:

$$\lim_{t \to +\infty} |x_i(t)| = 0, \qquad i = 1,2, \dots, n$$

also holds, then the stability is asymptotic.

A stationary point $x_i = 0$, $i = 1,2, \dots, n$ is unstable if for an arbitrarily small $\delta > 0$ condition (2.6) does not hold for at least one solution $x_i(t)$, $i = 1,2, \dots, n$.

Examples 2.1.1

Example 1:

Proceeding from the definition of Lyapunov stability investigates for stability the solution of the equation:

$$\frac{dx}{dt} = 1 + t - x,$$

satisfying the initial condition $x(0) = 0$.

Solution:

The equation

$$\frac{dx}{dt} = 1 + t - x,$$

is a non-homogeneous Linear equation. Its general solution is:

$$x(t) = ce^{-t} + t,$$

The initial condition $x(0) = 0$ satisfied by the solution $\varphi(t) = t$ of equation:

$$\frac{dx}{dt} = 1 + t - x,$$

The initial condition $x(0) = x_0$ satisfied by the solution:

$$x(t) = x_0 e^{-t} + t,$$

we consider the difference of solution $x(t) = x_0 e^{-t} + t$, and $\varphi(t) = t$ of equation:

$$\frac{dx}{dt} = 1 + t - x,$$

and write it as:

$$x(t) - \varphi(t) = x_0 e^{-t} + t - t = (x_0 - 0)e^{-t},$$

Hence it is seen that for any $\varepsilon > 0$ there exists $\delta > 0$ (for example $\delta = \varepsilon$), such that for any solution $x(t)$ of equation

$$\frac{dx}{dt} = 1 + t - x,$$

whose initial values satisfy the condition $|x_0 - 0| < \delta$ the inequality:

$$|x(t) - \varphi(t)| = |x_0 - 0|e^{-t} < \varepsilon,$$

hold for all $t \geq 0$, therefor the solution $\varphi(t) = t$ is stable. More over since:

$$\lim_{t\to+\infty} |x_i(t) - \varphi(t)| = \lim_{t\to+\infty} |x_0 - 0|e^{-t} = 0,$$

the solution $\varphi(t) = t$ is asymptotically stable. That solution $\varphi(t)$ is unbounded when $t \to +\infty$. (The above example shows that the stability of the solution of a differential equation does not imply the boundedness of the solution).

Example 2:

Consider the equation:

$$\frac{dx}{dt} = sin^2 x,$$

It has the obvious solutions:

$$x = k\pi, \qquad k = 0, \pm 1, \pm 2, \dots \dots \qquad *$$

we integrate equation $\frac{dx}{dt} = sin^2 x$:

$$\cot x = c - t, \qquad or \quad \cot x = \cot x_0 - t,$$

whence:

$$x = arc\cot(\cot x_0 - t), \qquad x \neq k\pi, \qquad **$$

All solution * and ** are bounded in $(-\infty, +\infty)$ the solution $x(t) \equiv 0$ is however, unstable when $t \to +\infty$.

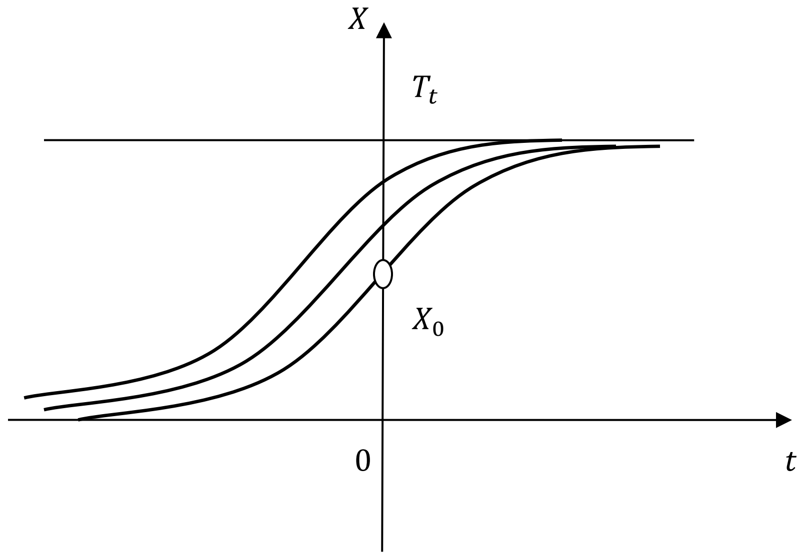


Figure 2.2

Since for any $x_0 \in (0, \pi)$ we have:

$$\lim_{t \to +\infty} x(t) = \pi,$$

(Therefore, the boundedness of solution of differential equation does not imply their stability (fig. 2)). This phenomenon is characteristic of non linear equation and systems.

Example 3:

Proceeding from the definition of Lyapunov stability show that the solution of system:

$$\frac{dx}{dt} = -y,$$

$$\frac{dy}{dt} = x,$$

satisfying the initial conditions $x(0) = 0, y(0) = 0$ is stable.

Solution:

The solution of this system satisfying the given initial conditions is $x(t) \equiv 0, y(t) \equiv 0$.

Any solution of the system satisfying the conditions $x(0) = x_0$, $y(0) = y_0$ is of the form:

$$x(t) = x_0 \cos t - y_0 \sin t\,, \quad y(t) = x_0 \sin t + y_0 \cos t,$$

we shall take an arbitrary $\varepsilon > 0$ and show that there exists $\delta(\varepsilon) > 0$ such that for:

$$|x_0 - 0| < \delta, \;\; |y_0 - 0| < \delta,$$

the inequalities:

$$|x(t) - 0| = |x_0 \cos t - y_0 \sin t| < \varepsilon,$$

$$|y(t) - 0| = |x_0 \sin t + y_0 \cos t| < \varepsilon,$$

hold for $t \geq 0$.

