Geometry and Trigonometry Module 1 Discrete Mathematics

Section 1.1 Graph Theory

Looking Back 1.1

Unlike Geometry, Algebra, or Calculus, discrete mathematics is not a separate course. However, discrete mathematics does involve many branches of mathematics that share the common feature of being discrete rather than continuous. The branches that are discrete include matrices, which have already been addressed, graph theory, counting (with combinatorics), social choice, and recursion. These latter topics are part of real-life situations, which are modeled in mathematics. The modeling occurs through technology, algorithmic and recursive thinking, and decision making. Modern computer science is built upon graph theory and combinatorics.

Below are some of the contrasts between continuous mathematics and discrete mathematics:

Continuous	**Discrete**
Natural Science	Computer and Management and Social Sciences
Uncountable	Countable
Infinite Limits	Finite Processes

A major method of validating algorithms and proving theorems in discrete mathematics is by using the principles of mathematical deduction. The following questions are questions that can be answered from the field of discrete mathematics:

1. How many traffic lights are needed at an intersection?

2. How many area codes are needed for a telephone system?

3. How can an estate be divided fairly?

4. What is the cheapest airfare when flying through multiple cities?

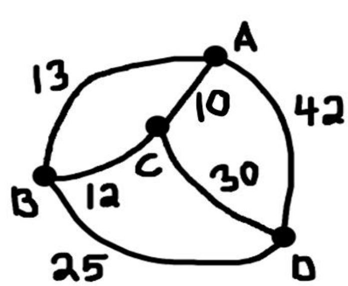
5. How can a fair voting system be established?

6. How can chemicals be stored and shipped efficiently when some contain hazardous substances and are not compatible with others?

Looking Ahead 1.1

In this section we will investigate the famous Traveling Salesman Problem.

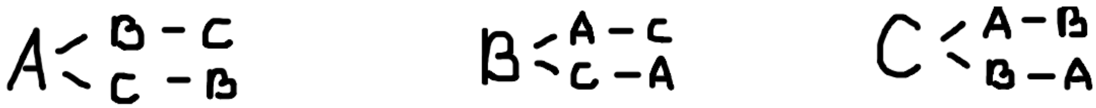
The traveling salesman needs to make visits to all of his/her customers at different cities. He/she needs to visit each customer at least once while traveling the shortest distance possible. The salesman must start and finish where he/she resides. Below is a visual representation of the problem. The numbers represent the distance between the different cities the salesman must visit.



It is easiest to let city A be the starting and ending point. The easiest way to solve the problem is to find all the $(n-1)!$ permutations of the cities and calculate the distance of each route. The minimum distance is the preferred route.

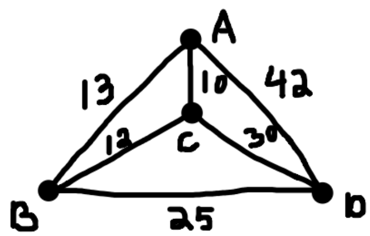
You learned about permutations in Pre-Algebra. A permutation is all possible arrangements of an object when order is important.

For example, there is only one arrangement of A, which is A, and two arrangements of A and B, which are AB and BA. The arrangements for ABC could be done with tree diagrams, as shown below:



There are six total arrangements for ABC, or $3! = 3 \cdot 2 \cdot 1$.

For four cities, the possible arrangements are $4!$ or $4 \cdot 3 \cdot 2 \cdot 1 = 24$. There are 24 possible arrangements. Each of these can be checked to find the shortest route. The graph can be drawn simpler, as shown below:



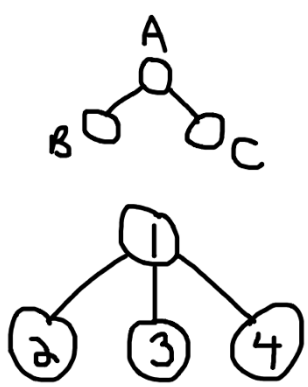
This is much more geometric, but not proportionally accurate. It is just a better view for problem-solving.

In electrical engineering, circuits used for conduction involve many terms, such as: branch, element, loop, component, node, etc. In discrete mathematics, understanding vocabulary such as "node" and "branch" will help in problem-solving. These terms will be defined as we progress through the module.

A graph is a set of points (vertices) and edges connecting them. The Traveling Salesman Problem has four points (A, B, C, and D) and six edges connecting them (AB, AC, AD, BC, BD, and CD). Please note that the edge AB is the same as BA, which is 13 miles either way.

The graph for the Traveling Salesman Problem is a weighted graph because it has values (distances assigned) to each edge. Edges frequently represent distance, time, or costs.

A node is each point of the graph that is a vertex. The degree of the node in a network (graphs) is the number of connections it has to other nodes through the edges.



Point A is an even degree node. It has two edges connecting it to other nodes. Point 1 is an odd degree node. It has three edges connecting it to other networks. (This will be helpful to know in the next two sections!)

As the vertices increase the possibilities become more complex. Another method to solve this problem may be the nearest-neighbor algorithm. Move from one vertex to the next that is the minimum distance each time until returning to the original destination.

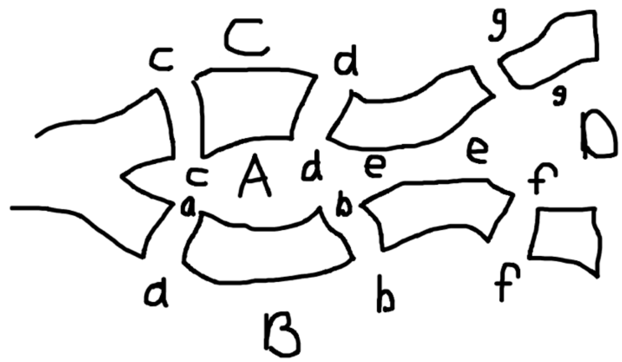
There will only be ten problems in each of the Practice Problem sections of this module as they require much time and creative thinking. The Traveling Salesman Problem is included in this section of Practice Problems.

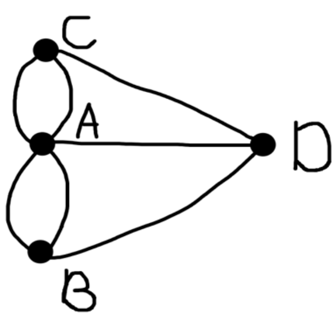
Section 1.2 Euler Paths and Circuits

Looking Back 1.2

In this section, we will investigate the famous Konigsberg Bridge Problem. In 1736, Leonard Euler solved the problem and a branch of discrete mathematics called Graph Theory was launched. Below is a description of the problem.

