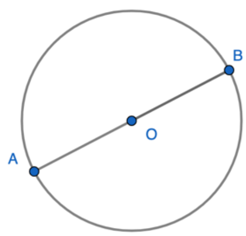
Geometry and Trigonometry Module 5 Circles

Section 5.1 Circumference and Diameter of Circles

Looking Back 5.1

In Module 4, you investigated perimeter, which is the distance around a polygon. A polygon is a many-sided shape. A circle, on the other hand, has no sides; it is circular in shape. The distance around a circle is called the circumference.

The distance from the center of a circle to any point on the circumference of the circle is the radius. The diameter extends from one point on the circle to the point on the opposite side through the center of the circle.



The circle to the left is named by its center: Circle O or ⊙O. The radii (plural for radius) are $\overline{AO}$ and $\overline{BO}$ (or $\overline{OA}$ and $\overline{OB}$). The diameter of the circle is $\overline{AB}$ or $\overline{BA}$. There are infinitely many diameters in a circle.

In a previous course, we tied string around tin cans of various sizes and then investigated the ratio of the circumference of the can to the diameter of the can. In each case, the ratio $\frac{C}{d}$ was approximately 3 or 3.14. The formula $C = \pi d$ can also be written as $C = 2\pi r$, since two radii are equal to one diameter ($d = 2r$) in a circle.

The invention of the wheel (a circular shape) can be traced back to the Egyptians. Even in ancient times, they understood that for a wheel to roll, the circumference of the wheel must be approximately 3.14 times the length of the diameter.

Example 1:	A bicycle tire has a circumference of 6 feet. What is the length of a spoke of the wheel?

Example 2: A tree adds a tree ring for each year of its growth. The average growth of a ring for a tree is 0.6 centimeters and it has a circumference of 325 centimeters. Approximately how old is this tree?

Looking Ahead 5.1

In this course, we will be using tin cans again to investigate the unit circle and radian measure. You will be introduced to radians in this module since we are investigating circles but will not try practice problems involving radians until later in this course.

We also previously used pie-shaped pieces (or sectors of a circle) to determine the formula for area of a circle. The sectors, when placed right-side up and upside down, approximated a parallelogram. The base of the parallelogram was one-half the circumference, and the height was the radius of the circle. We can simplify the formula using substitution:

$$\mathrm{A} = b \cdot h$$

$$\mathrm{A} = \frac{1}{2}\mathrm{C} \cdot r$$

$$\mathrm{A} = \frac{1}{2}(2\pi r) \cdot r$$

$$\mathrm{A} = \pi r^2$$

Example 3: A semicircle is one-half of a circle. Find the area of the shaded region of the circle (semicircle) if the diameter is 8 centimeters.

The area is measured is in square units or in units squared. In Example 3, the radius squared is $(4\text{ cm.})\cdot(4\text{ cm.})$, which is 16 cm.^2 or 16 sq. cm.

Example 4: The curves at each end of a track are two semicircles. The radius is the distance from the soccer goal to the outer rim of the track and is approximately 44 m. The distance between the soccer goals is 84 m. What is the total distance around the track if you run in the outside lane?

Section 5.2 Parts of a Circle

Looking Back 5.2

We know that parallel lines are coplanar lines that do not intersect. All the points on parallel lines are equidistant from one another.

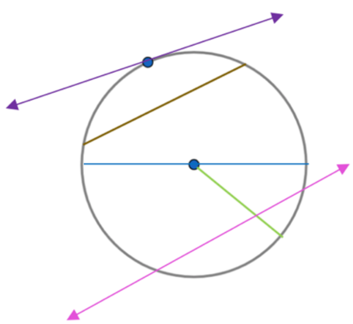
The definition of a circle is similar, but it is a set of points that are equidistant from a given point.

Any segment whose endpoints are on a circle is called a chord. A diameter is a special chord that passes through the center of a circle.

A line segment that connects the center of a circle to any point on the circle is a radius.

A secant line intersects a circle in two points. A tangent line intersects the circle in exactly one point.

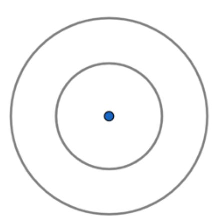
Example 1: Given the definitions above, label the parts of the circle: chord, radius, diameter, secant, and tangent.



The symbol for circle A is ⨀ A. The one point on the circle where the tangent line touches it is called the point of tangency.

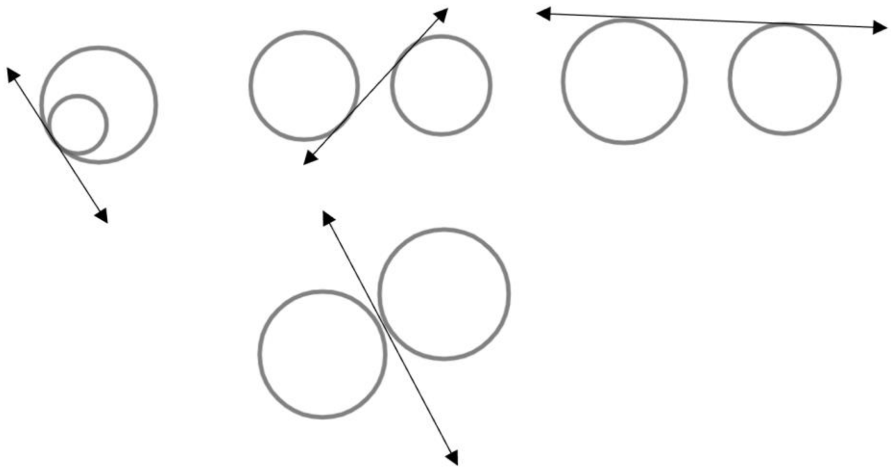
Two circles can intersect in no points, one point, or two points.

Coplanar circles that do not intersect but have a common center are called concentric circles.



A common internal tangent intersects the line segment that is between the center of two circles. A common external tangent does not intersect the segment that is between two circles.

Example 2: Tell if the tangent is a common internal tangent or common external tangent.



Example 3: Use your compass to draw a circle. Draw a line that appears tangent to the circle. Connect the radius from the point of tangency to the center of the circle. Measure the angles where the radius meets the tangent line. Do this for several tangent lines around the circle. What conjecture can you make about the lines that are the radius and the tangent to it?