This exactly means according to the definition that the zero solution $x(t) \equiv 0, y(t) \equiv 0$ of system:

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x,$$

is a Lyapunov stable solution.

Obviously we have:

$$\begin{array}{l} |x_0 \cos t - y_0 \sin t| \leq |x_0 \cos t\,| + |y_0 \sin t| \leq |x_0| + |y_0| \\ |x_0 \sin t - y_0 \cos t| \leq |x_0 \sin t\,| + |y_0 \cos t| \leq |x_0| + |y_0| \end{array}, \qquad *$$

for all t. Therefore if:

$$|x_0| + |y_0| < \varepsilon$$

then so much the more

$$|x_0 \cos t - y_0 \sin t| < \varepsilon, \quad |x_0 \sin t - y_0 \cos t| < \varepsilon, \qquad **$$

for all t.

Consequently if we take for example $\delta(\varepsilon) = \frac{\varepsilon}{2}$ then by * inequalities ** will hold for all $t \geq 0$ when $|x_0| < \delta$ and $|y_0| < \delta$, i.e. the zero solution of system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x,$$

is indeed a Lyapunov stable solution, but its stability is not asymptotic.

2.2 Definitions

Definition 2.2.1

A solution $x = X(t)$ of $\dot{x} = f(t, x)$, $(\dot{x} = \frac{dx}{dt})$, is said to be stable if given any $\varepsilon > 0$ and any $t_0 \geq 0$ there exists a $\delta = \delta(\varepsilon, t_0)$ such that:

$$|x(t_0) - X(t_0)| < \delta \Rightarrow |x(t) - X(t)| < \varepsilon \quad \forall\, t \geq t_0 \geq 0,$$

for any solution $x(t)$ of $\dot{x} = f(t, x)$.

Definition 2.2.2

A solution $x = X(t)$ of $\dot{x} = f(t, x)$ is said to be uniformly stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ independent of t_0 such that:

$$|x(t_0) - X(t_0)| < \delta \;\Rightarrow |x(t) - X(t)| < \varepsilon \quad \forall\, t \geq t_0 \geq 0,$$

is satisfied for all $t_0 \geq 0$.

Definition 2.2.3

A solution $x = X(t)$ of $\dot{x} = f(t,x)$ is said to be unstable if it is not stable.

Definition 2.2.4

A solution $x = X(t)$ of $\dot{x} = f(t,x)$ is said to be asymptotically stable if it is stable and for any $t_0 \geq 0$ there exists a positive constant $c = c(t_0)$ such that:

$$|x(t_0) - X(t_0)| < c \;\Rightarrow x(t) - X(t) \to 0 \;\; as \;\; t \to \infty,$$

for any solution $x(t)$ of $\dot{x} = f(t,x)$.

Definition 2.2.5

A solution $x = X(t)$ of $\dot{x} = f(t,x)$ is said to be uniformly asymptotically stable if it is uniformly stable and there exists appositive constant c independent of t_0 such that for every $\eta > 0$ there exists $T = T(\eta) > 0$ such that for all $t \geq 0$:

$$|x(t_0) - X(t_0)| < c \;\Rightarrow |x(t) \; - X(t)| < \eta \;\; \forall\, t \geq t_0 + T(\eta),$$

for any solution $x(t)$ of $\dot{x} = f(t,x)$.

Definition 2.2.6

A solution $x = X(t)$ of $\dot{x} = f(t, x)$ is said to be globally uniformly asymptotically stable if it is uniformly stable with $\delta(\varepsilon)$ satisfying:

$$\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty,$$

and for all positive η and c there exists $T = T(\eta, c) > 0$ such that for all $t \geq 0$:

$$|x(t_0) - X(t_0)| < c \;\Rightarrow |x(t) - X(t)| < \eta \;\; \forall \; t \geq t_0 + T(\eta, c),$$

for any solution $x(t)$ of $\dot{x} = f(t, x)$.

2.3 Stability for Linear system

The problem of stability of solutions of the linear system:

$$\dot{x} = A(t)x, \tag{2.7}$$

will first be considered.

Here $x = x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ is an unknown vector function and the matrix $A(t) = (a_{ij}(t))$ is continuous for $t_0 \leq t < \infty$. Recall that the solution of (2.7) satisfying $x(t_0) = x_0$, then defined for $t \geq t_0$ and given by $x(t; t_0, x_0) = \varphi(t)x_0$ where $\varphi(t)$ is the fundamental matrix satisfying $\varphi(t_0) = I$ we will need the notion of the norm of matrix.

Definition 2.3.1

Given the $n \times n$ matrix $A = (a_{ij})$ then $||A||$ the norm of A, is defined by:

$$||A|| = \sum_{i,j=1}^{n} |a_{ij}|,$$

Evidently $||\quad||$ is a real valued nonnegative function defined on the set of $n \times n$ matrixes and if $A = A(t)$ is continuous then $||A(t)||$ is continuous.

In addition it satisfies the properties:

i. $||A + B|| \leq ||A|| + ||B||$, $||AB|| \leq ||A||\ ||B||$.
ii. $||cA|| \leq |c|\ ||A||$ for all any scalar c.
iii. $||A(t)x|| \leq ||A||\ ||x||$ for all any vector x.

as may be easily verified.

In general, the notions of stability of a solution and boundedness of a solution are independent, for example the solution $x = t + x_0$ of $\dot{x} = 1$ are stable but unbounded.

Theorem 2.3.1

All solutions of $\dot{x} = A(t)x$ are stable if and only if they are bounded.

Proof:

If all solutions of $\dot{x} = A(t)x$ are bounded then there exists a constant M such that $||\varphi(t)|| < M$, where $\varphi(t)$ is the fundamental matrix of $\dot{x} = A(t)x$ satisfying $\varphi(t_0) = I$.