In the 18th century, citizens who took walks on Sundays wondered if they could traverse the seven bridges of Konigsberg over the Pregel River in a single trip without walking over the same bridge twice and ending in the same place they began.

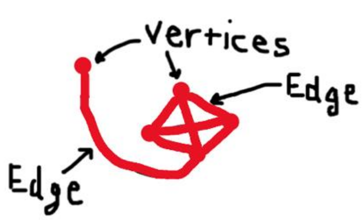




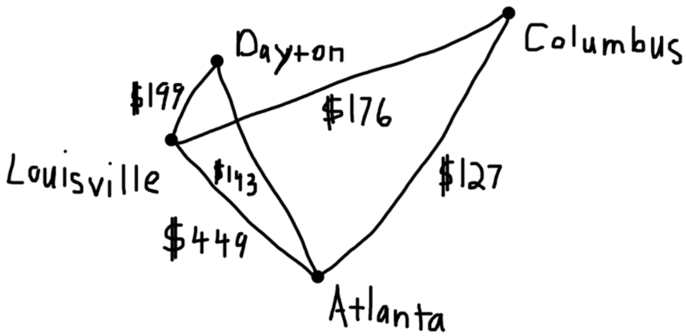
First, a multigraph must be drawn with four nodes and seven edges. A node is a point at which lines or pathways on the diagram intersect or branch on the graph. A graph is a set of points (vertices) and the edges and that connect them. The points that form a multigraph are A, B, C, and D, and they are central connecting points. See if you can start at D and trace every line with your pencil once and once only, not tracing over any line twice and ending back at D.

The citizens of the city of Konigsberg could not walk the path given the stipulations, just as you could not trace it. Leonard Euler, a notable Swiss mathematician, and author of 500 books, was intrigued by the problem when the mayor of Danzig asked him to solve it in 1736. Euler knew it could not be solved by using geometry, algebra, or counting principles. Instead, he began by using a sequence of letters to represent the region and bridge. What Euler ended up proving was that the problem could not be solved. The only solution could be one that began and ended at different places. This became known as an Euler path, which is a path that begins and ends at different vertices. Another term, Euler circuit, was coined, which was used if the path could begin and end at the same vertex.

Example 1:	How many vertices does the graph have? How many edges does the graph have?



Example 2: What does the weight of each edge of the graph below represent? What is the cheapest way to fly from Louisville to Atlanta?

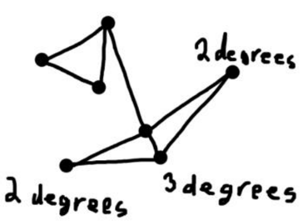


The number of edges that extend from a vertex determines the degree of the vertex. An Euler path uses every edge of a graph exactly once, but starts and ends at different vertices. An Euler circuit uses every edge of a graph exactly once as well, but starts and end at the same vertex.

Simple Graph

(Only one path to follow)

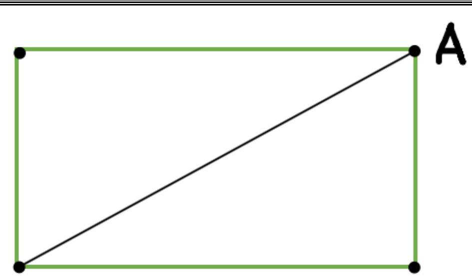
Multigraph



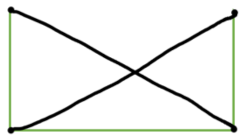
(Multiple paths to follow)

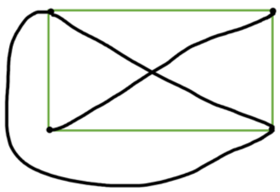
Every time a circuit passes through a vertex it doubles its degree. One degree is entering the vertex and the other degree is exiting the vertex. This makes it an even degree. The Konigsberg Bridge Problem has odd degrees at vertices and therefore, a circuit cannot exist.

Example 3: Does the diagram below show an Euler path or Euler circuit? Can you begin at A, cover every edge, and end at A without lifting your pen/pencil?

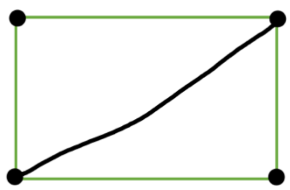


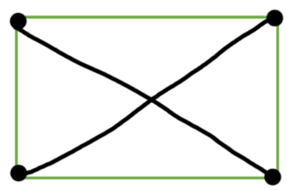
Example 4: Do the diagrams below show Euler paths, Euler circuits, or neither?

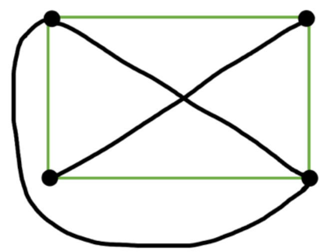




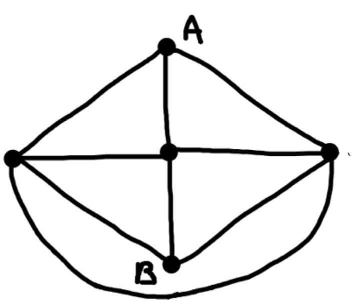
Example 5: What are the degrees of each node shown below? How do you know none are circuits?







Example 6: An Euler path can have at most two odd vertices. Does the diagram below show an Euler path, a Euler circuit, or neither?



Section 1.3 Hamiltonian Paths and Circuits

Looking Back 1.3

In the 19th century, Sir William Rowan Hamilton, a mathematician, and an imaginative genius, invented the Icosian game. The graph in Problem 10 of Section 9.1 with 20 vertices, each which represented a major city in Europe, is an example of this game. The object of the game is to visit each vertex once and once only.

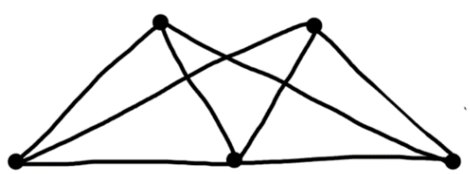
Sir William Rowan Hamilton also did extensive work with the telescope and laid the groundwork for quantum mechanics with his discovery of quaternions. Quaternions have four dimensions and consist of four scalar numbers, one real dimension, and three imaginary dimensions.