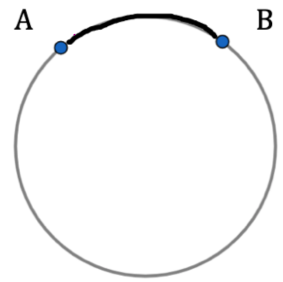
Example 4: Use a compass to draw a circle. Place a point on the paper outside the circle. Draw two rays extending from the point tangent to the circle. Measure each line from the external point to the point of tangency. What conjecture can you make about the length of the line segments?

Section 5.3 Arcs and Arc Measures

Looking Back 5.3

The arc of a circle is on the circumference (C) of the circle. It is a continuous unbroken part of the circle between two points called endpoints.

You can write an arc two ways using either words or symbols.

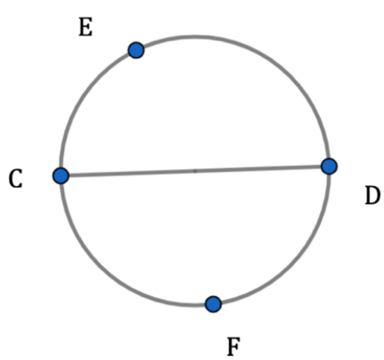


The circumference (C) is the distance around the circle. It is $C = \pi\, d$ or $C = 2\pi\, r$, where d is the diameter and r is the radius of the circle. You have reviewed this in the previous section.

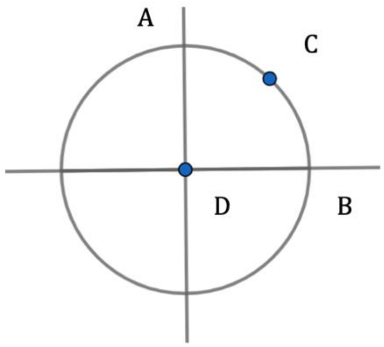
Looking Ahead 5.3

Arcs are one of three types: minor arcs, major arcs, or semicircles. A semicircle has endpoints whose endpoints are the same as those of the diameter. A major arc is larger than a semicircle and a semicircle is larger than a minor arc.

Example 1:	Using the given circle, name two semicircles, two major arcs, and two minor arcs.



Example 2: The central angle has a vertex at the center of the circle. What conjecture can you make about the central angle and the minor arc of the circle created by its endpoints?

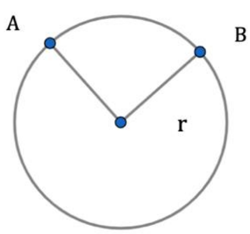


An intercepted arc is created when segments intersect parts of the circle. Two radii of a central angle intersect the circumference of the circle and create an intercepted arc. The measure of the intercepted arc is the same as the measure of its central angle and is calculated in units of degree.

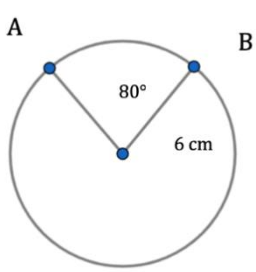
Arc length is not the same as the arc measure. The arc length depends on the size of the circle. The arc length is some fraction of the circumference of the circle and is calculated in units of distance.

A portion of the circumference of the circle is the arc length. The ratio of the length of a given arc to the circumference of a circle is equal to the ratio of the arc length to $360°$ degrees.

The proportion below allows you to calculate the arc length in linear units.



Example 3: Find the arc length of AB in the given circle.



Section 5.4 Areas of Circle Sectors

Looking Back 5.4

In a previous course, you learned that the area of a circle is $A = \pi r^2$; in this course, you reviewed this formula in Section 5.1. Now, we will be using our review to find areas of circle sectors.

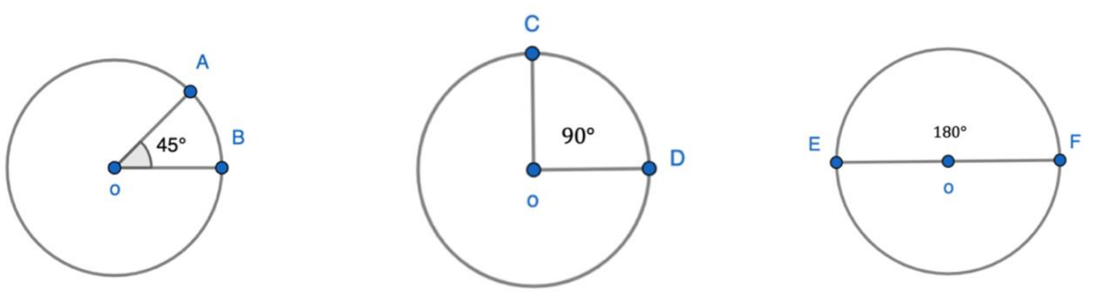
A slice of pizza represents a circle sector.

Circle Sector

The sector is the region between two radii and the arc of the circle between them.

A semicircle is a one-half circle sector. Half of a semicircle is a one-fourth circle sector.

Example 1: Show that the ratio of the area of the circle sector to the whole circle is equal to the ratio of the measure of the intercepted arc to $360°$ degrees. Let the radius be 2 inches for all three circles. Find the area of each circle sector.

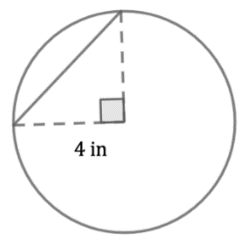


To find the area of a circle sector, use the formula for area of a circle and multiply it by the portion of the circle that is the sector. Let a be the arc measure of the circle in degrees, then multiply $\frac{a}{360°}$ by the area of a circle (πr^2) to get the area of the sector.

Example 2:	Find the area of a circle sector that has an arc measure of 45° and a radius of 3 inches.

Looking Ahead 5.4

To find areas of regions of a circle beyond a chord, find the area of the sector and subtract out the area of a triangle formed by the center of the circle and the endpoints of the chord. This area is called a segment of the circle.



The $A_{\text{sector}} = \frac{90}{360} \cdot \pi(4)^2$

$$= \frac{16}{4} \cdot \pi$$

$= 4\pi$ square inches (12.56)

The $A_{\text{triangle}} = \frac{1}{2} b \cdot h$

$$= \frac{1}{2} 4 \cdot 4$$

$$= \frac{16}{2}$$

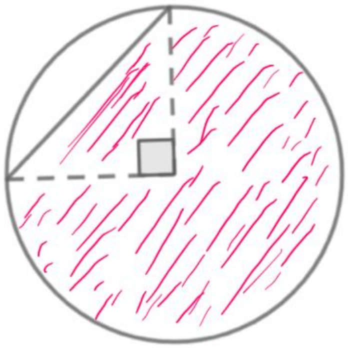
$= 8$ square inches

$$A_{\text{chord (segment)}} = A_{\text{sector}} - A_{\text{triangle}}$$

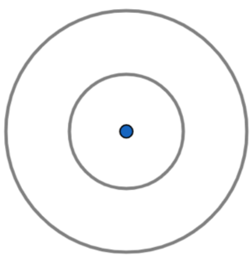
$$= 4\pi - 8$$

≈ 4.56 square inches

Example 3: Find the area of the non-shaded region given the radius of the circle is 2 inches.