Given any $\varepsilon > 0$ then $||x_0 - x_1|| < \varepsilon/_{M}$ implies that:

$$||x(t;t_0,x_0) - x(t;t_0,x_1)|| = ||\varphi(t)(x_0 - x_1)|| \leq M||x_0 - x_1|| < \varepsilon,$$

and hence all solution are stable.

Conversely if all solutions are stable then the solution $x(t; t_0, x_1) \equiv 0$ is stable, therefore given $\varepsilon > 0$ there exists $\delta > 0$ such that $||x_1|| < \varepsilon$ implies:

$$||0 - x(t; t_0, x_1)|| = ||\varphi(t)x_1|| < \varepsilon,$$

in particular we can let x_1 be the vector with $\frac{\delta}{2}$ in the ith place and zero elsewhere. Then:

$$||\varphi(t)x_1|| = ||\varphi_i(t)||\frac{\delta}{2} < \varepsilon,$$

where $\varphi_i(t)$ is the ith column of $\varphi(t)$ and hence

$$||\varphi(t)|| < 2n\varepsilon\delta^{-1} = k,$$

Therefore for any solution we have

$$||x(t; t_0, x_0)|| = ||\varphi(t)x_0|| < k||x_0||,$$

and hence all solutions are bounded.

Theorem 2.3.2

If the characteristic polynomial of A is stable then every solution of $\dot{x} = Ax$ is asymptotically stable.

Proof:

If the characteristic polynomial is stable then there exist positive constants R and α such that:

$$||\varphi(t)|| \leq Re^{-\alpha t}, \qquad t \geq t_0 \geq 0,$$

where $\varphi(t)$ is fundamental matrix satisfying $\varphi(t_0) = I$ since $Re^{-\alpha t}$ is a decreasing function, given $\varepsilon > 0$, then $||x_0 - x_1|| \leq \varepsilon R^{-1} e^{-\alpha t_0}$ implies:

$$||x(t; t_0, x_0) - x(t; t_0, x_1)|| \leq ||\varphi(t)|| \, ||x_0 - x_1|| \leq Re^{-\alpha t} ||x_0 - x_1||,$$

The right side is less then ε for $t \geq t_0$ and furthermore approaches zero as t approaches ∞ so all solutions are asymptotically stable.

2.4 Stability in the First Approximation

Let the following system of differential equation be given:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n), \qquad i = 1,2, \dots, n \tag{2.8}$$

and let $x_i \equiv 0, i = 1,2, \dots, n$ be a stationary point of system (2.8). i.e. $f_i(0,0, \dots ,0) = 0, i = 1,2, \dots, n$ we shall assume that functions $f_i(x_1, x_2, \dots, x_n)$ can be differentiated a sufficiently large number of times at the origin of coordinates.

We expand the functions f_i in the *Taylor series* of x in the neighborhood of the origin of coordinates:

$$f_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} a_{ij} x_j + R_i(x_1, x_2, \dots, x_n),$$

Here $a_{ij} = \frac{\partial f_i(0,0,\ldots,0)}{\partial x_j}$ and R_i are terms of the second order of smallness with respect to $x_1, x_2, \ldots, x_n$.

The original system (2.8) will then be written as:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + R_i(x_1, x_2, \ldots, x_n), \quad (i = 1,2, \ldots, n), \tag{2.9}$$

Instead of system (2.9) we shall consider the system:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j, \quad (i = 1,2, \ldots, n)\big(a_{ij} + \cos t\big), \tag{2.10}$$

Called the system of equation of the first approximation for system (2.8).

Theorem 2.4.1

The following propositions hold:

1. If all roots of the characteristic equation:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0, \tag{2.11}$$

have negative real parts, then zero solutions $x_i \equiv 0, i = 1,2, \ldots, n$ of system (2.10) and system (2.9) are asymptotically stable.

2. If at least one root of the characteristic equation (2.11) has a positive real part then the zero solution of system (2.10) and system (2.9) is unstable.

It is said that investigation for stability in the first approximation is possible in cases 1 and 2.

In critical cases when the real parts of all roots of the characteristic equation (2.11) are non positive with the real part of at least one root being zero investigation for stability in the first approximation is in general impossible (nonlinear terms R_i starting to exert influence).

Example 2.4.1

Investigate the stationary point $x = 0, y = 0$ of the system:

$$\begin{aligned} \dot{x} &= 2x + y - 5y^2 \\ \dot{y} &= 3x + y + \frac{x^3}{2} \end{aligned} \qquad \left(\dot{x} = \frac{dx}{dt}\,, \qquad \dot{y} = \frac{dy}{dt}\right),$$

for stability in the first approximation.

Solution:

The system of the first approximation is;

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 3x + y \end{aligned}$$

the nonlinear terms satisfy the necessary conditions their order being greater than or equal to two.

We set up the characteristic equation for system;

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 3x + y \end{aligned}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = 0 \quad or \quad \lambda^2 - 3\lambda - 1 = 0,$$

The roots of the characteristic equation $\lambda^2 - 3\lambda - 1 = 0$:

$$\lambda_1 = \frac{3+\sqrt{13}}{3}\,, \qquad \lambda_2 = \frac{3-\sqrt{13}}{3}\,,$$

are real and $\lambda_1 > 0$.

There for the zero solution $x = 0, y = 0$ of system:

$$\dot{x} = 2x + y - 5y^2$$
$$\dot{y} = 3x + y + \frac{x^3}{2},$$

is unstable.

Chapter 3

Stability of Zero Solution

3.1 The Simplest Types of Stationary Points

3.2 Stability of Fixed Points

3.3 Zero Stability

Stability of zero solution

These chapters discuss the concept of stability of zero solution, the simplest types of stationary points, stability of fixed points, zero stability and consider some basic definitions and theorems about them.