Moreover, the Hamiltonian path, which visits each vertex of a graph once and once only and ends at a vertex different from the one it begins at, was named in honor of Sir Hamilton. The Hamiltonian circuit was named after William Rowan Hamilton as well. A Hamiltonian circuit also visits each vertex of a graph once and once only, but it begins and ends at the same vertex.

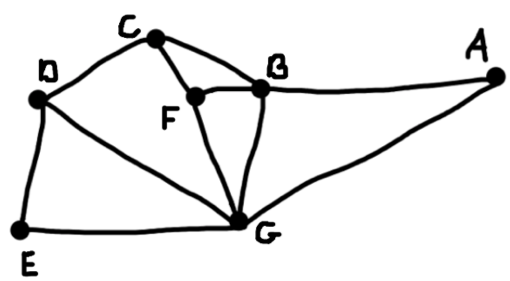
Looking Ahead 1.3

The Traveling Salesman Problem (which we reviewed in Section 1.1) represents an Euler path or circuit in terms of distances between cities (represented by the edges). The Traveling Salesman Problem also represents a Hamiltonian path or circuit, but in terms of cities visited (represented by the vertices).

Example 1: Is the diagram below a Hamiltonian path, circuit, or neither?



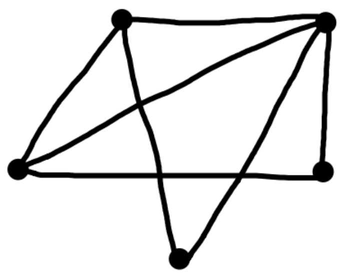
Example 2: In the diagram below, a mailman must deliver mail to each mailbox that is connected by the sidewalks. Is it possible for the mailman to do this? Can he start and return to his mail truck, which is parked at A, if he visits each mailbox once and once only?



Example 3: Although there is no general solution for Hamiltonian circuits, there exists a theorem that guarantees the existence of a Hamiltonian circuit:

If a connected graph has n vertices, where $n > 2$ and each vertex has a degree of at least $\frac{n}{2}$, then the graph is a Hamiltonian circuit.

Is the graph below is a Hamiltonian circuit?



Section 1.4 Graph Coloring and Planarity

Looking Back 1.4

There is a famous map coloring problem that asks how many colors are required to color a map of England so that no two adjacent regions are given the same color.

Let us try the problem with just one region of the United States. Using colored pencils, color the Western region of the map of the US (shown below) with the least possible number of colors so that no adjacent states have sides touching that are the same color. This excludes meeting at a single point.

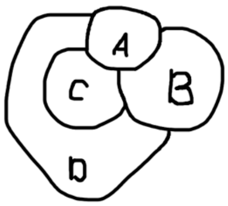
What is the minimum number of colors needed so that no two neighboring states share the same color?

This problem began in the mid-19th century when Francis Guthrie, a former student at University College London (who would go on to be the first professor of mathematics in Africa) was asked by his brother Frederick how many colors were needed to color a map of England so that every common boundary was a different color. Guthrie had studied under Augustus De Morgan, the first professor of mathematics at University College London, so he relayed the question to him in 1852.

In turn, De Morgan proposed the question to William Rowan Hamilton as this problem is a quaternion of color. However, Hamilton replied that he did not have the time to investigate it. Still, De Morgan continued looking for someone to solve the problem.

Finally, the problem was solved and when it became known that four colors worked for the problem, it was quickly conjectured that four colors would work for any map. A minimum of four colors was needed to color any map in the world so that regions sharing a common boundary do not share the same color.

Years earlier, in 1840, mathematician August Ferdinand Mobius posed a different but similar problem. Mobius posed this problem: If a king is on his death bed, how can he divide his land among his five sons so that each brother's land shares a border with each of the others?

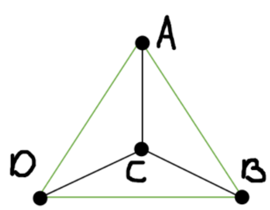


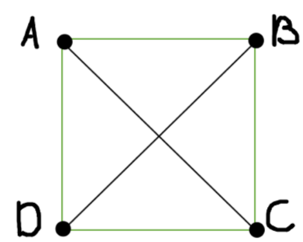
This problem was found to have no solution, as five neighboring regions cannot exist. At most, four neighboring regions can exist. (If you add a fifth region to the diagram it is impossible to neighbor it with C.)

August Ferdinand Mobius also invented the Mobius strip (also known as the Mobius band). Take a one-inch strip of paper and mark a line through the middle from one end to the other. Give the strip a half twist and tape the ends together. An insect could travel along the pencil mark and crawl the full length of the strip without ever crossing an edge. This concept, along with color mapping, often marks the beginning of topology, which is the study of geometric properties and spatial relations.

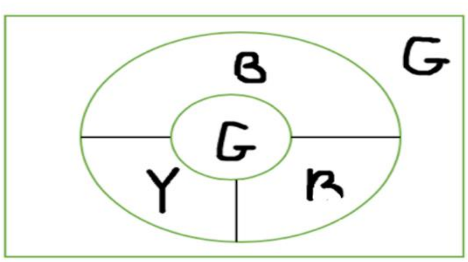
<u>Looking Ahead 1.4</u>

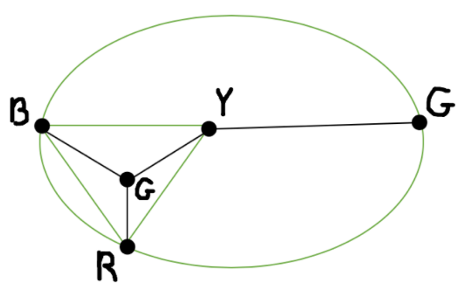
A map can be on a sphere or on a plane. A graph is planar if it can be drawn on a plane so that no edges cross. The graphs below are planar since both have four vertices connected by the same edges. (In the shape to the right, the center is not an intersection because there is no vertex.)



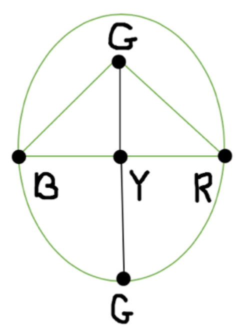


The first map coloring below can be drawn as a planar graph in multiple ways.





Or…



Over the years, the four-color conjecture became a problem and a puzzle. In 1879, Alfred Bray Kempe wrote a proof submitted to the American Journal of Mathematicians, which was later disproved by PJ Heawood, who worked for over 60 years on the four-color proof using Euler's characteristic or number, which will be investigated in the next section. In 1898 he proved that if the number of edges around a region is divisible by 3 then the regions require four colors. Wolfgang Haken (1954) and Kenneth Cybil (1976) used computers to confirm the four-color proof. After that, the Department of Mathematics at the University of Illinois used a postmark that read: FOUR COLORS SUFFICE. And Lewis Carroll, who wrote Alice in Wonderland, turned it into a game for two people. You will be playing a four-color game in the Practice Problems.