The annulus of a circle is the region between two concentric circles. To find the area of the annulus, find the area of the larger circle and subtract from it the area of the smaller circle.

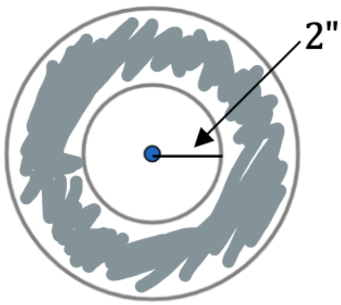


R = radius of larger circle

r = radius of smaller circle

$A_{\text{annulus}} = \pi R^2 - \pi r^2$

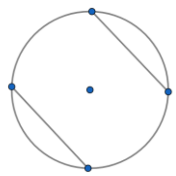
Example 4: Find the area of the shaded region in the circle below. (Suppose the width of the shaded region is 3 inches.)



Section 5.5 Angles Inscribed in a Circle

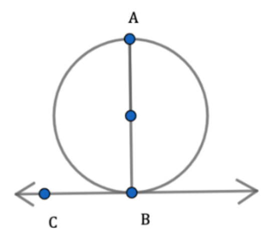
Looking Back 5.5

Central angles in a circle have their vertex at the center of the circle. If you connect each end of the two congruent chords to the center of the circle, what conjectures can you make about the intercepted arc lengths and the central angles?



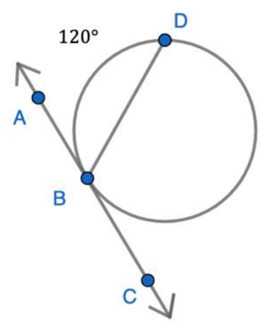
If you measure the two central angles, you will see they are congruent. Since the intercepted arcs are equal to the central angles, then the intercepted arcs are also congruent.

A line through a point tangent to the circle is perpendicular to the radius (and diameter, which runs through the radius). The angle formed is 90°. How does this angle compare to the semicircle?



The semicircle is 180°. The right angle of 90° at the point of tangency is one-half of the semicircle. This holds true for any tangent and chord that intersect at a point on the circle. The measure of each of the two angles formed will be one-half the measure of the intercepted arc.

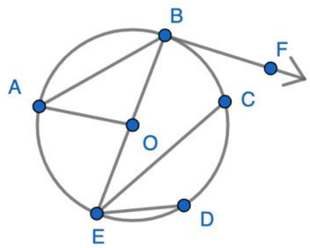
Example 1:	Find the measures of ∠ABD and ∠DBC if $\overleftrightarrow{AB}$ is tangent to the circle.



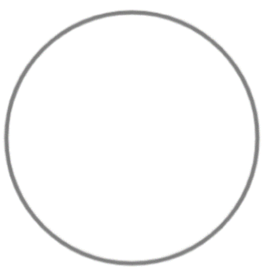
Looking Ahead 5.5

An inscribed angle has its vertex on the circle. The sides of an inscribed angle are chords of the circle.

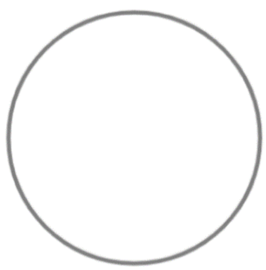
Example 2:	Determine which angles of circle O are inscribed and which angles are not inscribed.



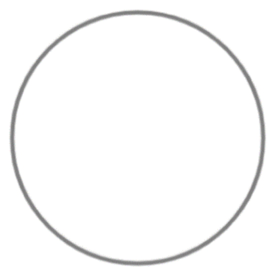
If two non-parallel lines intersect a circle, there are three places the lines can intersect. Show all three cases in the following circles:



Outside the circle

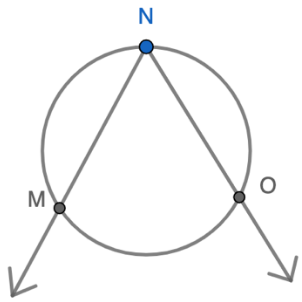


On the circle



Inside the circle

An arc that is between two segments, rays, or lines is called an intercepted arc (as has been previously stated for chords). If the endpoints of a chord or arc lie on the sides of the inscribed angle, then the chord or arc subtends the angle.

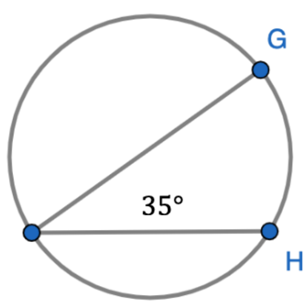


Arc OM is the intercepted arc of ∠N

Arc OM subtends ∠N

If two chords intersect at one point on the circle, then they form an inscribed angle. The inscribed angle is one-half of the intercepted arc.

Example 3:	Find the measure of arc GH.



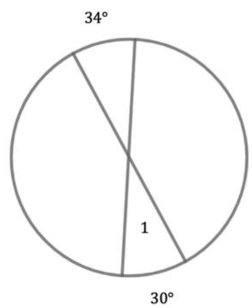
Section 5.6 Circumscribed Angles of a Circle

Looking Back 5.6

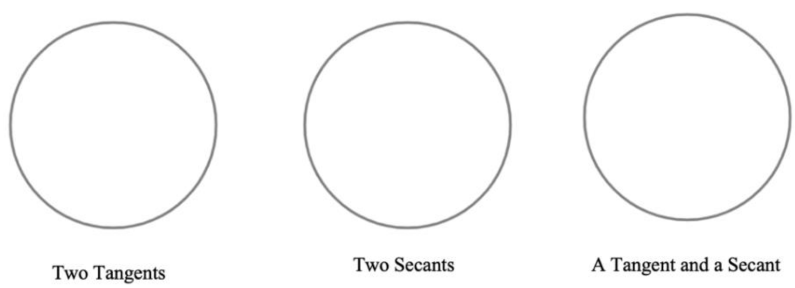
In the last section, we investigated inscribed angles of a circle. When chords intersect on a circle, they form an inscribed angle.

Two chords can also intersect inside a circle. When this happens, the measure of each angle formed is one-half the sum of the measures of the arcs intercepted by its angle and its vertical angle.