The materials in this chapter taken from the following references [4], [6], [8]and [9].

3.1 The Simplest Types of Stationary Points

Consider a system of two homogeneous linear differential equations with constant coefficients;

$$\begin{aligned} \frac{dx}{dt} &= a_{11}x + a_{12}y, \\ \frac{dy}{dt} &= a_{21}x + a_{22}y, \end{aligned} \tag{3.1}$$

with

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix} \neq 0,$$

A point $x = 0, y = 0$ which the right-hand sides of the equations of system (3.1) vanish called a stationary point of system (3.1).

In order for a stationary point of system (3.1) to be investigated it is necessary to set up the characteristic equation:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{21} - \lambda \end{vmatrix} = 0, \tag{3.2}$$

and find its roots λ_1 and λ_2.

The following cases are possible:-

1- The roots λ_1, λ_2 of the characteristic equation (3.2) are real and distinct:

a) $\lambda_1 < 0,\ \lambda_2 < 0$ The stationary point is asymptotically stable (a stable node).

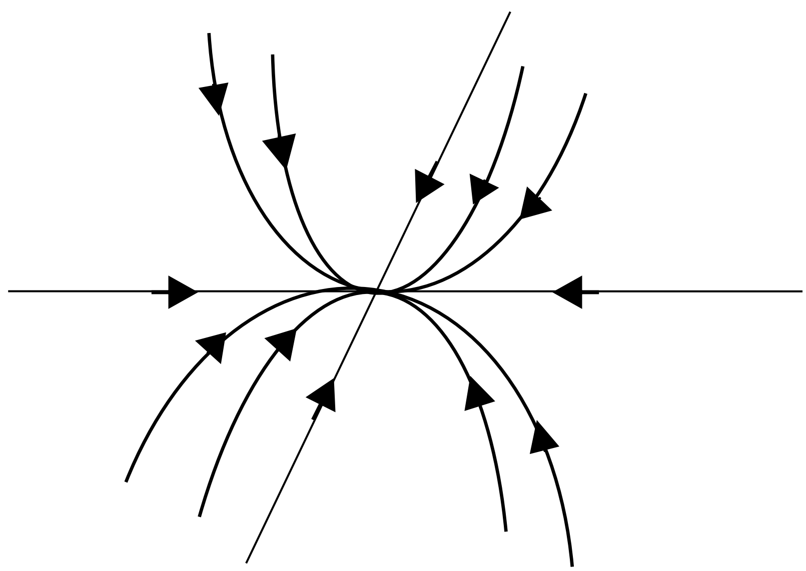


Figure 3.1(A stable node)

b) $\lambda_1 > 0,\ \lambda_2 > 0$ The stationary point is unstable (an unstable node).

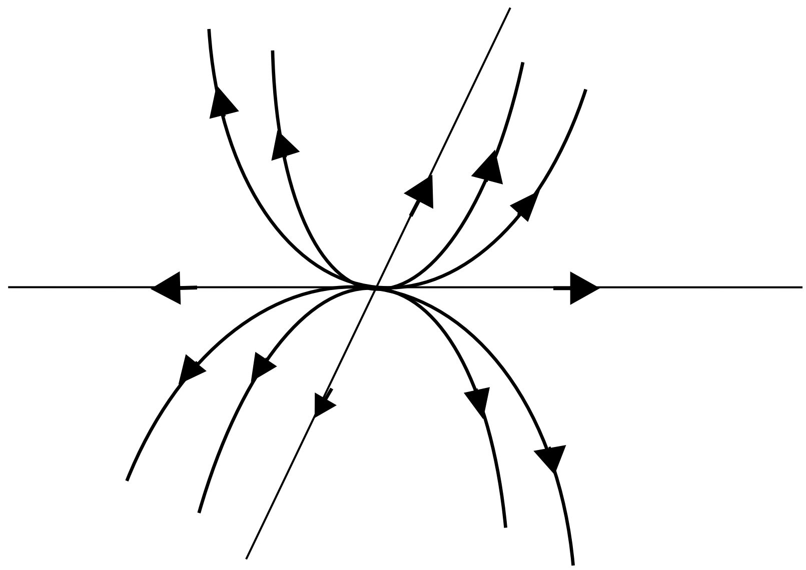


Figure 3.2 (An unstable node)

c) $\lambda_1 > 0, \lambda_2 < 0$ The stationary point is unstable (a saddle point).

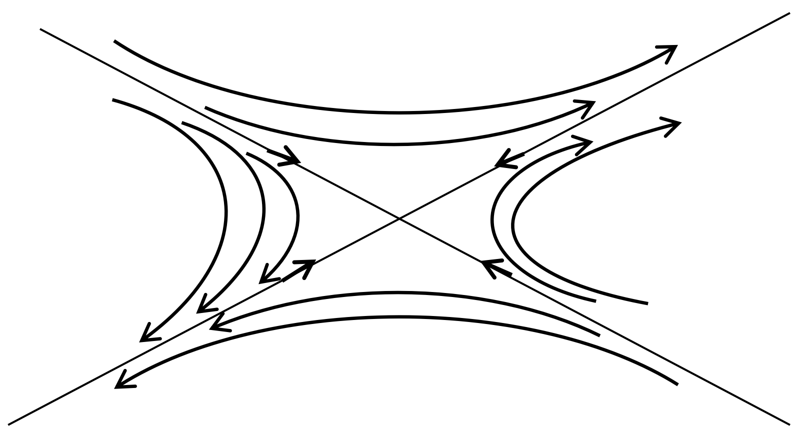


Figure 3.3(A saddle point)

2- The roots of the characteristic equation (2) are complex $\lambda_1 = p + iq$, $\lambda_2 = p - iq$:

a) $p < 0,\ q \neq 0$ The stationary point is asymptotically stable (a stable focus).