Section 1.5 Polyhedral Formula and Doodle Drawings

Looking Back 1.5

In the last section, we discussed Euler's polyhedron. The word "polyhedron" comes from the Greek word *poly*, which means "many," and the Indo-European word *hedron*, which means "seat." A polyhedron is a closed, solid shape that is three-dimensional. It has flat faces and straight edges. A cube is a polyhedron because it meets the requirements that make a shape a polyhedron. A cylinder is not a polyhedron because it has curved surfaces. If all the faces are of one shape and all the edges are the same length, and the angles between them are congruent, the shape is called a regular or platonic polyhedron. The plural form of polyhedron is polyhedra or polyhedrons.

Minerals are made of crystals that are built on atoms: protons, neutrons, and electrons. Crystals are three-dimensional and are formed by linking polyhedra into a chain or a framework. If you look at sugar crystals under a microscope you will see cubes. The formal name for these cubes is hexahedrons. Hexahedrons are the building blocks of the universe formed by our orderly Creator. This is another example of Intelligent Design. A hexahedron is one of five regular polyhedra. The five platonic solids are convex regular polyhedral. All the regular faces are identical and the regular figures at each vertex are identical.

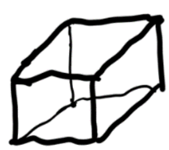
Looking Ahead 1.5

Example 1: Complete the table below for the first three columns of the five regular polyhedrons.

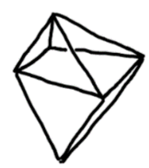
Polyhedron	**Vertices**	**Faces**	**Edges**	**Number of Polygons (*p*) at each Vertex**	**Number of Vertices (*n*) of each Polygon**
Tetrahedron					
Hexahedron					
Octahedron					
Dodecahedron					
Icosahedron					

Tetrahedron

Hexahedron



Octahedron



Dodecahedron

Icosahedron

Do you notice any relationship between the vertices and faces compared to the edges?

$$(V + F) - E = 2$$

Or

$$(V + F) - 2 = E$$

This holds true for any regular polyhedron.

Think about triangles. The interior angle of a regular triangle is $60°$ degrees. On a regular polyhedron, only 3, 4, or 5 triangles can meet at a vertex. The number of triangles that can meet at a vertex for three of the five polyhedrons is shown below:

Tetrahedron: 3 triangles
Octahedron: 4 triangles
Icosahedron: 5 triangles

If there are more than 6 triangles that meet at a vertex in a polyhedron the angles add up to more than $360°$. The ancient Greeks realized that any interior angle meeting at a vertex of a polyhedron must be less than $360°$. The number of shapes that meet at a vertex for the last two of the five polyhedrons is shown below:

Hexahedron: 3 squares
Dodecahedron: 3 pentagons

Any polygon larger than a hexagon has an interior angle of $120°$ degrees. These cannot form regular polyhedrons.

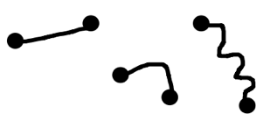
If we add two more columns to our table for the number of polygons (p) at each vertex and number of vertices (n) at each polygon, we gain two more formulas.

Dodecahedron: $e = \frac{nf}{2}$ $\quad e = \frac{5 \cdot 12}{2} = \frac{60}{2} = 30$

Icosahedron: $v = \frac{nf}{p}$ $\quad v = \frac{3 \cdot 20}{5} = \frac{60}{5} = 12$

In a planar graph, a link is an edge that leaves a vertex. A loop is a link that connects back to the original vertex.

<u>Link</u> <u>Loop</u>



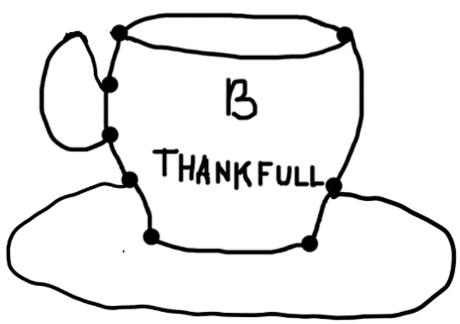
Drawings on a plane consist of vertices (or points) connected by links or loops (or edges).

Example 2: Count the vertices, faces, and edges in the two letters of the alphabet shown below. Does Euler's relationship for vertices, faces and edges in a polygon hold true for links and loops?

There seems to be one less edge than in Euler's formula. The relationship between the vertices, faces and edges for links and loops is $(V + F) - 1 = E$. Let us check this relationship using doodle drawings that consist of links and loops.

Example 3: Count the vertices, faces, and edges in the diagram shown below. Does $(V + F) - 1 = E$?

Example 4: Count the vertices, faces, and edges in the diagram shown below. Does $(V + F) - 1 = E$?



So $(V + F) - E = 2$ or $(V + F) - E = 1$ or $(V + F) - E = x$ where *x* is called the Euler characteristic.

Leonhard Euler was a brilliant man and perhaps the greatest mathematician during the Age of Reason. He contributed greatly to the field of mathematics, but he always asserted that the works of the Creator were infinitely greater than any reasoning, knowledge, or skill of man. Psalm 139: 17b-18a confirms his belief when discussing the thoughts of God, "How vast is the sum of them. If I should count them, they would outnumber the sand."

Leonhard Euler said, "Nothing takes place in the world whose meaning is not of some maximum or minimum." He always tried to live toward the maximum. One example of this is given by E.T. Bell in *Men of Mathematics* (147).

Denis Diderot, the chief editor of the *Encyclopédie* and a French philosopher, who was a skeptic of the Christian faith, came to Russia to teach his philosophies during the Age of Enlightenment. The Queen encouraged Euler to debate him and give his algebraic reasoning for the existence of God. Diderot came to the court and Euler stated, " $a + \frac{bn}{n} = x$, hence God exists," to which Diderot made no reply and returned to France.

Let us investigate further if Euler's reasoning and characteristic hold true for links and loops. Does the formula $(V + F) - 1 = E$ hold true for any doodle drawing? Try the Practice Problems and see what you discover.