Example 1:	Find the measure of ∠1.

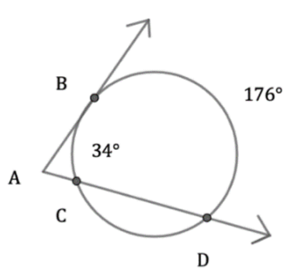


If angles lie outside a circle, then the two lines that intersect are two tangents, two secants, or a tangent and a secant (only the rays where the angles meet are needed here for all three cases). Show all three cases using the following circles:



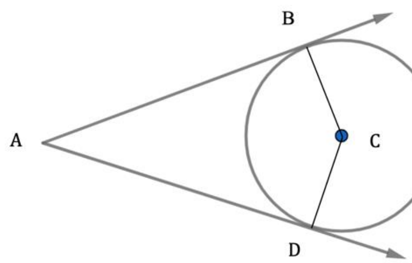
The measure of the angle formed outside the circle is one-half the difference of the measures of the intercepted arcs.

Example 2: The tangent $\overrightarrow{AB}$ and the secant $\overrightarrow{CD}$ intersect outside the circle, forming an angle. Find the measure of ∠DAB.



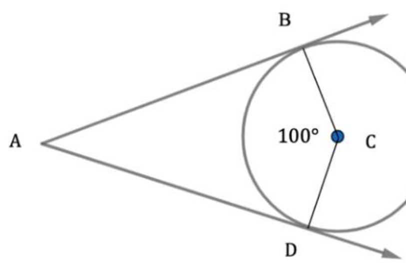
Looking Ahead 5.6

An angle is circumscribed when the circle is inside of it. That means the rays meet outside the circle and touch the circle at one point. Therefore, the sides of a circumscribed angle are tangent to a circle.

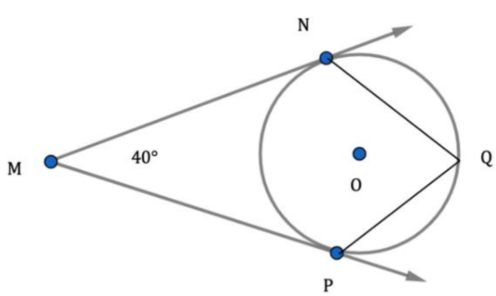


We have learned that the radii are perpendicular to the line that is tangent to a circle at the point of tangency and $\overline{AB}$ is congruent to $\overline{AD}$. The measure of the circumscribed angle is equal to 180° minus the measure of the central angle (or its intercepted arc).

Example 3:	Find the measure of ∠DAB.



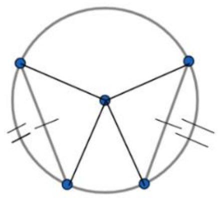
Example 4:	Find the measure of ∠NQP. Explain your reasons for the steps you take to find it.



Section 5.7 Chords and Chord Theorems

Looking Back 5.7

In earlier sections, we have stated that a chord is a segment whose endpoints are on a circle. We have investigated chords and central angles. There are several theorems in this section that come from the results of our investigations:



If two chords in a circle are congruent, then the central angles formed from the vertex at the center of the circle to the endpoints of the chords are also congruent. The intercepted arcs of the chords and their central angles are congruent as well.

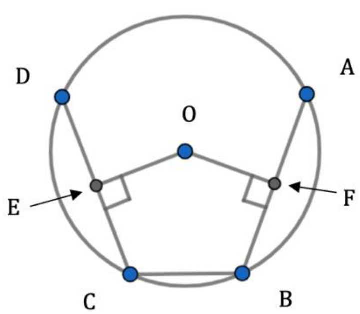
Looking Ahead 5.7

Draw a perpendicular line from the center of the circle (shown above to the left) to each chord. If you measure the length of the segment on each side of the perpendicular line, you will see they are equal. The conjecture we could make is that the perpendicular line from the center of a circle to a chord bisects the chord. This is not a formal proof as proving every theorem would be exhaustive and we only cover four modules of geometry in this course; however, there are many formal proofs, most of which you could view online on your own time.

If you measure the distance from the center of the circle (shown above to the left) to the chords along the perpendicular lines, you will see that they are also equal lengths. The theorem states that two congruent chords of a circle are equidistant from the center of a circle. Sometimes, this is called the Chord Distance to the Center Theorem, and sometimes, it is called the Equidistant Chords Theorem. Rather than remembering the theorem by name, I find it easier to remember the theorem itself and just write out the explanation in the "Reasons" part of a formal proof.

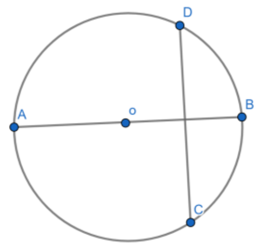
In this module, we will simply discover and use most of the formal proofs. We have done several formal proofs in previous sections and will do more as we progress through the next module, which covers triangles.

Example 1:	Find the measure of $\overline{\mathrm{DE}}$, $\overline{\mathrm{BF}}$, and $\overline{\mathrm{AF}}$ given that $\overline{\mathrm{AB}} \cong \overline{\mathrm{CD}}$ and $\mathrm{CE} = 5$ in.



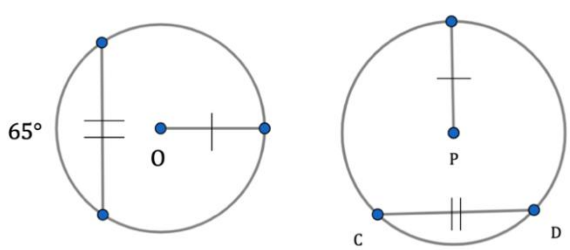
Example 2:	In Example 1, if the measure of OE is 8 in., what segment is congruent to it and why?

Another proof involving chords is as follows: If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord and the arc that is intercepted by the chord. If $\overline{AB}$ is a diameter of circle O and $\overline{AB} \perp \overline{CD}$, then $\overline{CE} \cong \overline{DE}$ and arc BC $\cong$ arc BD. Let point E be the intersection of $\overline{AB}$ and $\overline{CD}$.



Example 3:	Write the converse of the above theorem and use letters and symbols to explain it.

Example 4:	Find the measure of arc CD and explain your reasoning.



Section 5.8 Tangent Theorems

Looking Back 5.8

Given the theorems of chords we learned in the previous section, there is one more that makes logical sense and has a logical name that includes the hypothesis:

"If a perpendicular line inside a circle bisects a chord, then the line passes through the center of the circle."

What is the hypothesis?