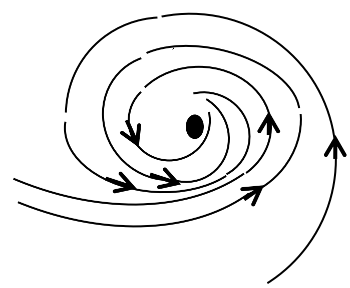


Figure3.4(A stable focus)

b) $p > 0, q \neq 0$ The stationary point is unstable (an unstable focus).

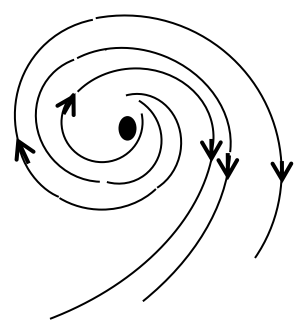


Figure 3.5 (An unstable focus)

c) $p = 0, q \neq 0$ The stationary point is stable (a mid point).

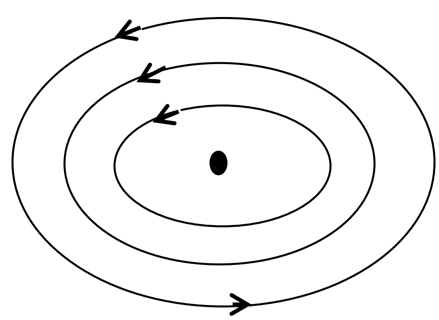


Figure 3.6(A mid point)

3- The roots $\lambda_1 = \lambda_2$ are multiple:

a) $\lambda_1 = \lambda_2 < 0$ The stationary point is asymptotically stable (a stable node).

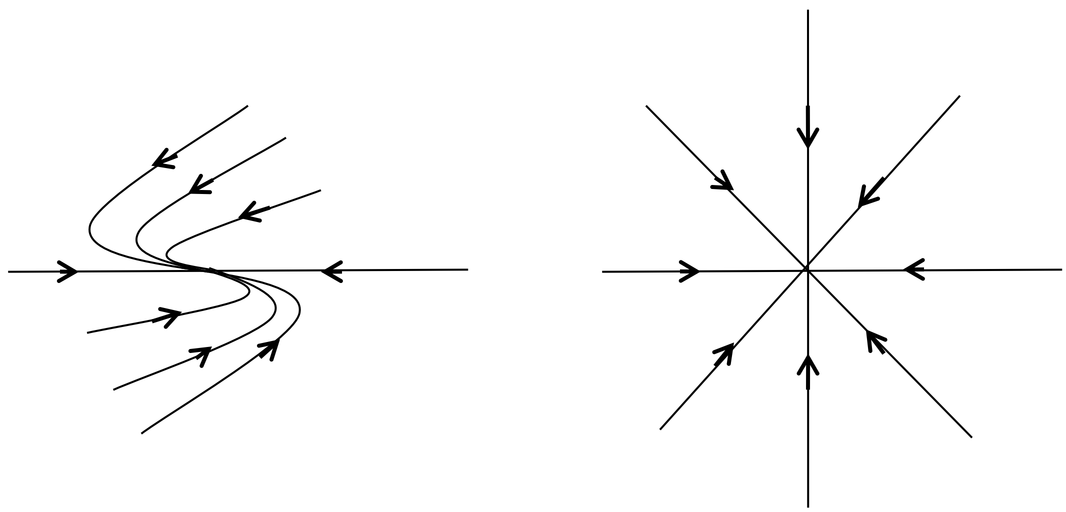


Figure 7(A stable node)

b) $\lambda_1 = \lambda_2 > 0$ The stationary point is unstable (an unstable node).

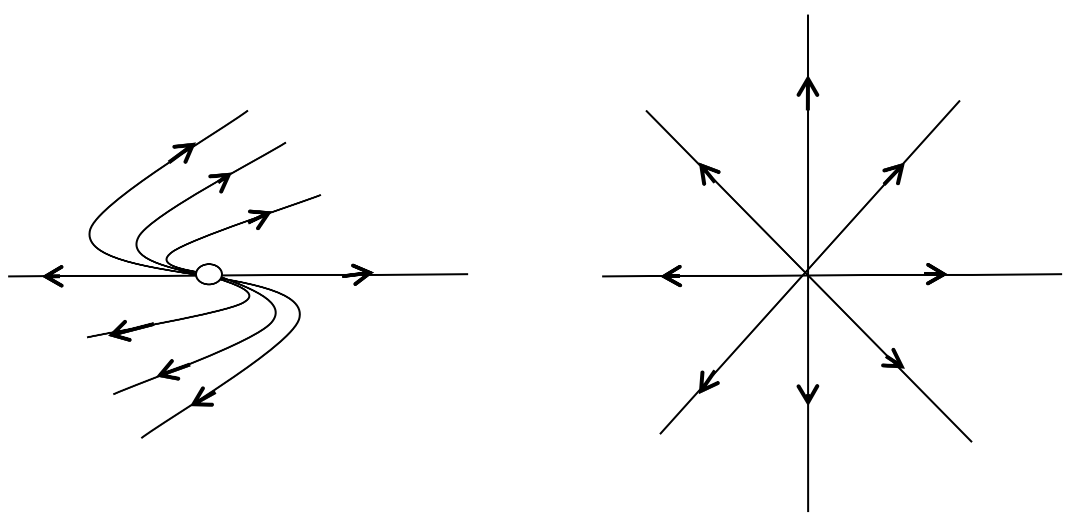


Figure 8(An unstable node)

Example 3.1.1

Determine the character of the stationary point $(0,0)$ of the system:

$$\frac{dx}{dt} = 5x - y,$$

$$\frac{dy}{dt} = 2x + y,$$

Solution:

We set up the characteristic equation:

$$\begin{vmatrix} 5-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \quad or \quad \lambda^2 - 6\lambda + 7 = 0,$$

its roots $\lambda_1 = 3 + \sqrt{2} > 0$, $\lambda_2 = 3 - \sqrt{2} > 0$ are distinct and positive.