Section 1.6 Spanning Trees

Looking Back 1.6

A spanning tree is a subset of the given graph it is connected to. It is a non-directed graph that connects all the vertices with the minimum number of edges. A spanning tree does not have cycles, nor can it be disconnected.

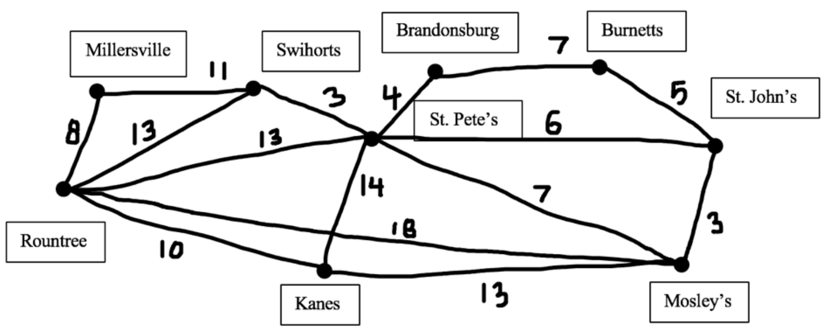
Sometimes, graphs are weighted and either a minimum spanning tree or a maximum spanning tree is the most expedient solution to a problem. Sometimes, spanning trees do not exist as is the case when the graph is not connected. This will all make more sense as we get deeper into the section.

Looking Ahead 1.6

One algorithm we can use to create a spanning tree is by starting at a vertex and coloring the edges that connect it to adjacent vertices. Then, from those vertices, we can connect to other adjacent vertices using another color. If there are several choices, randomly pick one. Continue this process until all the vertices are colored. If all the vertices are colored at the end of edges, then a spanning tree exists. Try the algorithm for the situation described and displayed below.

If a state experiences a natural disaster and needs to clear the roads so that travelers can get to and from each city, not all the roads in the state would be cleared. As time is crucial, only the roads that directly connect the big cities would be cleared.

Example 1: Use the algorithm described above to see if you can create a spanning tree from the diagram below. Start with yellow for Kanes, use blue for St. Pete’s, green for Swihorts/Brandonsburg, and red for Burnetts to St. John’s or from Mosley’s to St. John’s.

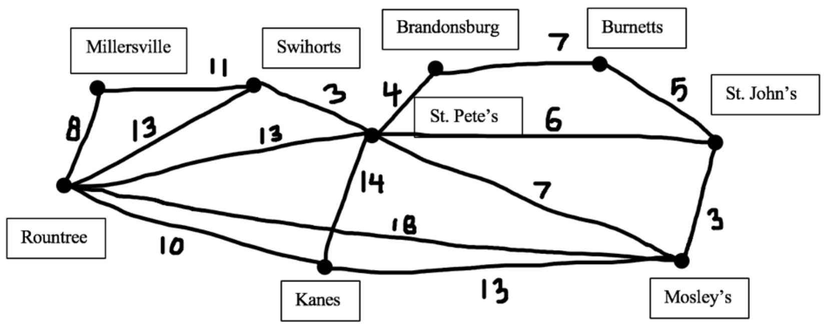


There is an algorithm for minimum spanning trees written by Joseph Kruskal, who studied mathematics at the University of Chicago. (Fun fact: Kruskal’s mother helped popularize origami in America in the 1950s!) Kruskal’s minimum spanning tree algorithm is shown as follows:

1. List the edges in order from least to greatest. If two are the same, they can be ordered randomly.
2. Color the first edge listed.
3. Go to the next edge listed. If it does not form a cycle with the colored edge, then color it.
4. Let n represents vertices. Continue Step 3 until $n - 1$ of the edges of the graph have been colored.

(Note: There will not be a minimum spanning tree if the graph is not connected.)

Example 2: Find the minimum spanning tree for the graph of cities affected by the natural disaster.



Section 1.7 Decision-Making

Looking Back 1.7

People in the business world organize tasks in an efficient manner so they are completed on time. Computer programs follow step by step algorithms to solve problems. Teams use flow charts to plan projects so every part of an extension job is completed along the way so the next step can be completed. Many tasks, like these, require prerequisite tasks to make the goal achievable.

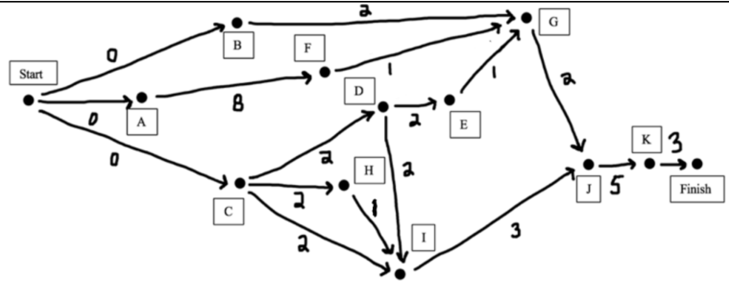
If a company is producing a campus guide for a university, each activity must be completed in order to get the guide ready for the school year on time. The table below shows the tasks needed in the order of completion and the duration of time for each activity in days. It also shows the prerequisite activity needed in order for the main activity to be performed.

Activity	**Duration**	**Prerequisite Activity**
A. Department Interviews	8	None
B. President Introduction	2	None
C. Campus and Department Photos	2	None
D. Campus Map	2	C
E. History of Community	1	D
F. Campus Statistics	1	A
G. Article Edits	2	B, E, F
H. Cover Artwork	1	C
I. Interior Artwork	3	C, D, H
J. Layout of Design	5	G, I
K. Print Guides	3	J

The Department Interviews, President Introduction, and Campus and Department Photos are all tasks that can be done individually as they have no prerequisites. However, they are prerequisites for other activities so they must be performed before those activities that rely on them. Photos are needed to make a community and campus sketch. The sketch itself may include historic buildings. These historic buildings require an explanation in the guide. All the activities described must be edited and error-free before the cover is designed. The cover design will not be included in the interior unless it is the map as well so the cover must be done before the interior artwork is designed. Finally, once the editing and artwork is done, the layout can be finished (though it does take some time). At last, when the guide is completed, it must be printed.

Looking Ahead 1.7

Example 1: If activities are represented by the vertices and prerequisites are represented by the edges, draw a diagram that shows the relationship among the tasks. Let each edge show the number of days it takes to complete an activity. Draw activities with the same prerequisite(s) parallel to one another in the directed graph. Draw the directed graph for the campus guide project.



The points B and C are not connected to A but are beside it as they start within the 8-day time frame.