It is "a perpendicular line inside a circle bisects a chord." A logical name for the theorem based on the hypothesis is: "The Perpendicular Bisector of a Chord Theorem." Again, I think it is easier just to state the theorem, but this may help some.

Now, let us investigate another new theorem. Draw a circle. Place a ruler across the circle. Trace a chord along either side of the ruler inside the circle whose endpoints are on the circle. Two parallel lines will be created. Two arcs will be created as well. Fold one side of the paper onto the other so the arcs created by the ruler width are aligned. What do you notice?

They are congruent.

If the chords of a circle are parallel, then their intercepted arcs are congruent. This could be called: "Parallel Lines' Intercepted Arcs Congruence Theorem."

Looking Ahead 5.8

Now, let us use a geometer's utility to verify the theorems involving tangents and secants.

Example 1: Draw a circle A. Construct a tangent and call it line B. Construct a radius of circle A perpendicular to tangent line B. Write the theorem as a conditional statement. Write the biconditional statement as well. Is the converse also true?

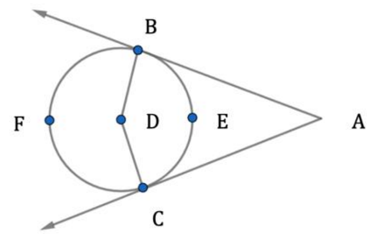
Example 2: Use a geometer's utility to demonstrate that tangent segments to a circle constructed from a common point outside of the circle are congruent.

Example 3: Construct a chord of a circle and construct a tangent to the circle that intersects one of the endpoints of the chord. Use a geometer's utility to demonstrate that the angle formed that is one-half the measure of the intercepted arc would hold true for the circle and the angles.

Recall that a tangent is a line, so a ray and a segment are parts of the tangent line. A tangent segment lies on a tangent line. A tangent to a circle lies in the plane for these theorems to be true.

Example 4: Complete the steps of the proof for the Circumscribed Angles Theorem to show that the measure of a circumscribed angle is equal to 180° minus the measure of the central angle that intercepts the arc formed by the common external points where the tangent rays intersect.

Given: $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are tangent to circle D Prove: $m\ \angle A = 180° - m\ \angle D$



Steps	Reasons
1. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are tangent to circle D	1.
2. $m(\text{arc BEC}) = 360° - m(\text{arc BFC})$	2.
3. $m\ \angle D = m(\text{arc BEC})$	3.
4. $m\ \angle D = 360° - m(\text{arc BFC})$	4.
5. $m\ \angle A + m\ \angle B + m\ \angle C + m\ \angle D = 360°$	5.
6. $m\ \angle B = 90°$ and $m\ \angle C = 90°$	6.
7. $m\ \angle A + 90° + 90° + m\ \angle D = 360°$	7.
8. $m\ \angle A + m\ \angle D = 360° - 180°$	8.
9. $m\ \angle A = 180° - m\ \angle D$	9.

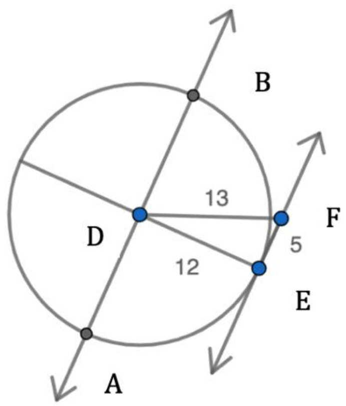
Section 5.9 Secant Theorems

Looking Back 5.9

We have reviewed the tangent theorems. Now, we will review the secant theorems. In the next section, we will explore tangent segment theorems and secant segment theorems.

Recall that a secant intersects a circle in two points. The line segment that lies in the circle with endpoints where the secant crosses the circle is a chord.

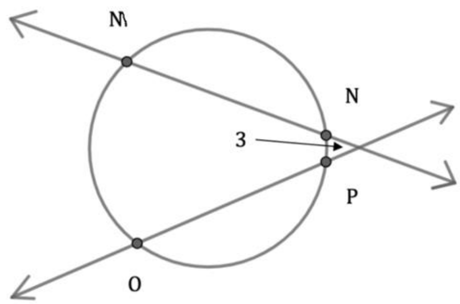
Example 1: What is the name of the circle below? What is the length of the radius? Name a secant. What is the name of the chord that lies on the secant and what is its distance in units?



Example 2: Name the tangent of the circle from Example 1. Verify it is a tangent.

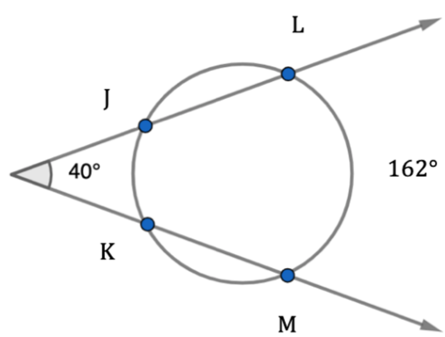
Looking Ahead 5.9

If two secants of a circle intersect outside the circle, then the angle formed where the two secants intersect is one-half the difference of the intercepted arcs.

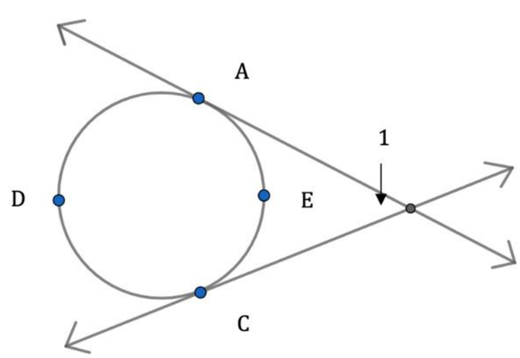


$$m\,\angle 3 = \frac{1}{2}(m(\text{arc MO}) - m(\text{arc NP}))$$

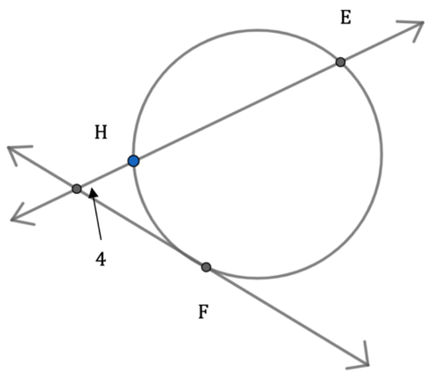
Example 3:	Find the measure of arc JK.



The Angles Outside of a Circle Theorem holds true for two secants, two tangents, or a tangent and a secant.