Therefore the stationary point $(0,0)$ is an unstable node.

Definition 3.1.1

Given an autonomous system:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n), \qquad i = 1,2, \dots, n \tag{3.3}$$

Assume that $(a_1, \dots, a_n)$ is a set of numbers such that $f_i(a_1, \dots, a_n) = 0$, $i = 1,2, \dots, n$.

Then the system of functions $x_i(t) \equiv a_i,\ i = 1,2, \dots, n$ is solution of system (3.3). The point $(a_1, a_2, \dots, a_n)$ of the phase space $(x_1, x_2, \dots, x_n)$ is called a rest point of the given system.

Definition 3.1.2

We say that the rest point $x_i = 0,\ i = 1,2,\dots,n$ of system (3.3) is stable if for any $\varepsilon > 0$ $(0 < \varepsilon < R)$ there is $\delta = \delta(\varepsilon) > 0$ such that any trajectory of the system which begins the initial moment $t = t_0$ at the point $M_0 \in S(\delta)$ remains all the time in the sphere $S(\varepsilon)$.

The rest point is asymptotically stable if:

1- It is stable.

2- There is $\delta_1 > 0$ such that every trajectory of the system which begins at the point M_0 of the domain $S(\delta_1)$ approaches the origin when the time t increases indefinitely.

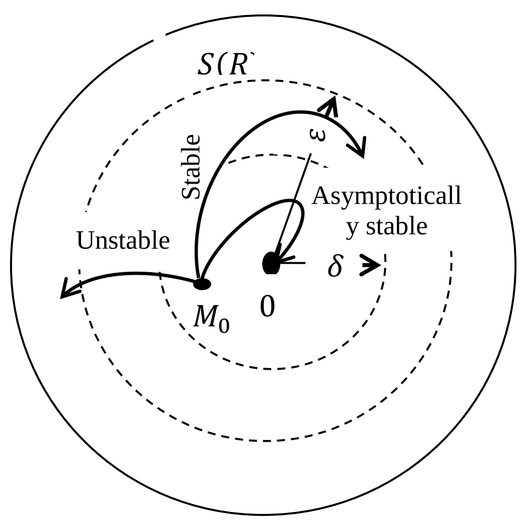


Figure 9 (The rest point)

3.2 Stability of Fixed Points

Definition 3.2.1

A fixed point x_0 of $f(x)$ is called stable if for any given neighborhood $U(x_0)$ the exists another neighborhood $V(x_0) \subseteq U(x_0)$ such that any solution starting in $V(x_0)$ remains in $U(x_0)$ for all $t \geq 0$.

A fixed point which is not stable will be called unstable.

Definition 3.2.2

A fixed point x_0 of $f(x)$ is called asymptotically stable if it is stable and if there is a neighborhood $U(x_0)$ such that:

$$\lim_{t \to \infty} |\varphi(t, x) - x_0| = 0, \quad for\ all\ x \in U(x_0), \tag{3.4}$$

Definition 3.2.3

A fixed point x_0 of $f(x)$ is called exponentially stable if there are constants $\alpha, \delta, c > 0$, such that:

$$|\varphi(t, x) - x_0| \leq c e^{-\alpha t} |x - x_0|, \qquad |x - x_0| \leq 0, \tag{3.5}$$

Example 3.2.1

Consider $\dot{x} = ax$ in R' then $x_0 = 0$ is stable if and only if $a \leq 0$ and exponentially stable if and only if $a < 0$.

More generally, suppose the equation $\dot{x} = f(x)$ in R' has a fixed point x_0. Then it is not hard to see that x_0 is stable if;

$$\frac{f(x)-f(x_0)}{x-x_0} \le 0, \quad x \in U(x_0)\backslash\{x_0\}, \tag{3.6}$$

For some neighborhood $U(x_0)$ and asymptotically stable if strict inequality holds. It will be exponentially stable if:

$$\frac{f(x)-f(x_0)}{x-x_0} \le -\alpha, \qquad 0 \le |x-x_0| \le \delta, \tag{3.7}$$

In fact (3.5) with $c = 1$ follows from a straightforward sub/super solution argument by comparing with solutions of the linear equation $\dot{y} = -\alpha y$.

In particular if $f'(x_0) \neq 0$ the stability can be read of from the derivative of f at x_0 alone.

Theorem 3.2.1 (Exponential Stability Via Linearization)

Suppose $f \in c^1$ has a fixed point x_0 and suppose that all eigenvalues of the Jacobian matrix at x_0 have negative real part, then x_0 is exponentially stable.

However if $f'(x_0) = 0$ no information on the stability of the nonlinear system can read off from the linearized one as can be seen from the following example.

Examples 3.2.2

1- The equation $\dot{x} = \mu x^3$ is asymptotically stable for $\mu < 0$, stable for $\mu \le 0$ and unstable for $\mu > 0$.

2- The system $\dot{x} = \mu x - x^3$ has one stable fixed point for $\mu \leq 0$ which becomes unstable and splits off two stable fixed points at $\mu = 0$.

3- The system $\dot{x} = \mu x + x^2$ has one stable and one unstable fixed point for $\mu < 0$ which collide at $\mu = 0$ and vanish.

4- The system $\dot{x} = \mu x - x^2$ has two stable fixed points for $\mu \neq 0$ which collide and exchange stability at $\mu = 0$.