If the graph were simple it would be easy to determine the shortest time to make the campus guides and when they would be delivered. This graph is complex, and the path of targeted tasks needed to finish the project in the shortest time is called the critical path.

First, calculate the earliest start time (EST) for each task so that the activity's prerequisite can begin as soon as possible. Label each vertex with the shortest possible time needed before each task. Put these in parenthesis.

Example 2: Label each vertex in the diagram above with the minimum number of days needed before the task can begin.

Example 3: Task D is not on the critical path. G is on the critical path and must be started by day 9. What day must task D begin? This is called the latest start time (LST).

Section 1.8 Fair Division

Looking Back 1.8

There are many situations in life in which we are to divide items fairly among a number of recipients. For example, an estate might be left to be divided among a family of heirs or a group of friends might order a pizza to split.

In order to solve problems such as these, we can use fair division. Fair division is often looked at as a game, with rules and players. Fair division methods ensure that each player in the game will receive what they consider a "fair share." A fair division game has a set of N players ($P1$, $P2$, … PN) and set of goods, S. The goods, S, are divided into N shares ($S1$, $S2$, … SN) so that each player gets a fair share of S. A fair share, in the opinion of the player receiving it, is worth $\frac{1}{N}$ of the total value of S. Any player is capable of deciding whether his/her share is fair. The player can assign sure values to S and any part of S.

A fair division game assumes the parties are non-cooperative and non-communicative. No party has knowledge of the likes and dislikes of the other parties. Fair division must occur without the intervention of an outside party. The game also assumes that the players are fair and do not manipulate the situation or take advantage of the game.

Different fair division games can be used in different situations. The example of an estate being divided among heirs would require a discrete fair division game. A discrete fair division game includes items that cannot be divided, such as a house or a chair. The example of a pizza being split among friends would require a continuous fair division game. A continuous fair division game includes items that can be divided, such as pizza or cake.

In a continuous fair division game, when players are dividing an item, they will use the divider-chooser method. In the divider-chooser method, the divider splits up the item and the chooser selects the piece he/she would like to receive. For example, suppose a set of triplets are sharing a cake. If Ian cuts the cake into three equal pieces (in his mind) and gives each sibling a piece (including himself), he is the divider and the chooser. If Duncan cuts the cake into thirds and Caelan picks a piece and then Ian picks a piece, Duncan is the divider and Caelan and Ian are the choosers. If Caelan cuts the cake into thirds and Duncan chooses who gets what piece, then Caelan is the divider and Duncan is the chooser.

Looking Ahead 1.8

Example 1: Suppose a cake is half chocolate and half vanilla. Gracie likes all chocolate and Taryn likes both chocolate and vanilla. Taryn thinks the half-chocolate, half-vanilla cake is worth \$10. Gracie values the cake at \$4, since the vanilla half is worthless to her. Taryn cuts the cake in half so on one half, $\frac{1}{4}$ of the cake is chocolate and $\frac{3}{4}$ is vanilla, and on the other half, $\frac{3}{4}$ is chocolate and $\frac{1}{4}$ is vanilla. Remember, in a fair division game she does not know the likes and dislikes of the other players. If Gracie and Taryn both get a half of the cake, do they receive their fair share?

Example 2: Sara and David split a bag of candy at a party. There are 10 Hershey bars, 10 Snickers bars, and 10 Reese's cups. David values the 10 Hershey® bars at \$3.00 (\$0.30 each), the 10 Snickers® bars at \$2.00 (\$0.20 each), and the 10 Reese's® cups at \$1.00 (\$0.10 each).

Sara divides the pile in half. One half has 4 Hershey® bars, 4 Snickers® bars, and 7 Reese's® cups. If she gives this pile to David, is it a fair share given what he has valued the candy at?

To divide an estate fairly, there is an algorithm such that each heir receives a sum that is considered larger than their fair share. The algorithm is described below.

1. Each heir submits a closed bid on each estate item. (Money is not an item; it will be divided equally among the heirs.)
2. The fair share is determined for each heir by finding the sum of all his/her bids and dividing the sum by the number of heirs.
3. The heir who bids the highest on the item receives it.
4. The cash received by each heir for the estate is the fair share (from Step 2) less the amount of bids for items received. If this number is negative, it is deducted from the cash. If there is no cash, then the heir pays the estate and it is divided among the other heirs.
5. The remaining cash is divided equally among all the heirs.

Example 3: Kim and Raphine are heirs to an estate that consists of a cabin, a boat, and a motorcycle. Their bids are summarized in the matrix below. Who received each item and how much money did each receive if $50,000 cash was also part of the estate?

	Cabin	**Boat**	**Motorcycle**
Kim	$30,000	$5,000	$2,000
Raphine	$25,000	$7,000	$3,000

There is not time to go over every fair division method in one section, but if you are interested in learning more about them there are plenty of resources online. If you are particularly interested in politics and legal voting methods, then you might want to research the Appointment Algorithm! We will investigate a few of these in the final section of today’s Practice Problems.

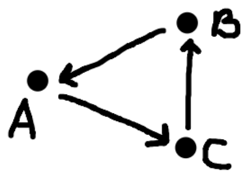
Section 1.9 Recursive Thinking

Looking Back 1.9

Recursive thinking is used to define patterns that occur when processes are repeated. Mathematical induction is also involved with recursive thinking as it generalizes patterns of solutions by extending the group to one bigger than the previous number.

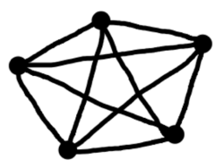
If a cake can be divided fairly among N people so that each gets $\frac{1}{N}$, then this can be extended to show that a cake can be divided fairly among $N+1$ people so each gets $\frac{1}{N+1}$ of the entire cake.

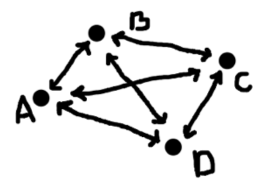
If you are in a room with two strangers, how many handshakes will occur until each of you have met? The illustrations below show how to answer this question using a graph (to the left) and a table (to the right).



Number of People	Number of Handshakes
1	0
2	1
3	3
4	6
5	10

What if you are in a room with four strangers? Five strangers?





When a new person enters the room, they must shake hands with each of the people already in there. Thus, the number of total handshakes for n people is the number of handshakes for $n-1$ people and the number of people previously in the room, $H_n = H_{n-1} + (n-1)$. This is a recursive equation; it relates each solution to the previous one for the number of handshakes. The explicit equation is $H_n = \frac{n(n-1)}{2}$, which gives us the direct answer for any number of people.