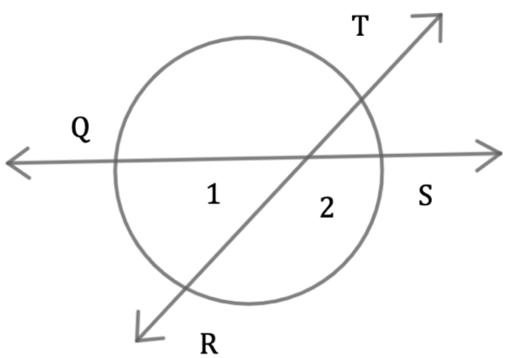


$$m\,\angle 1 = \frac{1}{2}(m(\text{arc ADC}) - m(\text{arc AC}))$$



$$m\angle 4 = \frac{1}{2}(m(\text{arc EF}) - m(\text{arc FH}))$$

If two secants intersect inside a circle, then the measure of each angle is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.



$$m\angle 1 = \frac{1}{2}(m(\text{arc QR}) + m(\text{arc TS}))$$

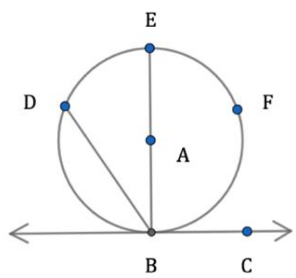
$$m\angle 2 = \frac{1}{2}(m(\text{arc QT}) + m(\text{arc RS}))$$

Example 4: In the figure above, if $m(\text{arc QT})$ is $152°$ degrees and $m(\text{arc RS})$ is $128°$ degrees, find the measure of $\angle 1$ and $\angle 2$.

Section 5.10 Chords, Tangents, and Secants

Looking Back 5.10

In Problem 20 from the practice problems for Section 5.8, the arc addition postulate was the reason for the statement for Step 9: $m(\text{arc DE}) + m(\text{arc EFB}) = m(\text{arc DFB})$ This is true because point E is on arc DEB and between points D and B.

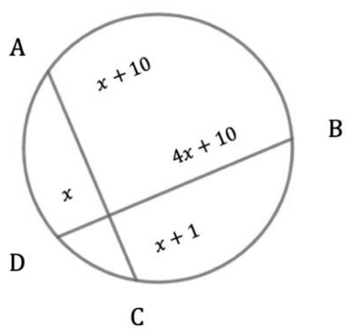


Another use of the Arc Addition Postulate given the diagram to the left is $m(\text{arc BD}) + m(\text{arc DE}) = m(\text{ arc BDE})$. You can see the Arc Addition Postulate is like the Angle Addition Postulate and assumed to be true without proof.

The diagram above has chord $\overline{\text{BD}}$, tangent $\overleftrightarrow{\text{BC}}$, and chord $\overline{\text{BE}}$ that is the diameter of the circle. Now, we will explore three final theorems for chords, tangents, and secants. A secant is a line while a chord is a segment.

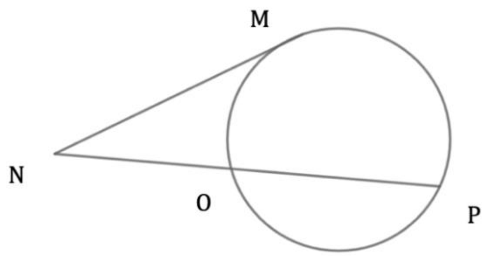
The Segments of a Chord Theorem states that if two chords intersect inside a circle, then the product of the lengths of the two segments of one chord is equal to the product of the lengths of the segments of the other chord.

Example 1:	Find the lengths of AC and BD in the figure below.



Looking Ahead 5.10

A tangent segment is a segment that is tangent to a circle at its endpoint which is on the circle. A tangent segment in the following diagram is $\overline{\text{NM}}$. A secant segment has one endpoint on the circumference of the circle, which contains a chord of the circle and has the other endpoint outside of the circle. The part of the secant segment that is not the chord and lies outside the circle is an external segment.



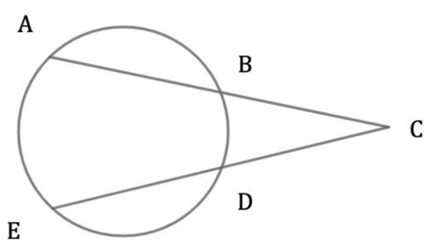
A secant segment is $\overline{NP}$

An external segment is $\overline{NO}$

If a secant segment and a tangent segment meet at a common endpoint outside a circle, then the product of the lengths of the secant segment and its external segment is equal to the square of the length of the tangent segment.

$$NM^2 = NP \cdot NO$$

Example 2:	If $NM = 8$ and $OP = 12$ in the figure above, find the length of NO.



If two secant segments meet at a common endpoint outside of a circle, then the product of the lengths of one secant segment and its external segment is equal to the product of the lengths of the other secant segment and its external segment.

$$AC \cdot BC = EC \cdot DC$$

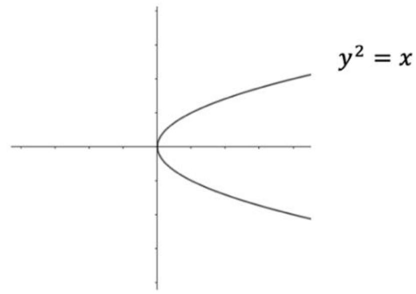
Example 3:	If $AB = 9$, $BC = 3$, and $ED = 5$ in the figure above, find the length of DC.

Section 5.11 Standard Equation of a Circle

Looking Back 5.11

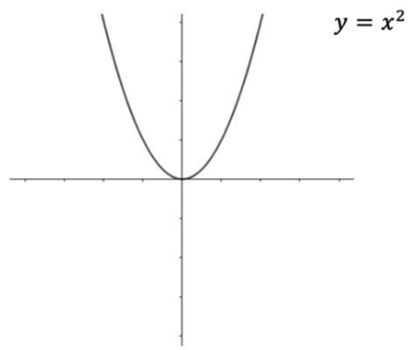
In Algebra 2, you investigated three types of symmetry for reflectional symmetry.

The graph of $y^2 = x$ has reflectional symmetry in the x-axis. If you fold the graph in half through the middle of the x-axis, the top of the graph will align with the bottom of the graph.



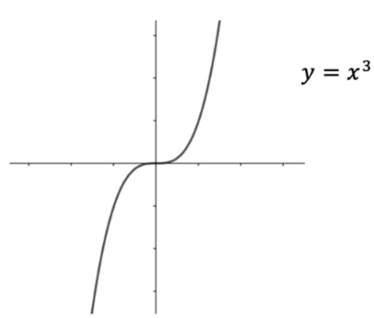
A pair of equations for the graph is $y = \sqrt{x}$ and $y = -\sqrt{x}$. The graph is symmetric about the x-axis because every point $(x, -y)$ lies on the graph of $y^2 = x$ whenever the point (x, y) lies on the graph.