3.3 Zero Stability

Definition 3.3.1

A linear k-step method for the ordinary differential equation $y' = f(x, y)$ is said to be zero stable if there exists a constant k such that for any two sequences (y_n) and $(\hat{y}_n)$ which have been generated by the same formula, but different initial data $y_0, y_1, \dots, y_{k-1}$ and $\hat{y}_0, \hat{y}_1, \dots, \hat{y}_{k-1}$ respectively we have:

$$|y_n - \hat{y}_n| \leq k\ Max\,\{|y_0 - \hat{y}_0|, |y_1 - \hat{y}_1|, \dots, |y_{k-1} - \hat{y}_{k-1}|\}, \qquad (3.8)$$

$$for\ \ x_n \leq X_n\ \ and\ as\ h\ tends\ to\ 0,$$

We shall prove later on that whether or not a method is zero stable can be determined by merely considering its behavior when applied to the trivial differential equation $y' = 0$ corresponding to $y' = f(x, y)$ with $f(x, y) \equiv o$ it is for this reson that the kind of stability expressed in definition(3.3.1) is called zero stability.

While definition(3.3.1) is expressive in the sense that it conforms with the intuitive notion of stability where by “small perturbations at input give rise to small perturbations at output" it would be a very tedious exercise to verify the zero-stability of linear multi-step method using definition(3.3.1) only, thus we shall next formulate an algebraic equivalent of zero stability known as the root condition which will simplify this task.

Theorem 3.3.1

A linear multi-step method is zero-stable for any ordinary differential equation of the form $y' = f(x, y)$ where f satisfies the Lipchitz condition $|f(x, y) - f(x, z)| \leq L|y - z|$ if and only if its first characteristic polynomial has zeros inside the closed unit disc with any which lie on the unit circle being simple.

The algebraic stability condition contained in this theorem namely that the roots of the first characteristic polynomial lie in the closed unit disc and those on the unit circle are simple is often called the root condition.

Chapter 4

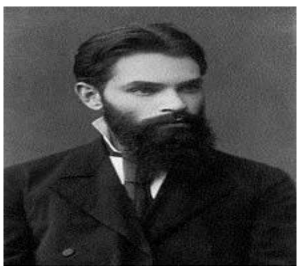
Stability in the Sense of Lyapunov

Stability in the sense of Lyapunov

This chapter discusses the concept of stability in the sense of Lyapunov theory and examines some theories related to these concepts.

The materials in this chapter taken from the following references [5], [7], [8] and [9].

4.1 History



Aleksander Mikhailovich Lyapunov Russian Citizenship was born in Yaroslavl, the Russian Empire in the June 6, 1857 and died in Odessa, People’s Republic of Ukraine at the age of 61 on November 3, 1918 and specialized in the field of applied mathematics.

Lyapunov stability is named after Aleksandr Lyapunov who published his book the General problem of stability of Motion in 1892.

Lyapunov is known for his development of the theory of a dynamical system as well as for his many contributions to mathematical physics and probability theory.

4.2 Definitions

Given differential equation:

$$\frac{dx}{dt} = f(t, x), \tag{4.1}$$

where the function $f(t, x)$ is defined and continuous for $t \in (a, +\infty)$ and x from a certain domain D and possesses a bounded partial derivative $\frac{\partial f}{\partial x}$.

Assume that the function $x = \varphi(t)$ is a solution of equation (1) which satisfies the initial condition $x|_{t=t_0} = \varphi(t_0), t_0 > a,$ we assume furthermore that the function $x = x(t)$ is a solution of the same equation which satisfies another initial condition $x|_{t=t_0} = x(t_0)$ it is assumed that the solutions $\varphi(t)$ and $x(t)$ are defined for all $t \geq t_0$.

i.e. can be extended indefinitely to the right.

Definition 4.2.1

The solution $x = \varphi(t)$ of equation (4.1) is said to be stable in the sense of Lyapunov as $t \to +\infty$ if, for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for every solution $x = x(t)$ of that equation the inequality

$$|x(t_0) - \varphi(t_0)| < \delta, \tag{4.2}$$

Yields an inequality:

$$|x(t) - \varphi(t)| < \varepsilon, \tag{4.3}$$

for all $t \geq t_0$ (we can always assume that $\delta \leq \varepsilon$).

This means that solutions that are close to the solution $x = \varphi(t)$ as concerns the initial values remain close for all $t \geq t_0$ as well.

In terms of geometry this means the following the solution $x = \varphi(t)$ of equation (4.1) is stable if however narrow the ε –strip containing the cure $x = \varphi(t)$ all the integral curves $x = x(t)$ of the equation which are sufficiently close to the strip at the initial moment $t = t_0$ lie entirely in the indicated ε –strip for all $t \geq t_0$.