If the formula above is for n people in a group, the formula for one more person (or $n+1$ people) in a group is as follows: $H_{n+1} = \frac{n+1(n+1-1)}{2}$, which simplifies to $H_{n+1} = \frac{(n+1)n}{2}$.

If the original formula was $H_n = \frac{n(n-1)}{2}$ for n people, then adding one more person requires n more handshakes or $H_{n+1} = \frac{n(n-1)}{2} + n$, which can be simplified to $H_{n+1} = \frac{n(n+1)}{2}$. This is what was previously discovered, it is just a different method.

Looking Ahead 1.9

Example 1: Use the finite difference method to find the explicit formula for the table below.

Number of People	Number of Hands	First Difference	Second Difference
1	0	-	-
2	1	1	-
3	3	2	1
4	6	3	1
5	10	4	1

Value of n	**Value of Polynomial**	**First Difference**	**Second Difference**
1	$a + b + c$	-	-
2	$4a + 2b + c$	$3a + b$	-
3	$9a + 3b + c$	$5a + b$	$2a$
4	$16a + 4b + c$	$7a + b$	$2a$
5	$25a + 5b + c$	$9a + b$	$2a$

Example 2: Another method for finding the explicit formula is to use matrices to solve for a system of equations. You learned how to do this for three equations in Algebra 2 and then used the graphing calculator to confirm it. Use this method to find the explicit formula.

There are recursive patterns found throughout nature again and again because the God who created this Universe is an intelligent designer and a God of order. We will investigate these patterns in circles and star polygons in today's Practice Problems.

Section 1.10 Pascal's Triangle

Looking Back 1.10

Blaise Pascal was a mathematician in France in the 1600s. Not only did he make great achievements in his field, including discovering Pascal's Triangle (which we will learn about in this section) and laying the foundation for Probability Theory with Pierre de Fermat, but he was also a devout believer. He published a theological work, Les Provinciales, in 1657 to define his faith, and his Pensées (a thought or reflection in literary form) was first published in 1670.

Pascal's most famous discovery is Pascal's Triangle. In this section, we will investigate the various patterns that emerge from this shape.

Row
0
1
2
3
4
5
6
7

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

As you can see, each number in the triangle comes from the sum of the number above it to the left and the number above it to the right. Complete the bottom row of numbers for Pascal's Triangle.

Looking Ahead 1.10

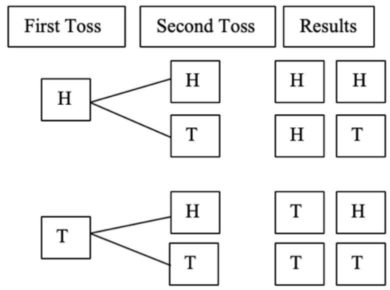
Example 1: Write the sequence for the sum of each of the rows of Pascal's Triangle. What is an expression for the sum of the rows if r is the row number? What would be the sum of each of the next three rows? What would be the sum of the 20^{th} row?

Example 2: Look at the diagonal along the left or right. What is $1 + 1$, then $1 + 1 + 1$, then $1 + 1 + 1 + 1$? Now look at the next diagonal to the left. What is $1 + 2$, then $1 + 2 + 3$, and $1 + 2 + 3 + 4$? Find the expression for the same diagonal on the right?

Find the sums along the slanted diagonals from left to right through the triangle.

What is the name of this sequence?

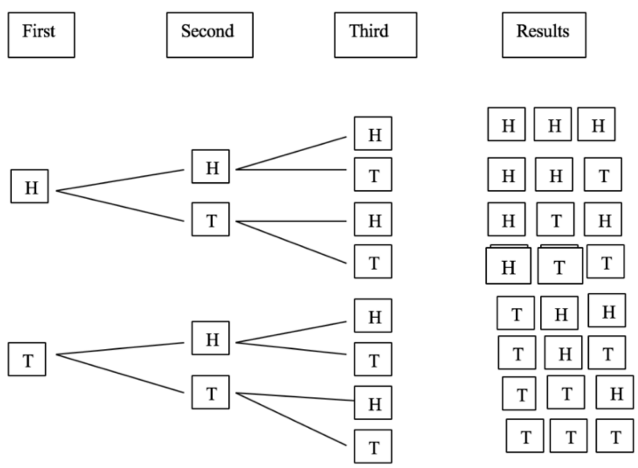
Example 3:	Use Pascal's Triangle to investigate coin tosses.



If a coin is tossed twice, how many ways can each of the following tosses occur?

2 heads: 1 head: 0 heads:

How does the number of occurrences relate to Pascal's Triangle?



If a coin is tossed three times, how many ways can each of the following tosses occur?

3 heads: 2 heads: 1 head: 0 heads:

How does the number of occurrences relate to Pascal's Triangle?

If a coin is tossed 4 times, complete the ways to get 4, 3, 2, 1, or 0 heads.

4 heads: 3 heads: 2 heads: 1 head: 0 heads:

How did you do it? Did you use Pascal's Triangle?

Example 4:	Expand the given binomial. Where do the coefficients come from? What patterns do you notice?

$$(x+y)^0 = 1$$

$$(x+y)^1$$

$$(x+y)^2$$

$$(x+y)^3$$

$$(x+y)^4$$

Section 1.11 Counting Techniques

Looking Back 1.11

In Pre-Algebra, you were introduced to the Fundamental Principle of Counting (or Fundamental Counting Principle). If one event can occur in m ways, and another event can occur in n ways, then both events can occur in $m \cdot n$ ways. This is also called the Multiplication Rule of Probability.

Example 1: If there are three events that occur in m, n, and p ways, respectively, then all three events can occur in $m \cdot n \cdot p$ ways.

In the United States, how many three-digit area codes are possible for a given region? (Numbers can be re-used; however, 0 or 1 may not be used for the first digit as they are used for other purposes.)

How many local phone numbers of 7 digits are possible if a 0 or 1 cannot be used for the first digit?

Looking Ahead 1.11

There is another important principle of counting called the "addition principle." The addition principle states that "if events M and N can occur in m and n ways, respectively, then either event M or N can occur in $m + n$ ways."

Example 2: The Boy Scouts and Girl Scouts of America have nineteen national council members. Ten of the members are young men and nine of the members are young women. The office of president is to be held by one member, either a young man or young woman. If only one person is to be selected to hold office as president, how many ways are there to make the selection?

Example 3:	How many ways can the letters of the word HEAR be arranged?