The graph of $y = x^2$ has reflectional symmetry in the y-axis. If you fold the graph in half through the middle of the y-axis, the right half of the graph will align with the left half of the graph.



The graph is symmetric about the y-axis because every point $(-x, y)$ lies on the graph $y = x^2$ whenever the point (x, y) lies on the graph.

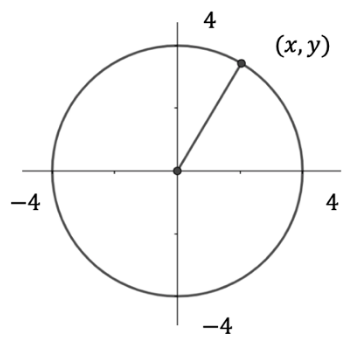
The graph of $y = x^3$ has symmetry about the origin because every point $(-x, -y)$ lies on the graph of $y = x^3$ whenever the point (x, y) lies on the graph.



Example 1:	Is $y = x^3 + 2x$ symmetric about the x-axis, the y-axis, or about the origin?

Looking Ahead 5.11

If a point (x, y) lies in a coordinate plane and is 4 units from the origin, and all points of the graph lie 4 units from the origin, then- by definition of a circle- the graph is a circle.



Using the distance formula: $\sqrt{(x-0)^2 + (y-0)^2} = 4$ and $\sqrt{x^2 - y^2} = 4$

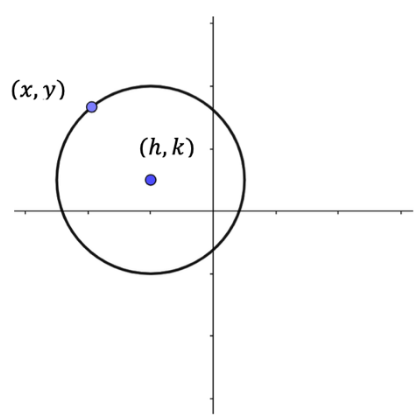
Squaring both sides: $(\sqrt{(x^2 - y^2)})^2 = 4^2$ and $x^2 - y^2 = 16$

The relation is the equation of a circle with a radius of 4 units. If a point (x, y) lies in a coordinate plane on a circle with a radius r and center (h, k), then using the distance formula:

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

and

$$(x-h)^2 + (y-k)^2 = r^2$$



Example 2: Find the equation of a circle with a radius of 7 and a center of $(-5,-3)$.

Example 3: What is the radius of the circle and the center of the circle:

$$(x+7)^2+(y+6)^2=25$$

In Algebra 1 and 2, you learned to complete the square as a method of factoring to find the graphing or vertex form of a quadratic equation. Completing the square can also be used to find the center and radius of a circle whose equation is given.

Example 4: Complete the square to find the radius and center of the circle given the equation:

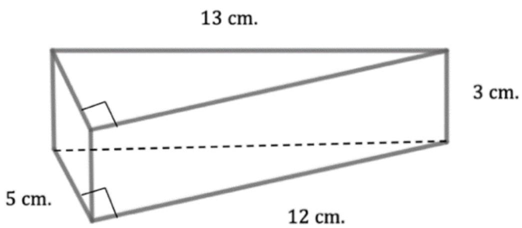
$$x^2-8x+y^2+6y=-9$$

Section 5.12 Surface Area and Circles

Looking Back 5.12

The surface area of a prism is the sum of the surface area of all the lateral faces of a prism. Its bases are the same polygon.

The surface area of a triangular prism is the sum of the area of the two triangles that are the bases and the three rectangles that are the lateral faces of the sides of the prism.



The area of a triangle is: $\mathrm{A} = \frac{1}{2}b \cdot h$; the area of a rectangle is: $\mathrm{A} = b \cdot h$. Therefore, the area of the triangular prism is:

$$\mathrm{S.A.} = 2\left(\frac{1}{2} \cdot 5 \cdot 12\right) + 5 \cdot 3 + 12 \cdot 3 + 13 \cdot 3$$

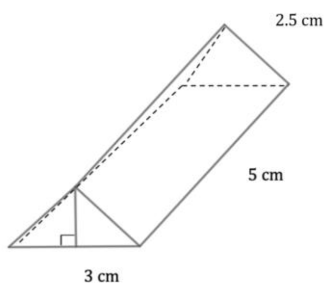
$$\mathrm{S.A.} = 2(6 \cdot 5) + 3(5 + 12 + 13)$$

$$\mathrm{S.A.} = 2(30) + 3(30)$$

$$\mathrm{S.A.} = 60 + 90$$

$$\mathrm{S.A.} = 150\ \mathrm{cm}^2$$

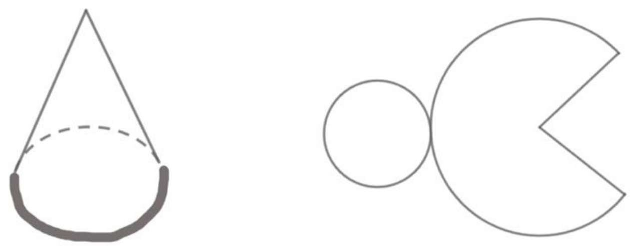
Example 1: A triangular prism has the given dimensions. If the surface area of the triangular prism is $46\ \mathrm{cm}^2$, what is the height of the triangular base? Draw and label the net of the triangular prism.



In the previous Example, a net for a 3-dimensional figure is a 2-dimensional pattern of the figure. The 2-dimensional pattern can be folded into the 3-dimensional figure.

Looking Ahead 5.12

A cone, or a circular cone, has a circular base and a vertex that is in a different plane. The height is the altitude and is the perpendicular distance from the vertex to the base. If the height meets the circular base in the center, then the cone is a right cone. The slant height is the distance from the vertex of the cone to any point on the edge of the circular base.