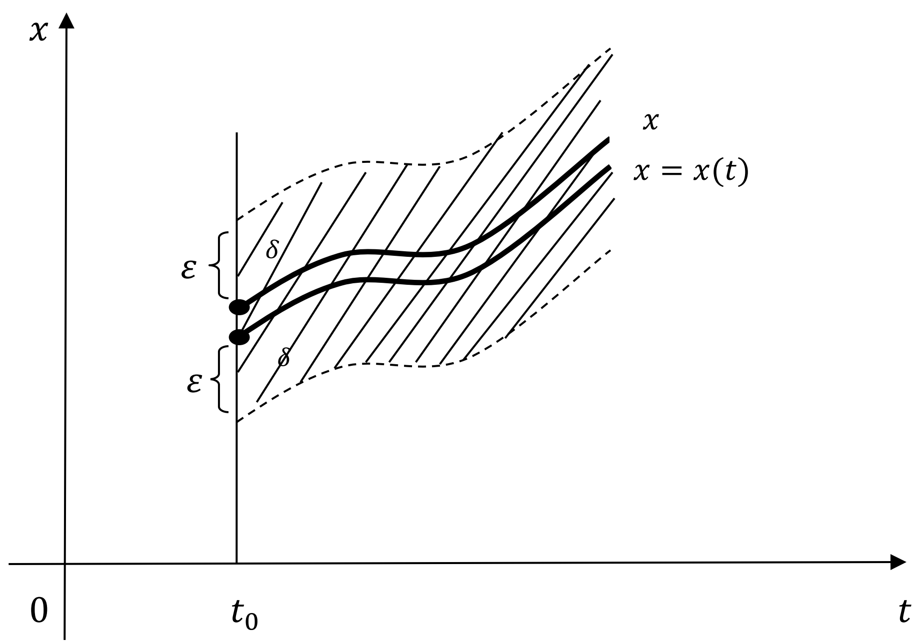


Figure 4.1

If for an arbitrarily small $\delta > 0$ inequality (4.3) does not hold for at least one solution $x = x(t)$ of equation (4.1) then the solution $x = \varphi(t)$ of that equation is said to be unstable.

That solution which cannot be extended to the right $t \to +\infty$ must be considered to be unstable.

Definition 4.2.2

The solution $\varphi_i(t), i = 1,2, \dots, n$ of system:

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), \qquad i = 1,2, \dots, n \tag{4.4}$$

is said to be stable in the sense of Lyapunov as $t \to +\infty$ if for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for every solution $x_i(t),\ i = 1,2, \dots, n$ of that system, whose initial values satisfy the inequalities

$|x_i(t_0) - \varphi_i(t_0)| < \delta, i = 1,2, \dots, n$ the inequalities;

$$|x_i(t) - \varphi_i(t)| < \varepsilon, i = 1,2, \dots, n \tag{4.5}$$

are satisfied for all $t \geq t_0$.

i.e. the solutions close as concerns the initial values remain close for all $t \geq t_0$.

If for an arbitrarily small $\delta > 0$ inequalities (4.5) do not hold even for one solution $x_i(t),\ i = 1,2, \dots, n$ then the solution $\varphi_i(t)$ is unstable.

4.3 Lyapunov's theorem

The method of Lyapunov functions is to investigate directly the stability of the equilibrium position of the system

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), \qquad i = 1,2, \dots, n$$

with the help of a suitably selected function $v(t, x_1, x_2, \dots, x_n)$ the Lyapunov function this being done without finding beforehand any solutions of the system.

We restrict ourselves to the consideration of autonomous system;

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n), \qquad i = 1,2, \dots, n \tag{4.6}$$

for which $x_i = 0, i = 1,2, \dots, n$ is a stationary point.

Theorem 4.3.1 (Lyapunov's Stability Theorem)

If for a system of differential equations (4.6) there exists a function of fixed sign $v(t, x_1, x_2, \dots, x_n)$ (a Lyapunov function) whose total derivative $\frac{dv}{dt}$ with respect to time composed by virtue of system (4.6) is a function of constant signs of sign opposite to that of v or identically equal to zero then the stationary point $x_i = 0,\ i = 1,2, \dots, n$ of system (4.6) is stable.

Example 4.3.1

Consider the system:

$$\begin{array}{l} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{array}, \tag{4.7}$$

we choose the function $v = x^2 + y^2$ as the function $v(x, y)$ it is positive definite.

The derivative of the function v is by virtue of system (4.7) equal to:

$$\frac{dv}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2xy - 2xy \equiv 0,$$

It follows from (Theorem 4.3.1) that the stationary point $O(0,0)$ of system (4.7) is stable.

Theorem 4.3.2 (Lyapunov's Asymptotically Stability Theorem)

If for a system of differential (4.6) there exists a function of fixed sign $v(t, x_1, x_2, \ldots, x_n)$ whose total derivative with respect to time composed by virtue of system (4.6) is also a function of fixed sign of sign opposite to that of v then the stationary point $x_i \equiv 0$ of system (4.6) is asymptotically stable.

Example 4.3.2

Consider the system:

$$\begin{aligned} \frac{dx}{dt} &= y - x^3 \\ \frac{dy}{dt} &= -x - 3y^3 \end{aligned}, \tag{4.8}$$

Taking $v(x, y) = x^2 + y^2$ we find that,

$$\frac{dv}{dt} = 2x(y - x^3) + 2y(-x - 3y^3) = -2(x^4 + 3y^4),$$

Thus $\frac{dv}{dt}$ is a negative definite function by (Theorem 4.3.2) the stationary point $O(0,0)$ of system (4.8) is asymptotically stable.

Theorem 4.3.3 (Lyapunov's Instability Theorem)

Let there exist for the system of differential equations (4.6) a function differentiable in the neighborhood of the origin of coordinates $v(t, x_1, x_2, \dots, x_n)$ such that $v(0,0,\dots,0) \equiv 0$.

If its total derivative $\frac{dv}{dt}$ composed by virtue of system (4.6) is a positive definite function and arbitrarily close to the origin of coordinates there are points in which the function $v(x_1, x_2, \dots, x_n)$ takes positive values, then the stationary point $x_i = 0, i = 1,2, \dots, n$ is unstable.

Example 4.3.3

Investigate the stationary point $x = 0, y = 0$ of the system:

$$\begin{array}{c} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y \end{array},$$

for stability.

Solution:

Take the function $v(x, y) = x^2 - y^2$, then;

$$\frac{dv}{dt} = \frac{\partial v}{\partial x} . \frac{dx}{dt} + \frac{\partial v}{\partial y} . \frac{dy}{dt} = 2x^2 + 2y^2,$$

is a positive definite function.

Since arbitrarily close to the origin of coordinates there are points which $v > 0$ (for example $v = x^2 > 0$ along the straight line $y = 0$) all the conditions of (Theorem 4.3.3) hold and the stationary point $O(0,0)$ is unstable (a saddle point).

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