Example 4:	Given the word HEART, how many arrangements can be made with only three letters?

Example 5:	How many ways can the word OHIO be arranged?

Section 1.12 Probability and Combinatorics

Looking Back 1.12

In this module, we have been investigating permutations. A permutation is a counting technique in which the order of events matters. A combination is another counting technique, but with it the order of events does not matter.

Let's look at the word H-E-A-R-T again.

Example 1: How many ways can three letters be chosen from HEART if order is not important?

Looking Ahead 1.12

Following is a list of possible counting techniques:

1. Lists
2. The multiplication principle
3. The addition principle
4. Tree diagrams
5. Permutations
6. Combinations

Example 2: If four cards from a deck of 52 are randomly dealt, what is the probability that the four cards will be honor cards of the same suit?

Example 3: Let's revisit Pascal's Triangle to find binomial coefficients again. Using the information below, find the coefficient of the third monomial term in the binomial expansion (series) for $(x+y)^{10}$.

								1									Row 0
							1		1								Row 1
						1		2		1							Row 2
					1		3		3		1						Row 3
				1		4		6		4		1					Row 4
			1		5		10		10		5		1				Row 5
		1		6		15		20		15		6		1			Row 6
	1		7		21		35		35		21		7		1		Row 7
1		8		28		56		70		56		28		8		1	Row 8

Row 0: $(x+y)^0 = \mathbf{1}$

Row 1: $(x+y)^1 = \mathbf{1x + 1y}$

Row 2: $(x+y)^2 = \mathbf{1x^2 + 2xy + 1y^2}$

Row 9: 1 9 36 84 126 126 84 36 9 1

Row 10: 1 10 45 120 210 252 210 120 45 10 1

Example 4:	Simplify the binomial coefficient formula when $m \neq n$ and m is not zero.

Example 5:	Use Pascal's Triangle to find the following eight coefficients for the expansion $(x+y)^7$.

$$\binom{7}{0}\binom{7}{1}\binom{7}{2}\binom{7}{3}\binom{7}{4}\binom{7}{5}\binom{7}{6}\binom{7}{7}$$

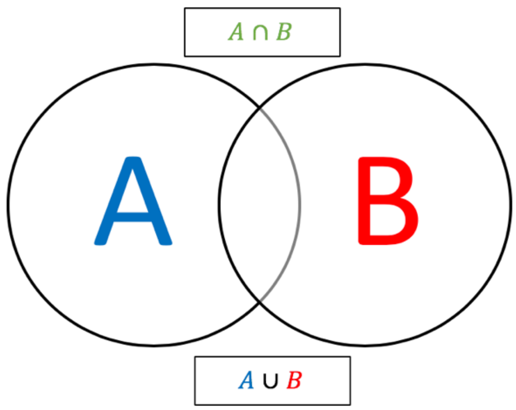
Section 1.13 Dependent and Independent Events

Looking Back 1.13

The union of two events A and B is written $A \cup B$. These are all the outcomes that are in A or in B or in both A and B.

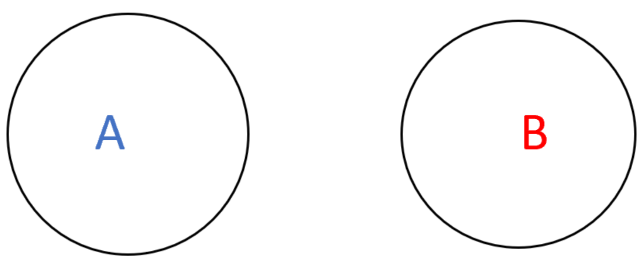
The intersection of two events A and B is written $A \cap B$. These are all the outcomes that are in A and B.

If A and B are in the same sample space, then the probability of A and B occurring is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



This section investigates:
Union/Intersection
And/or
Dependent Events/Independent Events

If $A \cap B$ is an empty set, then $P(A \cup B) = P(A) + P(B)$ and A and B are called mutually exclusive events.

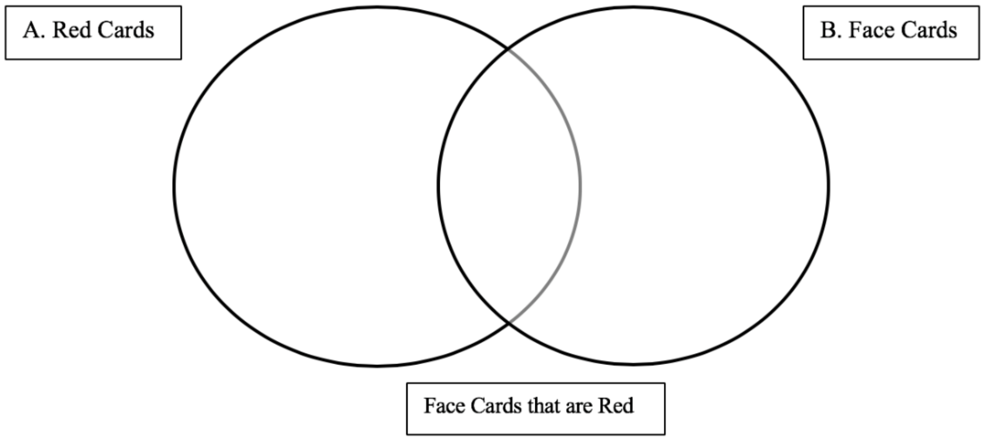


Mutually Exclusive Events

Independent Events

Looking Ahead 1.13

Example 1: What is the probability that a card drawn from a deck of 52 will be a red card or a face card?



Example 2: On a student council, three students are running for the office of president. If the probability that Candidate A will win is 0.3 and the probability that Candidate B will win is 0.6, what is the probability that Candidate C will win?

Example 3: In 2020, a factory had employees work overtime or hired temporary workers for 20 weeks to sterilize face masks for the COVID19 worldwide virus. Employees worked overtime for 12 of those weeks and temporary workers were hired for 10 of those weeks. If the factory foreman randomly reviews a safety check for one of those weeks at the end of the year, what is the probability that both employees worked overtime and temporary workers were hired that week?

1) What are you being asked to find in terms of probability?
2) Substitute the given information into the formula.
3) Solve for the quantity you are being asked to find.

Example 4: If two events occur and one has no effect on the other, then the two events are independent. If one event occurs and it does affect the other, then the two events are dependent.

If A and B are independent events, then the probability that both events occur is $P(A \cap B) = P(A) \cdot P(B)$.

If you roll a die twice, what is the probability that both rolls land on even numbers?