The surface area of the circular base is πr^2 and the lateral surface of the cone is a sector of a circle. The area of the sector is as follows:

$$\frac{\text{Area of the Sector}}{\text{Area of the Circle}} = \frac{\text{Arc Length of the Sector}}{\text{Circumference of the Circle}}$$

$$\frac{\text{Area of the Sector}}{\pi l^2} = \frac{2\pi r}{2\pi l}$$

$$\text{Area of the Sector} = \frac{2\cancel{\pi} r}{2\cancel{\pi} l} \cdot \pi l^2$$

$$\text{Area of the Sector} = \pi r l$$

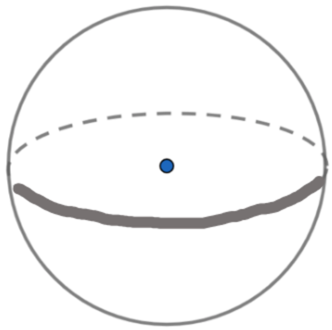
The surface area of a right cone is as follows:

$$\text{S.A.} = \pi r^2 + \pi r l$$

(where r is the radius and l is the slant height)

Example 2:	Find the surface area of a right cone with a radius of 7 inches and a slant height of 21 inches.

A circle is the set of all points in a plane that are equidistant from a given point. A sphere is the set of all points in space equidistant from a given point. The point is the center of the sphere. A chord of a sphere is a segment whose endpoints lie on the sphere. A diameter is a chord that runs through the center of the sphere.



A plane can intersect a sphere in a single point if it just touches it at one point. Hold a ball with one hand and a piece of cardboard with the other. Move them close to one another until they touch so you can visualize this.

A plane can also intersect a sphere in a circle. The circle at the intersection is called a great circle if the plane contains the center of the sphere. The circumference of the sphere is the circumference of the great circle. The great circle separates the sphere into two congruent hemispheres, each being half of the sphere.

If you uncover a baseball and lay the leather covering out flat, it forms two figure eight shapes with four congruent circles.

The surface area of a sphere is as follows:

$$\text{S.A.} = 4\pi r^2$$

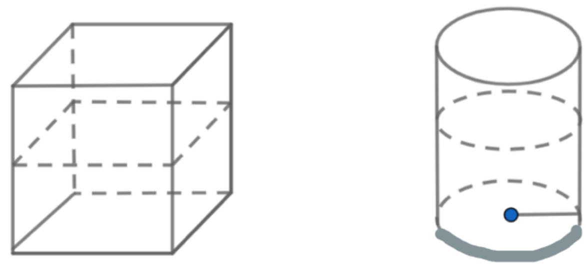
Example 3: Find the surface area of a sphere with a circumference of 16π m. Round the answer to the nearest tenths place.

Section 5.13 Volume and Circles

Looking Back 5.13

The surface area is the number of square units on the outside of a solid. The volume is the number of cubic units on the inside of a solid.

The volume of a rectangular prism that has the same cross-sectional area as a cylinder has the same volume.



The area of the base of the cube is πr^2 and the area of the cylinder is πr^2. If we stack up the areas until the height is reached, we get a volume of $V = \pi r^2 h$ for the cube and the cylinder.

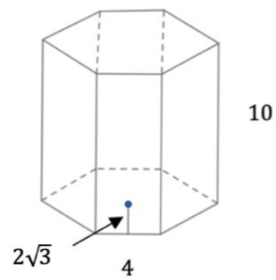
Bonaventura Cavalieri was an Italian mathematician during the 1600s who discovered that different solids with the same cross-sectional areas from top to bottom have the same volume. This led to integral calculus and finding areas under the curve, which we will study in the next and final course in this series.

For now, we will find the volume of a cylinder using the formula above.

Example 1: How many cubic centimeters of beans can be stored in a can with a radius of 1.5 inches and a height of 4 inches?

So, the volume of any prism is the area of the base, B, times the height, h.

Example 2: Find the volume of a hexagonal prism with an apothem of $2\sqrt{3}$ inches, side length of 4 inches, and a height of 10 inches. Then find a formula for the volume of a right hexagonal prism.



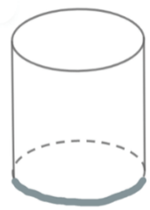
You learned about central angles and apothems with the mirror investigation in Section 4.6 and now you have applied what you know about 2-dimensional polygons to 3-dimensional prisms. Now, let us talk about oblique prisms.

If you stack up 5 pennies, you have a cylinder. If you move each penny a little to the right so the stack does not fall, then the shape will change but the volume will stay the same. An oblique cylinder has the same volume as a right cylinder. The space has shifted so the height of the oblique is the perpendicular height from the top of the stack to the bottom. In an oblique cone or cylinder, the axis is slanted and no longer perpendicular to the base. In an oblique prism, the lateral edges are not perpendicular to the bases.

Example 3: Find the volume of the oblique hexagonal prism with a base of 625 square centimeters and a perpendicular height of 8 centimeters.

There is a relationship between cones and cylinders with congruent bases and the same height, and prisms and pyramids with congruent bases and the same height.

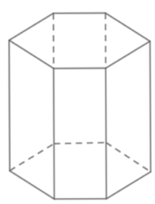
If you take a cone and fill it with sand or water, and then pour it into a cylinder with a congruent base and the same height, it will only fill it one-third of the way up. You must fill the cone and pour it into the cylinder three times for the cylinder to be full.



Cylinder: $\mathrm{V} = \mathrm{B}h$

Cone: $\mathrm{V} = \frac{1}{3}\mathrm{B}h$

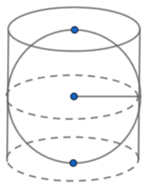
The same holds true for a pyramid and prism that have a congruent base and equal height.



Hexagonal Prism: $\mathrm{V} = \mathrm{B} \cdot h$

Hexagonal Pyramid: $\mathrm{V} = \frac{1}{3}\mathrm{B} \cdot h$

The pyramid must be filled and poured into the prism three times to fill it completely.



A relationship can be made between the volume of a hemisphere with a radius r (of the base) and a height $2r$. This is the smallest cylinder that encloses a sphere.

If you fill the hemisphere with rice or birdseed and pour it into the cylinder, it only fills one-third of the cylinder. Two hemispheres or one full hemisphere of rice/birdseed only fills two-thirds of the cylinder.

The volume of the cylinder in terms of the common radius is shown as follows:

$$\text{V} = \pi r^2 \cdot 2r \text{ or } \text{V} = 2\pi r^3$$

The volume of the sphere is only two-thirds of the volume of the cylinder:

$$\text{V} = \frac{2}{3} \cdot 2\pi r^3 \text{ or } \text{V} = \frac{4}{3}\pi r^3$$

Example 4: A giant egg on display is to be filled with mini chocolate eggs. The egg on display is open and only the bottom of it will be filled. Each mini chocolate egg that will be used to fill it is 3 cubic centimeters. How many mini eggs are needed to fill the bottom of the giant egg, which has a diameter of 12 inches? (2.54 cm. = 1 in.)